Stability Analysis of Reaction-Diffusion Equations with Double Diffusivity System

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ABSTRACT

Stability analysis for steady state solution of reaction-diffusion equations with double diffusivity discuss and arise in the solution of problems of flow of homogeneous liquids and heat conduction involving air-temperature $u(x,t)$ and a grain-temperature $v(x,t)$, the resulting of this analysis shows that the system is stable when:
$Dk^4 + a_1'Lk^2 + b_2'Lk^2 + b_2'a_1'L^2 - b_1'a_2'L^2 > 0$.

Keywords: Stability Analysis, Reaction-Diffusion Equation, Double Diffusivity System.

تحليل استقرارية معادلات الانتشار – التفاعل ذات نظام الانتشار المزدوج

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الملخص

تم دراسة استقرارية الحل اللازمني لنموذج من معادلات الانتشار–التفاعل ذات الانتشار المزدوج (double diffusivity)، والتي تظهر في حل مشاكل جريان الموائع المتجانسة وفي توصيل الحرارة المتضمن درجة حرارة الهواء $u(x,t)$ ودرجة حرارة حبة القمح $v(x,t)$، وقد تبين من تحليل النتائج أن هذا النظام مستقر إذا كان:

$Dk^4 + a_1'Lk^2 + b_2'Lk^2 + b_2'a_1'L^2 - b_1'a_2'L^2 > 0$

الكلمات المفتاحية : تحليل الاستقرارية، معادلات الانتشار–التفاعل ، الانتشار المزدوج.

1. Introduction

Reaction-diffusion give rise to texture synthesis based on the simulation of a process of local nonlinear interactions, which has been proposed as a model of biological pattern formation. A chemical mechanism that was first proposed by Alan [10] (1952) to account for pattern formation in biological morphogenesis, he postulated that patterning is governed primarily by concurrently operating processes: diffusion of morphogens through the tissue and chemical reactions that produce and destroy morphogens at a rate that depends, among other things, on their concentrations. Such mechanisms are called reaction-diffusion (RD) systems [11].

Antic and Hill (2000) [1] are studied a mathematical model for heat transfer in grain store microclimates, this model is "double diffusivity" such that they are used the heat-balance integral method to transform the coupled partial differential equations to the coupled ordinary differential equations and solved it numerically by using the Fehlberg fourth-fifth order Range-kutta method.

Aggarwala and Nasim [2] derived the solution of reaction-diffusion equations with double diffusivity by laplace technique and fourier transforms which appear to be simpler and more direct.

Chow Tanya [4] are studied the derivation of similarity solutions for one-dimensional coupled systems of reaction-diffusion equations, these solutions are obtained by means of one-parameter group methods.

Molz [7] used a coupled system of a model for water transport through plant tissue and Rubinsein [9] is able to derive a coupled system in a one-dimensional case includes heat conduction in heterogeneous media.

In this paper, we will study the stability analysis of steady state solution of double diffusivity model.

2. Model of equations:

The one-dimensional case of reaction-diffusion equations with double-diffusivity is given by [2]

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} - a_1 u(x,t) + b_1 v(x,t) \quad \ldots(1a)$$

$$\frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + a_2 u(x,t) - b_2 v(x,t) \quad \ldots(1b)$$

where $u(x,t)$ and $v(x,t)$ denote the air-temperature and a grain-temperature respectively, the self-diffusivities D_1, D_2, a_1, b_1, a_2 and b_2 are positive constants, with initial conditions:

$$u(x,0) = u_0$$

$$v(x,0) = v_0$$

where u_0 and v_0 are constant interior solutions of (1a)-(1b) and zero flux boundary conditions on u and v [2]:

$$\frac{\partial u}{\partial x} = 0 \quad \text{at } x=0 \quad \ldots(2a)$$

$$\frac{\partial v}{\partial x} = 0 \quad \text{at } x=0 \quad \ldots(2b)$$

when $a_1 = a_2$ and $b_1 = b_2$ this system represent mathematical model for heat transfer in grain store microclimates[1].

For dimensionless form, we introduce the following dimensionless quantities:

$$x' = \frac{x}{L},\ t' = \frac{tD_2}{L^2},\ u' = \frac{u}{U},\ v' = \frac{v}{U},\ D = \frac{D_1}{D_2},\ a_1 = \frac{a_1' D_2}{L},\ a_2 = \frac{a_2' D_2}{L},\ b_1 = \frac{b_1' D_2}{L},\ b_2 = \frac{b_2' D_2}{L}$$

substitute these non-dimensional quantities into equations (1a) , (1b) and the conditions (2a),(2b) we get:

$$\frac{\partial u'}{\partial t'} = D \frac{\partial^2 u'}{\partial x'^2} - a_1' L u'(x',t') + b_1' L v'(x',t') \quad \ldots(3a)$$

$$\frac{\partial v'}{\partial t'} = \frac{\partial^2 v'}{\partial x'^2} + a_2' L u'(x',t') - b_2' L v'(x',t') \quad \ldots(3b)$$

$$\frac{\partial u'}{\partial x'} = 0 \quad \text{at } x=0 \quad \ldots(3c)$$

$$\frac{\partial v'}{\partial x'}=0 \quad \text{at} \quad x=0 \qquad \ldots(3d)$$

3. Stability analysis:

Assume that the value of the concentrations u and v has the following from [5]:

$$u'(x',t')=u_1'(x')+u_2'(x',t')$$
$$v'(x',t')=v_1'(x')+v_2'(x',t') \qquad \ldots(4)$$

where $u_1^{'}(x^{'})$ and $v_1^{'}(x^{'})$, denote the steady state case, $u_2^{'}(x^{'},t^{'})$ and $v_2'(x',t')$ denote the disturbance case.
If we substituted (4) in equations (3a)-(3d), we get the following systems:
The steady state system:

$$D\frac{d^2u_1'}{dx'^2}-a_1'Lu_1'(x')+b_1'Lv_1'(x')=0 \qquad \ldots(5a)$$

$$\frac{d^2v_1'}{dx'^2}+a_2'Lu_1'(x')-b_2'Lv_1'(x')=0 \qquad \ldots(5b)$$

with the boundary conditions:

$$\frac{du_1'}{dx'}=0 \quad \text{at} \quad x=0 \qquad \ldots(5c)$$

$$\frac{dv_1'}{dx'}=0 \quad \text{at} \quad x=0 \qquad \ldots(5d)$$

the second system:

$$\frac{\partial u_2'}{\partial t'}=D\frac{\partial^2 u_2'}{\partial x'^2}-a_1'Lu_2'(x',t')+b_1'Lv_2'(x',t') \qquad \ldots(6a)$$

$$\frac{\partial v_2'}{\partial t'}=\frac{\partial^2 v_2'}{\partial x'^2}+a_2'u_2'(x',t')-b_2'Lv_2'(x',t') \qquad \ldots(6b)$$

with the related boundary conditions:

$$\frac{\partial u_2'}{\partial x'}=0 \quad \text{at} \quad x=0 \qquad \ldots(3c)$$

$$\frac{\partial v_2'}{\partial x'}=0 \quad \text{at} \quad x=0 \qquad \ldots(3d)$$

4. Solution of the steady state case:

To find the solution to (5a)-(5b) with the boundary conditions (5c)-(5d), we shall use the technique which Benson. D. and sherratt. J. [3] are used for the solution the linearized model of coupled ordinary differential equations as follows:

$$\frac{d^2u_1'}{dx'^2}-\frac{a_1'L}{D}u_1'(x')+\frac{b_1'L}{D}v_1'(x')=0 \qquad \ldots(7a)$$

$$S\frac{d^2v_1'}{dx'^2}+Sa_2'Lu_1'(x')-Sb_2'Lv_1'(x')=0 \qquad \ldots(7b)$$

adding (7a) to (7b) gives:

$$\left(\frac{d^2u_1'}{dx'^2}+S\frac{d^2v_1'}{dx'^2}\right)+\left(Sa_2'L-\frac{a_1'L}{D}\right)u_1'+\left(\frac{b_1'L}{D}-Sb_2'L\right)v_1'=0$$

$$\Rightarrow\left(\frac{d^2(u_1'+Sv_1')}{dx'^2}\right)+\left(Sa_2'L-\frac{a_1'L}{D}\right)\left(u_1'+\frac{\left(\frac{b_1'L}{D}-Sb_2'L\right)}{\left(Sa_2'L-\frac{a_1'L}{D}\right)}v_1'\right)=0 \qquad \ldots(8)$$

we choose (S) such that:

$$S=\frac{\left(\frac{b_1'L}{D}-Sb_2'L\right)}{\left(Sa_2'L-\frac{a_1'L}{D}\right)}$$

which is a quadratic equation for (S) and thus:

$$Da_2'LS^2+\left(Db_2'L-a_1'L\right)S-b_1'L=0$$

$$S=\frac{-\left(Db_2'L-a_1'L\right)\pm\sqrt{\left(Db_2'L-a_1'L\right)^2+4Da_2'L^2b_1'}}{2DLa_2'}$$

which given:

$$S_1=\frac{-\left(Db_2'L-a_1'L\right)-\sqrt{\left(Db_2'L-a_1'L\right)^2+4Da_2'L^2b_1'}}{2DLa_2'}$$

$$S_2=\frac{-\left(Db_2'L-a_1'L\right)+\sqrt{\left(Db_2'L-a_1'L\right)^2+4Da_2'L^2b_1'}}{2DLa_2'}$$

Equation (8) becomes:

$$\frac{d^2\left(u_1'+S_jv_1'\right)}{dx'^2}+\left(S_ja_2'L-\frac{a_1'L}{D}\right)\left(u_1'+S_jv_1'\right)=0\text{, for j=1} \qquad \ldots(9)$$

The general solution of (9) is:

$$u_1'+S_jv_1'=A_j\cos\left(\sqrt{\lambda}x\right)+B_j\sin\left(\sqrt{\lambda}x\right)\text{, for j=1,2} \qquad \ldots(10)$$

here A_j and B_j are constants of integration, and

$$\lambda_j=\left(S_ja_2'L-\frac{a_1'L}{D}\right)\text{, for j=1,2}$$

where:

$$\lambda_1=\left(S_1a_2'L-\frac{a_1'L}{D}\right)$$

$$\lambda_1=\frac{-\left(Db_2'L+a_1'L\right)-\sqrt{\left(Db_2'L-a_1'L\right)^2+4Da_2'L^2b_1'}}{2D}$$

and the other solution is:

$$\lambda_2=\left(S_2a_2'L-\frac{a_1'L}{D}\right)$$

$$\lambda_2=\frac{-\left(Db_2'L+a_1'L\right)+\sqrt{\left(Db_2'L-a_1'L\right)^2+4Da_2'L^2b_1'}}{2D}$$

Then by applying zero flux boundary conditions (5c)-(5d),we get:

$$u_1'(x')=S_2\cos\left(\sqrt{\lambda_1}x'\right)-S_1\cos\left(\sqrt{\lambda_2}x'\right)$$
$$v_1'(x')=\cos\left(\sqrt{\lambda_2}x'\right)-\cos\left(\sqrt{\lambda_1}x'\right)$$

5. Stability analysis (disturbance case):

Assume that the value of $u_2'(x',t')$ and $v_2'(x',t')$, has the following form ([6],[8]):

$$u_2'(x',t') = F_1 e^{ik(x'-ct')} \qquad \ldots(11)$$

$$v_2'(x',t') = F_2 e^{ik(x'-ct')}$$

here $(c = c_1 + ic_2)$ is an eigenvalue represent the speed of the wave, the functions F_1 and F_2 are the constant amplitudes, k is the wave number. The flow is stable if the linearized equation correspond to eigenvalue c with negative part $(c_2 < 0)$ for presented configurations.

Now, substitute (11) in the equations (6a)-(6b), we get respectively:

$$-ik(c_1 + ic_2)F_1 = -Dk^2 F_1 - a_1' L F_1 + b_1' L F_2$$

$$-ik(c_1 + ic_2)F_2 = -k^2 F_2 + a_2' L F_1 - b_2' L F_2$$

by separate the real part and imaginary part, we get:

$$kc_2 F_1 = -Dk^2 F_1 - a_1' L F_1 + b_1' L F_2 \qquad \ldots(12a)$$

$$kc_2 F_2 = -k^2 F_2 + a_2' L F_1 - b_2' L F_2 \qquad \ldots(12b)$$

from (12a) we get:

$$(kc_2 + Dk^2 + a_1' L)F_1 = b_1' L F_2$$

$$F_1 = \left(\frac{b_1' L}{(kc_2 + Dk^2 + a_1' L)} \right) F_2 \qquad \ldots(13)$$

we substitute (13) in the equation (12b), we get:

$$kc_2 F_2 = -k^2 F_2 + \frac{a_2' b_1' L^2}{(kc_2 + Dk^2 + a_1' L)} F_2 - b_2' L F_2$$

$$(k^2 c_2^2 + Dk^3 c_2 + a_1' L k c_2 + k^3 c_2 + k^4 D + a_1' L k^2 - a_2' L^2 b_1' + b_2' L k c_2 + b_2' L D k^2 + b_2' L^2 a_1') F_2 = 0$$

such that $F_2 \neq 0$, then we get:

$k^2 c_2^2 + (Dk^3 + a_1' L k + k^3 + b_2' L k)c_2 + (k^4 D + a_1' L k^2 - a_2' L^2 b_1' + b_2' L D k^2 + b_2' L^2 a_1' = 0$ when $a_1' = a_2'$ and $b_1' = b_2'$ [1] we get:

$$c_2 = \frac{-(Dk^3 + a_1' L k + k^3 + b_2' L k)}{2k^2} \mp \frac{\sqrt{(Dk^3 + a_1' L k + k^3 + b_2' L k)^2 - 4k^2(k^4 D + a_1' L k^2 + b_2' L D k^2)}}{2k^2}$$

then:

$$c_2 = \frac{-(Dk^2 + a_1' L + k^2 + b_2' L)}{2k} \mp \frac{\sqrt{(Dk^2 + a_1' L + k^2 + b_2' L)^2 - 4(k^4 D + a_1' L k^2 + b_2' L D k^2)}}{2k}$$

since $a_1' > 0, b_2' > 0, D > 0, L > 0$ and $k > 0$ then:

$$(Dk^2 + a_1' L + k^2 + b_2' L) > \sqrt{(Dk^2 + a_1' L + k^2 + b_2' L)^2 - 4(k^4 D + a_1' L k^2 + b_2' L D k^2)}$$

this system is stable always.

If $a_1' \neq a_2'$ and $b_1' \neq b_2'$ [2] we get:

$$c_2 = \frac{-(Dk^3 + a_1'Lk + k^3 + b_2'Lk)}{2k^2} \mp \frac{\sqrt{\left(Dk^3 + a_1'Lk + k^3 + b_2'Lk\right)^2 - 4k^2\left(k^4D + a_1'Lk^2 + b_2'LDk^2 - a_2'b_1'L^2 + a_1'b_2'L^2\right)}}{2k^2}$$

$$c_2 = \frac{-(Dk^3 + a_1'Lk + k^3 + b_2'Lk)}{2k^2} \mp \frac{\sqrt{\left(-Dk^3 - a_1'Lk + k^3 + b_2'Lk\right)^2 + 4k^2L^2a_2'b_1'}}{2k^2} \quad \ldots(14)$$

either:

$$c_2 = \frac{-(Dk^2 + a_1'L + k^2 + b_2'L)}{2k} - \frac{\sqrt{\left(-Dk^2 - a_1'L + k^2 + b_2'L\right)^2 + 4L^2a_2'b_1'}}{2k} \quad \ldots(15)$$

such that $L > 0, D > 0, k > 0, a_1' > 0, a_2' > 0, b_1' > 0, b_2' > 0$ and $c_2 < 0$ always, thus the system is stable.

or:

$$c_2 = \frac{-(Dk^2 + a_1'L + k^2 + b_2'L)}{2k} + \frac{\sqrt{\left(-Dk^2 - a_1'L + k^2 + b_2'L\right)^2 + 4L^2a_2'b_1'}}{2k} \quad \ldots(16)$$

and the system is stable when:

$$(Dk^2 + a_1'L + k^2 + b_2'L)^2 > \left(-Dk^2 - a_1'L + k^2 + b_2'L\right)^2 + 4L^2a_2'b_1'$$

and unstable otherwise.

The neutral stability curve, when $(c_2 = 0)$ in equation (16) is:

$$(Dk^2 + a_1'L + k^2 + b_2'L)^2 = \left(-Dk^2 - a_1'L + k^2 + b_2'L\right)^2 + 4L^2a_2'b_1'$$

$$\Rightarrow Dk^4 + a_1'Lk^2 + b_2'LDk^2 + a_1'b_2'L^2 - a_2'b_1'L^2 = 0$$

$$\Rightarrow Dk^4 + (a_1'L + b_2'LD)k^2 + (a_1'b_2'L^2 - a_2'b_1'L^2) = 0$$

$$k^2 = \frac{-\left(a_1'L + b_2'LD\right) \mp \sqrt{\left(a_1'L + b_2'LD\right)^2 - 4D\left(a_1'b_2'L^2 - a_2'b_1'L^2\right)}}{2D} \quad \ldots(17)$$

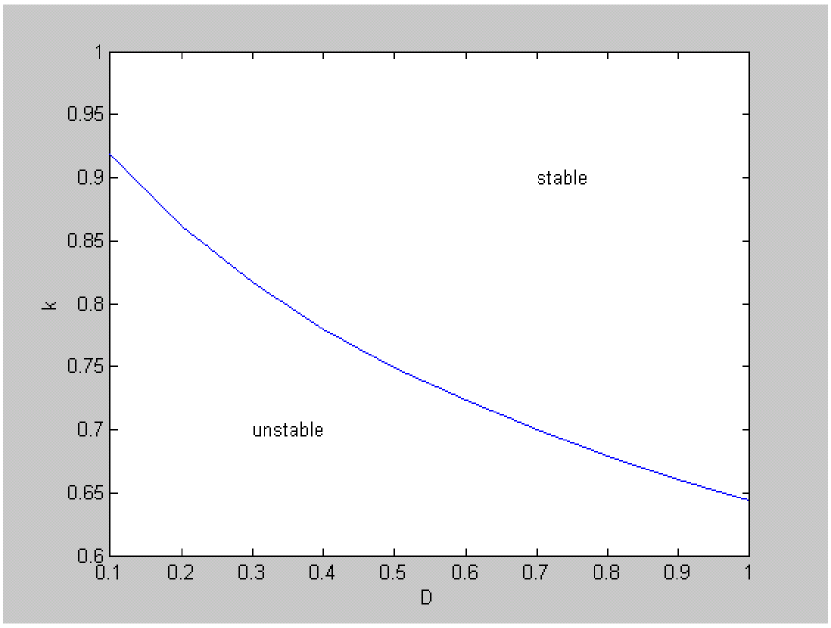
Table(1)

$a_1' = 1, b_2' = 1, L = 1$ $a_2' = 1, b_1' = 2$, D	$a_1' = 1, b_2' = 1, L = 1$ $a_2' = 1, b_1' = 2$, k	$a_1' = 1, b_2' = 1, L = 1$ $a_2' = 2, b_1' = 2$, D	$a_1' = 1, b_2' = 1, L = 1$ $a_2' = 2, b_1' = 2$,. k
0.0500	0.9554	0.0500	1.5962
0.1000	0.9189	0.1000	1.5040
0.1500	0.8880	0.1500	1.4342
0.2000	0.8612	0.2000	1.3780
0.2500	0.8376	0.2500	1.3312
0.3000	0.8165	0.3000	1.2910
0.3500	0.7974	0.3500	1.2559
0.4000	0.7801	0.4000	1.2247
0.4500	0.7641	0.4500	1.1968
0.5000	0.7494	0.5000	1.1714
0.5500	0.7357	0.5500	1.1483
0.6000	0.7229	0.6000	1.1270

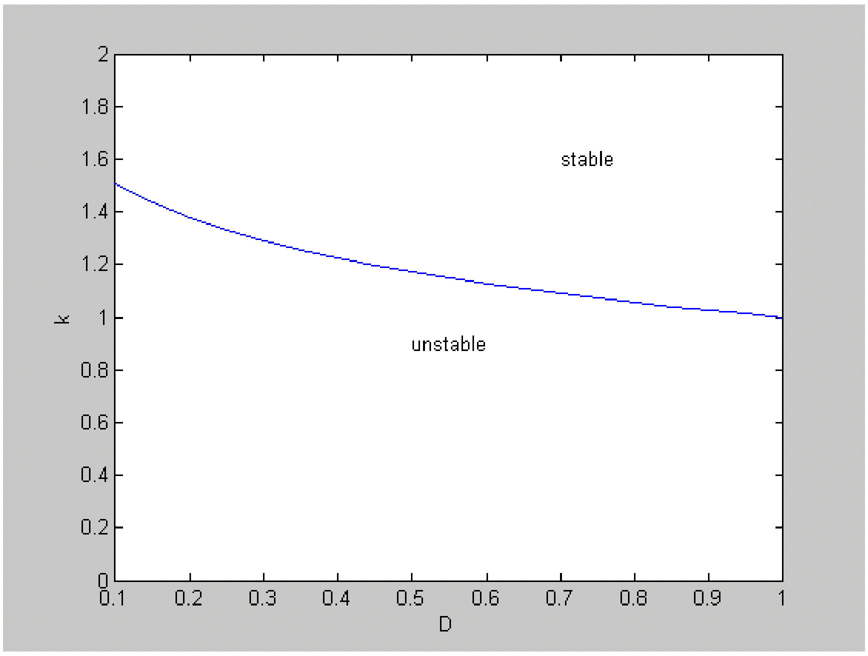
0.6500	0.7109	0.6500	1.1073
0.7000	0.6997	0.7000	1.0889
0.7500	0.6891	0.7500	1.0718
0.8000	0.6790	0.8000	1.0557
0.8500	0.6695	0.8500	1.0406
0.9000	0.6604	0.9000	1.0263
0.9500	0.6518	0.9500	1.0128
1.0000	0.6436	1.0000	1.0000

Table(2)

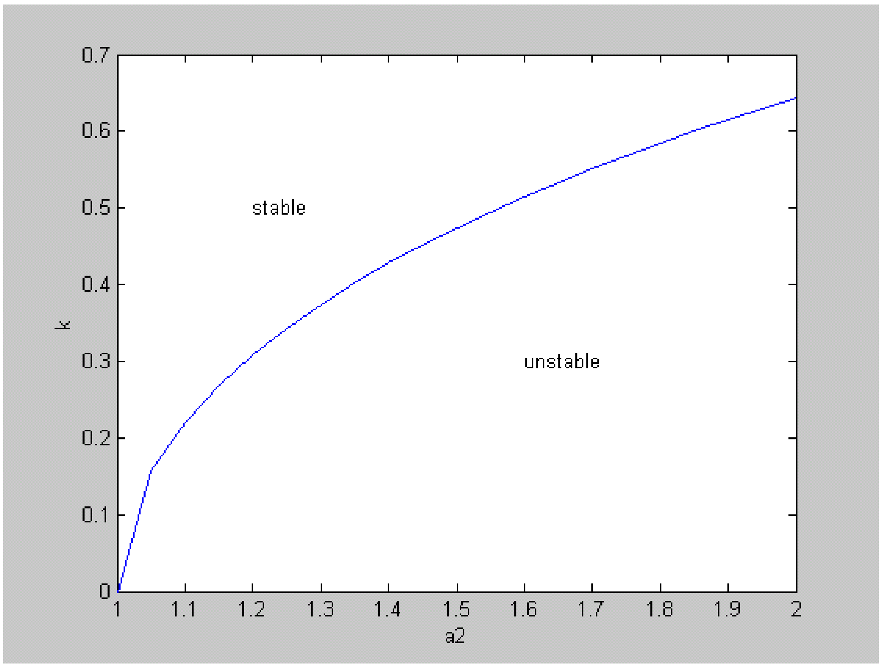
$a'_1 = 1, b'_2 = 1, L = 1$ $D = 1, b'_1 = 1$	$a'_1 = 1, b'_2 = 1, L = 1$ $D = 1, b'_1 = 1$
a'_2	k
1.0000	0.0000
1.0500	0.1571
1.1000	0.2209
1.1500	0.2690
1.2000	0.3089
1.2500	0.3436
1.3000	0.3744
1.3500	0.4024
1.4000	0.4280
1.4500	0.4518
1.5000	0.4741
1.5500	0.4950
1.6000	0.5147
1.6500	0.5334
1.7000	0.5512
1.7500	0.5682
1.8000	0.5845
1.8500	0.6001
1.9000	0.6151
1.9500	0.6296
2.0000	0.6436



Fig(1).The natural stability curve in (17) when $a'_1 = 1, a'_2 = 1, b'_2 = 1, b'_1 = 2, L = 1$



Fig(2).The natural stability curve in (17) when
$a'_1 = 1, a'_2 = 2, b'_2 = 1, b'_1 = 2, L = 1$



Fig(3).The natural stability curve in (17) when
$a'_1 = 1, D = 1, b'_2 = 1, b'_1 = 1, L = 1$

6-conclusion:

The main conclusion, and from the figures is that the system is stable if $Dk^4 + a'_1 Lk^2 + b'_2 Lk^2 D + b'_2 a'_1 L^2 - b'_1 a'_2 L^2 > 0$, however when the coefficients a'_2 and b'_1 are increase then the unstable region is increase as shown in figures (1) , (2) and (3) and table (1) and (2).

REFERENCE

[1] Antic, A. and Hill, J. M., (2000), A mathematical model for heat transfer grain store microclimates, Anziam J. 42(E), pp. c117-133.

[2] Aggarwala, B. D. and Nasim, C., (1987), On the solution of reaction-diffusion equations with double diffusivity, Internet. J. Math. And Math. Sci., Vol. 10, No. 1, pp. 163-172.

[3] Benson, D. L., Sherratt, J. A. and Maini, P. K., (1993), Diffusion driven instability in an inhomogeneous domain, Bulletin of mathematical biology, Vol. 55, No. 2, pp. 365-384.

[4] Chow, T. L. M., (1996), System of partial differential equations and group methods, Thesis, University of western Sydney.

[5] Logan. J. D., (1987), Applied mathematics, Johon wiley and sons.

[6] Manaa, S. A., (2003), Stability analysis of thermal radiation effect in the MHD ekman layer on a porous plate, optically thich limit, Tikrit J. for pure sci., vol. 9, No. 1, pp. 112-119.

[7] Molz, F. J., (1976), Water transport through plant tissue: the apoplasim and symplasim pathways, J. of theoretical biology 59, pp. 277-292.

[8] Mosa, M. F. & Ibrahim, B. M.,(2001), Stability analysis and chaos in a bend duct, Raf. J. Sci., Vol. 12, No. 1, pp. 91-99.

[9] Rubinstein, L. I., (1948), Process of conduction of heat in heterogeneous media, Izvestiya akademii nauk sssr geogr., 12, pp. 27-45.

[10] Turing. A., (1952), The chemical basis of morphogenesis, Philosophical transactions of the royal society (B), 237: pp. 37-72.

[11] Witkin, A. and Kass, M.,(1991), Reaction-diffusion textures,computer graphics, 25(3).