Wnil-Injective Modules

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ABSTRACT

A right module M is called Wnil-injective if for any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism $f : a^nR \rightarrow M$ can be extended to $R \rightarrow M$. In this paper , we first give and develope various properties of right Wnil-injective rings, by which, many of the known results are extended. Also, we study the relations between such rings and reduced rings by adding some types of rings, such as SRB-rings, and other types of rings.

Keywords: Wnil-Injective Modules, n- regular rings, reduced rings, SRB-rings.

المقاسات الغامرة من النمط – Wnil

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الملخص

يقال للمقاس الأيمن M انه غامر من النمط Wnil إذا كان لكل $a \in N(R)$ يوجد عدد صحيح موجب n بحيث أن $a^n \neq 0$ وان أي هومومورفيزم $f : a^nR \rightarrow M$ يمكن توسعته الى $R \rightarrow M$. في هذا البحث أعطينا أولا خواصا متنوعة للحلقات الغامرة من النمط Wnil, كما قمنا بتوسيع عدد من النتائج المعروفة . كذلك درسنا العلاقة بين تلك الحلقات والحلقات المختزلة بإضافة بعض أنواع الحلقات ومنها, مثلا الحلقات SRB, وأنواع أخرى من الحلقات.

الكلمات المفتاحية : المقاسات الغامرة من النمط –Wnil , الحلقات المنتظمة من النمط –n , الحلقات المختزلة, الحلقات من النمط - SRB

1. Introduction :

Throughout this paper R is associative ring with identity, and R-module is unital. For $a \in R$, r(a) and l(a) denote the right annihilator and the left annihilator of a, respectively. We write J(R), Y(R) (Z(R)), N(R) and Soc(R_R) (Soc($_RR$)) for the Jacobson radical, the right (left) singular ideal, the set of nilpotent elements and right (left) socle of R, respectively.

(1) A right R-module M is called **nil-injective** [7] if for any $a \in N(R)$, any R- homomorphism $f : aR \to M$ can be extended to $R \to M$. Or equivalently, there exists $m \in M$ such that $f(x) = mx$ for all $x \in aR$. If R_R is nil-injective, then we call R is a right nil-injective ring. (2) A ring R is said to be **right NPP** if aR is projective for all $a \in N(R)$. (3) A ring R is (Von Neumann) **regular** [4] provided that for every $a \in R$ there exists $b \in R$ such that $a = aba$.
(4) A ring R is called **reduced** if it contains no non-zero nilpotent elements. (5) A ring R is called **n-regular** [7] if $a \in aRa$ for all $a \in N(R)$. Clearly, regular rings are n-regular, and reduced ring is right nil-injective. (6) According to Cohn [2], a ring R is called **reversible** if $ab = 0$ implies $ba = 0$ for $a,b \in R$. A ring R is called **strongly right bounded** (briefly SRB) [3] if every non-zero right ideal contains a non-zero two-sided ideal of R.(7) A ring R is called right **minsymmetric [7]** if kR minimal, $k \in R$, implies that Rk is minimal.

2. Wnil-Injective Rings

This section is devoted to study Wnil-injective rings with some of their basic properties. Also we give a relation between such rings with n-regular rings, and reduced rings.

<u>Definition 2.1 :</u> [7]

A right module M is called **Wnil-injective** if for any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism $f : a^nR \to M$ can be extended to $R \to M$. Or equivalently, there exists $m \in M$ such that $f(x) = mx$ for all $x \in a^nR$.

Clearly every right nil-injective module is right Wnil-injective. If R_R is Wnil-injective we call R is a right Wnil-injective ring.

<u>Lemma 2.2 :</u> [7]

A ring R is right nil-injective if and only if $lr(a) = Ra$ for all $a \in N(R)$. ■

We start the section with the following theorem which extends Lemma 2.2

Theorem 2.3 :

A ring R is a right Wnil-injective if and only if for any $0 \neq a \in N(R)$ there exists a positive integer n such that $a^n \neq 0$ and $Ra^n = lr(a^n)$.

Proof

Suppose that a ring R is right Wnil-injective. Then for every $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into R extends to endomorphism of R_R. It is clear $Ra^n \subseteq lr(a^n)$. Let $d \in lr(a^n)$, since $r(a^n) = r(l (r (a^n))) \subseteq r(d)$.We may define a right R-homomorphism $f : a^n R \rightarrow R$ by $f(a^nb) = db$ for all $b \in R$. Since R_R is Wnil-injective, there exists $y \in R$ such that $f(a^n) = ya^n$. Then $d = f(a^n) \in Ra^n$ which implies that $lr (a^n) \subseteq Ra^n$ and so that $lr(a^n) = Ra^n$.
Conversely, If $c \in N(R)$, there exists a positive integer n such that $Rc^n = lr(c^n)$. Let $f : c^nR \rightarrow R$ be any right R-homomorphism , then $r(c^n) \subseteq r(f(c^n))$ which implies $Rf(c^n) \subseteq lr(R(f (c^n))) \subseteq lr(c^n)=lr(Rc^n)=Rc^n$, and therefore $f(c^n) = dc^n$ for some $d \in R$. This shows that R is a right Wnil-injective ring . ■

Lemma 2.4 : [1]

Let R be a ring and $a \in R$. If $a^n - a^nr\, a$ is regular for some positive integer n and $r \in R$, then there exists $y \in R$ such that $a^n = a^nya$. ■

Lemma 2.5 : [4]

The following conditions are equivalent for a ring R
(1) R is a regular ring.
(2) every principal right ideal of R is generated by idempotent. ■

Next we prove the following result :

Theorem 2.6 :

The following conditions are equivalent
(1) R is n-regular.
(2) $N_1(R) = \{ 0 \neq x \in R \mid x^2 = 0 \}$ is regular.
(3) For any $a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and a^nR is generated by an idempotent element.

Proof :

It is clear that (1) $\Rightarrow$ (2) and (3)$\Rightarrow$(2).

(2) $\Rightarrow$ (3) : Let $a \in N(R)$ such that $a^{n+1} = 0$ and $a^n \neq 0$. Then $(a^n)^2 = 0$ so that $a^n \in N_1(R)$. This implies that a^n is regular element of R and so that there exists $b \in R$ such that $a^n = a^n b a^n$, which implies that $a^n R = eR$ by Lemma 2.5 and $a^n \neq 0$ for some $0 \neq e^2 = e \in R$.

(3) $\Rightarrow$ (1) : Let $a \in N(R)$, then there exits integer n such that $a^n R$ is generated by idempotent element. Hence $a^n = a^n ba$ for some $b \in R$.We shall show that a is regular. In fact, if n = 1, then it holds. Let n >1.

Put $d = a^{n-1}-a^{n-1}ba$, then ad =0. If d=0, then $a^{n-1} = a^{n-1}ba$. If $d\neq 0$ then $d^2 = (a^{n-1} - a^{n-1}ba)d = 0$.Since $N_1(R)$ is regular by (2), d is regular. Hence $a^{n-1} = a^{n-1}ya$ for some $y \in R$ by Lemma 2.4. Therefore we always have $a^{n-1} = a^{n-1}z_1a$ for some $z_1 \in R$. If n-1 > 1, then there exists $z_2 \in R$ such that $a^{n-2} = a^{n-2}z_2a$ by the proceeding proof. Repeating the above-mentioned process, we get that there is $x \in R$ such that a = axa, i.e. a is regular, which implies that R is n-regular ring. ■

The following theorem characterizes n-regular in terms of a right Wnil-injective rings.

Theorem 2.7 :

The following conditions are equivalent for a ring R.

(1) R is n-regular ring.

(2) R is a right Wnil-injective such that every cyclic singular right R-module is nil-injective.

Proof :

(1) $\Rightarrow$(2): It is clear.

Assume (2). Let $b \in N(R)$. There exists a positive integer n such that $b^n \neq 0$ and any right R-homomorphism of b^nR into R extends to endomorphism of R_R.

Then by Lemma 2.2 $Rb^n = lr(b^n) = lr(Rb^n)$. There exists a complement right ideal L of R such that $r(b^n) \oplus L$ is an essential right ideal. By hypothesis, the cyclic singular right R-module $R/(r(b^n) \oplus L)$ is nil-injective. Define a homomorphism $f : b^nR \rightarrow R/(r(b^n) \oplus L)$ by

$f(b^na) = a + (r(b^n) \oplus L)$ for all $a \in R$, then f is well defined. Indeed if $ba_1 = ba_2$ then $a_1-a_2 \in r(b^n) \subseteq (r(b^n) \oplus L)$ so that $f(b^na_1) = f(b^na_2)$. There exists $z \in R$ such that $f(b^na) = z b^na + (r(b^n) \oplus L)$. So that

$1 + (r(b^n) \oplus L) = f(b^n) = z b^n + (r(b^n) \oplus L)$ implies that $1-zb^n = u+v$, where $u \in r(b^n)$, $v \in L$.

For any $s \in r(Rb^n)$, s = us + vs and $vs = s-us \in r(b^n) \cap L = 0$ which yields $v(r(Rb^n)) = 0$ which implies that $v \in l(r(Rb^n)) = Rb^n$. If $v=cb^n$, for some $c \in R$, then $1\text{-}zb^nz=ucb^n$ which implies that $b^n\text{-}b^nzb^n = b^nu + b^ncb^n$; and therefore $b^n = b^n (z+c) b^n$ [since $b^nu = 0$]. This proves that for any $0 \neq b \in N(R)$, there exists a positive integer n such that $b^n \neq 0$ and b^nR is generated by an idempotent element, so by Theorem 2.6 R is an n-regular ring. ■

Corollary 2.8 :

Let R be a right Wnil-injective such that every cyclic singular right R-module is nil-injective, then $N(R)\cap J(R) = 0$

Proof :

If $a \in N(R) \cap J(R)$, then $a = aba$, $b \in R$, by Theorem 2.7. Hence $a(1\text{-}ba)=0$, because $a \in J(R)$. Hence $1\text{-}ba$ is invertable and so $a = 0$. Therefore $N(R)\cap J(R)=0$. ■

Theorem 2.9 :

The following conditions are equivalent for a ring R.
(1) R is a n-regular ring R.
(2) R is a right Wnil-injective a right NPP-ring.

Proof :

(1) $\Rightarrow$(2): It is clear.
(2) $\Rightarrow$ (1): Suppose that $0 \neq a \in N(R)$, then by Lemma 2.2 there exists a positive integer n such that $a^n \neq 0$ and $Ra^n = lr(a^n)$. Since R is a right NPP-ring, $r(a^n) = (1\text{-}e)R$, $0 \neq e^2 = e \in R$. Therefore $Ra^n = Re$ which implies that $a^nR = gR$ for some $0 \neq g^2 = g \in R$. So that, by Theorem 2.6, R is an n-regular ring. ■

3. Ring Whose Simple Singular R-module are Wnil-Injective.

In this section, we give an investigation of several properties for rings whose simple singular right R-modules are Wnil-injective due to J. Ch. Wei and J. H. Chen **[** 7 **]**. Also we study the relations between such rings, reduced rings, by adding some types of rings such as SRB-ring, and other types of rings.

Lemma 3.1 : [5]

If R is a semi-prime and SRB-ring, then R is reduced. ■
We begin with the following theorem.

Theorem 3.2 :

Let R be an SRB-ring whose every simple singular right R-module is Wnil-injective. Then R is a reduced ring.

Proof :

By Lemma 3.1, it is enough to show that R is semi-prime. Suppose that there exists a non-zero right ideal U of R such that $U^2 = 0$. Then there exists a non-zero elements $a \in U$ such that $a^2 = 0$. First observe that r(a) is an essential right ideal of R. If not, there exists a non-zero right ideal L of R such that r(a) $\oplus$ L is right essential in R. Since R is SRB, there is a non-zero ideal I of R such that $I \subseteq L$.Now $aI \subseteq aR \cap I \subseteq r(a) \cap L = 0$. Hence $I \subseteq r(a) \cap L = 0$. This is a contradiction. Thus r(a) must be proper essential right ideal of R. Hence there exists a maximal right ideal M of R containing r(a). Clearly M is an essential right ideal of R, and R / M is Wnil-injective. So any R-homomorphism of aR into R / M extends to one of R into R / M. Let $f : aR \to R / M$ be defined by f(ar) = r + M. Then f is well-defined R-homomorphism. Since R / M is Wnil-injective, so there exists $c \in R$ such that 1+M=f(a)=ca+ M which implies that $1-ca \in M$. Now $aca \in aRa \subseteq U^2 = 0$, hence $ca \in r(a) \subseteq M$, and so $1 \in M$, which is a contradiction. Therefore R must be semi-prime, and hence R is a reduced ring. ■

Following [6], a ring R is called a right **DS-ring** if every minimal right ideal of R is a direct summand.

Proposition 3.3 :

Let R be a ring whose every simple right R-module is Wnil-injective. Then

(1) $J(R) \cap Soc(R_R) = 0$

(2) J(R) is reduced ideal of R.

Proof :

(1) If $J(R) \cap Soc(R_R) \neq 0$, then there exists a minimal right ideal kR of R with $kR \subseteq J(R)$. If kR is direct summand then kR = eR for some $0 \neq e^2 = e \in R$ and we get $e \in J(R)$ which is a contradiction . So that $(kR)^2 = 0$. Since r(k) is a maximal right ideal of R, then R / r(k) is Wnil-injective. Set $f : kR \to R / r(k)$ defined by right R-homomorphism. Since R / r(k) is right Wnil-injective ring and $k \in N(R)$, there exists $c \in R$ such that f(kr) = ckr + r(k). Therefore 1 + r(k) = f(k) = ck + r(k) which implies $1-ck \in r(k)$. Since $k \in J(R)$, then $ck \in J(R) \subseteq r(k)$ which implies $1 \in r(k)$, which is also a contradiction. Therefore $J(R) \cap Soc(R_R) = 0$

(2) Let $0 \neq x \in J(R)$ such that $x^2 = 0$. Since $x \neq 0$, then $r(x) \subseteq M$ for some maximal right ideal M of R. Define a right R-homomorphism

$f : xR \to R / M$ such that $f(xr) = r + M$ for all $r \in R$. Then f is well-defined right R-homomorphism. Since R / M is a right Wnil-injective ring and $x \in N(R)$, there exists $c \in R$ such that $f(xr) = cxr + M$. Therefore
$1 + M = f(x) = cx + M$, which implies that $1-cx \in M$, and so $1 \in M$, which is a contradiction. Hence $J(R)$ is reduced. ■

Lemma 3.4 :[6]

Let R be a ring. Then R is a right DS-ring if and only if
$J(R) \cap Soc(R_R) = 0$. ■

Corollary 3.5 :

Let R be a ring whose every simple right R-module is Wnil-injective. Then R is right DS-ring. ■

Following [6] a ring R is called right **MC2-ring** if eRa=0 implies aRe=0, where a, $e^2 = e \in R$ and eR is minimal right ideal of R. Or equivalently, if $K \approx eR$ are minimal, $e^2 =e \in R$; then $K = gR$ for some $g^2=g \in R$.

Lemma 3.6 : [7]

Suppose that every simple singular right R-module is Wnil-injective. Then R is a right MC2 ring if and only if R is a right minsymmetric ring. ■

Theorem 3.7 :

Suppose that every simple singular right R-module is Wnil-injective. Then $Soc(R_R) \cap J(R) = Soc(R_R) \cap Y(R)$ if and only if R is a right minsymmetric ring.

Proof :

It is clear that $Soc(R_R) \cap Y(R) \subseteq Soc(R_R) \cap J(R)$. Now, to prove $Soc(R_R) \cap J(R) \subseteq Soc(R_R) \cap Y(R)$, let kR be a minimal right ideal of R, and $kR \subseteq Soc(R_R) \cap J(R)$. Since $r(k)$ is a maximal right ideal of R, and if $r(k)$ is not essential of R, then $r(k)$ is a direct summand of R. So $kR \approx R / r(k) \approx eR$, $e^2 = e \in R$. Hence kR is a direct summand of R. Since R is a right minsymmetric ring, then by Lemma 3.6, R is a right MC2. Then $kR = gR$, $g^2 = g \in R$ so that $g \in J(R)$, which is a contradiction. Hence $r(k)$ is essential in R and we have $kR \in Y(R)$. Therefore $kR \subseteq Soc(R_R) \cap Y(R)$.

Conversely, Assume that $Soc(R_R) \cap J(R) = Soc(R_R) \cap Y(R)$, if kR is a minimal right ideal of R, and $kR \approx eR$, $e^2 = e \in R$. If $kR \subseteq J(R)$, then $kR \subseteq Soc(R_R) \cap J(R) = Soc(R_R) \cap Y(R)$. Hence kR and eR are singular right ideals, a contradiction. Then $kR \not\subset J(R)$, and we get $(kR)^2 \neq 0$, so $(kR)^2 = kR = gR$, $g^2 = g \in R$.Therefore R is a right minsymmetric ring. ■

From Proposition 3.3 and Theorem 3.7 we conclude following corollary.

Corollary 3.8 :

Let R be a right minsymmetric ring and every simple singular right R-module is Wnil-injective. Then $Soc(R_R) \cap Y(R) = 0$. ■

Theorem 3.9 :

Let R be reversible and N(R) is an ideal of R. Then, the following conditions are equivalent :

(1) R is reduced
(2) R is n-regular
(3) R is right nil-injective and NPP.
(4) R is right Wnil-injective and NPP.
(5) every simple right R-module is Wnil-injective.
(6) every simple singular right R-module is Wnil-injective.

Proof :

It is obvious that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$(5) $\Rightarrow$(6).
(6) $\Rightarrow$ (1): Let $a \in R$ with $a^2 = 0$. If $a \neq 0$, then $r(a) \neq R$, so there exists a right ideal L of R such that $r(a) \oplus L$ is essential in R. If $r(a) \oplus L \neq R$, there exists a maximal right ideal M of R containing $r(a) \oplus L$. Clearly , M is an essential right ideal of R, and by hypothesis, R / M is Wnil-injective. So there exists $c \in R$ such that $1-ca \in M$. Since N(R) is an ideal of R, $ca \in N(R)$, so 1-ca is invertible. Hence M=R, which is a contradiction. This show that $r(a) \oplus L = R$.

Let $r(a) = eR$, $e^2 = e \in R$. Clearly, $a = ae = ea = 0$, which is a contradiction. So a = 0. ■

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