Using "Filter" Approach to Solve the Constrained Optimization Problems

Ban Ahmed Mitras
dr.banah.mitras@gmail.com
College of Computer Sciences and Mathematics
University of Mosul

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ABSTRACT

In this paper, the solution of constrained nonlinear programming problems by a Sequential Quadratic Programming (SQP) is considered. The aim of the present work is to promote global convergence without the need to use a penalty and Barrier functions in the mixed interior-exterior point method. Instead, a new concept of a "filter" that aims to minimize the objective function and its approach that allows appoint to be accepted if reduces the objective function and satisfies the constraint violation function. If that point is rejected a new point is tested.

Keyword: Filter approach, constrained Optimization, SQP method.

استخدام اسلوب المرشح لحل مسائل الأمثلية المقيدة

بان احمد متراس
كلية علوم الحاسوب والرياضيات
جامعة الموصل

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الملخص

في هذا البحث تم اقتراح خوارزمية جديدة لحل مسائل البرمجة غير الخطية المقيدة باستخدام البرمجة التربيعية المتدرجة. الهدف من هذا العمل هو تقديم حل للوصول إلى تقارب شامل للطريقة الداخلية والخارجية بدون الحاجة إلى استخدام كل من دالة الجزاء ودالة الحاجز وإنما استخدم بدل منها أسلوب جديد وهو"المرشح" والذي يهدف الى تقليل دالة الهدف وهو اسلوب يسمح للنقطة ان تقبل إذا تمكنت من تقليل دالة الهدف مع تحقق القيود واذا لم تتمكن فسوف تُرفض تلك النقطة ويتم اختبار نقطة جديدة. الاختبارات العددية على المدى الواسع لمسائل الاختبار كانت مشجعة جدا".

الكلمات المفتاحية: نهج تصفية ، الأمثل مقيدة ، طريقة SQP.

1. Introduction:

The ideas that are introduced in the *filter* approach are such that it is not obvious how to apply standard techniques for proving global convergence of penalty function methods (that is, convergence to a stationary point from an arbitrary initial estimate). At present the question of global convergence is open, although it has provoked some interesting discussions with other colleagues. In the context of our algorithm, heuristics are readily suggested that exclude obvious situations in which the algorithm might fail to converge. However we observe in practice that we can dispense with these heuristics and yet solve a substantial proportion of standard test problems using SQP algorithm without penalty function.

Consider the most general form of the problem

$$\text{Minimize} \quad f(x), \qquad x \in R^n . \qquad \ldots(1)$$

Subject to the general (possibly nonlinear) inequality constraints

$$c_j(x) \le 0 , \quad 1 \le j \le L , \qquad ...(2)$$

and (possibly nonlinear) equality constraints

$$c_j(x) = 0 , \; L+1 \le j \le m , \qquad \ldots(3)$$

with the simple bounds

$$L_i \le x_i \le u_i , \; 1 \le i \le n , \qquad \ldots(4)$$

where f and c_j are all assumed to be twice continuously differentiable and any of the bounded in eq.(4) may be infinite. [4]

1.1. The Exterior-Point Method (Penalty Function):

The exterior-point method is suitable for equality and in equality constraints. The new objective function $\phi(x,\mu_k)$ is defined by:

$$\phi(x,\mu_k) = f(x) + \frac{1}{\mu_k}\alpha(x) , \qquad \ldots(5)$$

where μ_k is a positive scalar and the remainder of the second term is the penalty function.

1.2. The Interior-Point Method (Barrier Function):

Interior-point method is suitable for inequality constraints. The new objective function $\phi(x,\mu_k)$ is defined by

$$\phi(x,\mu_k) = f(x) + \mu_k B(x) , \qquad \ldots(6)$$

where μ_k is a positive scalar and the reminder of the second term is the Barrier function [6].

Although both exterior and interior-point methods have many points of similarly, they represent two different points of view. In an exterior-point procedure, we start from an infeasible point and gradually approach feasibility. While doing so, we move away from the unconstrained optimum of the objective function. In an interior-point procedure we start at a feasible point and gradually improve our objective function, while maintaining feasibility. The requirement that we begin at a feasible point and remain within the interior of the feasible inequality constrained region is the chief difficulty with interior-point methods. In many problems we have no easy way to determine a feasible starting point, and a separate initial computation may be needed. Also, if equality constraints are present, we do not have a feasible inequality constrained region in which to maneuver freely. Thus interior-point methods cannot handle equalities.

We many readily handle equalities by using a “mixed” method in which we use interior-point penalty functions for inequality constraints only. Thus, if the first *m* constraints are inequalities and constraints *(m+1)* to *n* are equalities, our problem becomes:

$$\text{Minimize } \phi(x,\mu_k) = f(x) + \mu_k B(x) + \frac{1}{\mu_k}\alpha(x). \qquad \ldots(7)$$

The function $\phi(x,\mu_k)$ is then minimized for a sequence of monotonically decreasing $\mu_k > 0$ [2].

In this paper, a modification of mixed exterior-interior point method is proposed based on the use of filter approach.

2. An Nonlinear Programming Filter: [5]

This section describes the basic concepts that underpin the new approach towards inducing global convergence in SQP.

There are two competing aims in nonlinear programming. The first is the minimization of the objective function *f* and the second is the satisfaction of the constraints. Conceptually, these two conflicting aims can be written as

$$\text{Minimize } f(x), \qquad \ldots(8)$$

and

$$\text{Minimize } h(c(x)), \qquad \ldots(9)$$

where

$$h(c(x)) := \left\| c^+(x) \right\|_{L_1} := \sum_{j=1}^{m} c_j^+(x)$$

is the L_1 norm of the constraint violation.

Here $c_j^+ = \max(0, c_j)$. The problem of satisfiability has now been written as a minimization problem.

A penalty function combines eq.(8) and eq.(9) into a single minimization problem. Instead we prefer to view eq.(8), eq.(9) as separate aims, akin to a multi-objective optimization problem. However our situation is somewhat different from multi-objective optimization in that it is essential to find a point for which $h(c(x))=0$ if at all possible. In this sense the minimization of $h(c(x))$ has priority.
Nevertheless, it is convenient to make use of the concept of domination from multi-objective optimization. Let (f_k,h_k) denote values of $f(x)$ and $h(c(x))$ evaluated at x_k.

Definition 1:

A Pair (f_k,h_k) is said to *dominate* another pair (f_l,h_l) if and only if both $f_k \leq f_l$ and $h_k \leq h_l$.

In the context of nonlinear programming, this means that x_k is at least as good as x_l with respect to eq.(8) and eq.(9). With this concept it is now possible to define a filter, which will be used in a trust-region type algorithm as a criterion for accepting or rejecting a trail step.

Definition 2:

A *filter* is a list of pairs (f_l,h_l) such that no pair dominates any other. A point (f_k,h_k) is said to be acceptable for inclusion in the filter if it is not dominated by any point in the filter.

We stress that only two scalars are stored for each entry in the filter. No vectors such as x_l are stored and this has negative implications for the use of backtracking in the resulting algorithm. Each point in the filter generates a block of non-acceptable points and the union of these points represents the set of points that are not acceptable to the filter.

The idea is to use the filter as a criterion for accepting or rejecting a step in an SQP method. Starting with x_k, the solution of the quadratic problem produces a trial step d_k. Set $x_{k+1}=x_k+d_k$, the new trial point x_{k+1} is acceptable by the filter if the corresponding pair (f_{k+1},h_{k+1}) is not dominated by any point in the filter [5].

3. Mixed Exterior-Interior Point Methods:

We can solve the constrained problem given in eq.(1) to eq.(3) construct a new objective function $\phi(x,\mu_k)$ which is defined in eq.(7). Now our aim is to minimize the function $\phi(x,\mu_k)$ by starting from a feasible point x_0 and with initial value $\mu_0 = 1$ and the method reducing μ_k is simple iterative method such that:

$$\mu_{k+1} = \frac{\mu_k}{10}, \qquad \ldots(10)$$

where μ_k is a constant equal to *10* and the search direction d_k in this case can be defined

$$d_k = -H_k g_k, \qquad \ldots(11)$$

where H is a positive definite symmetric approximation matrix to the inverse Hessian matrix G^{-1} and g is the gradient vector of the function $\phi(x,\mu_k)$.

The next iteration is set to further point

$$x_{k+1} = x_k + \lambda_k d_k, \qquad \ldots(12)$$

where λ is a scalar chosen in such that $f_{k+1} < f_k$. We thus test $c_i(x_{k+1})$ to see that it is positive for all i. We find a feasible x_{k+1} and we can then proceed with the interpolation. Then the matrix H_k is updated by a correction matrix to get:

$$H_{k+1} = H_k + \phi_k, \qquad \ldots(13)$$

where ϕ_k is a correction matrix which satisfies quasi-Newton condition namely $(H_{k+1}y_k = \rho v_k)$ where v_k and y_k are difference vector between two successive points and gradients respectively and ρ is any scalar.

The initial matrix H_0 chosen to be identity matrix I. H_k is updated through the formula of BFGS update [1] [7].

4. Combined Barrier-Penalty Algorithm:

Step (1): Find an initial approximation x_0 in the interior of the feasible region for the inequality constraints i.e. $g_i(x_0)<0$.
Step (2): Set $k=1$ and $\mu_0 = 1$ is the initial value of μ_k.
Step (3): Set $\phi(x,\mu_k) = f(x) + \mu_k B(x) + \frac{1}{\mu_k}\alpha(x)$.
Step (4): Set $d_k=-H_k g_k$
Step (5): Set $x_{k+1} = x_k + \lambda_k d_k$, where λ is a scalar.
Step (6): Check for convergence i.e. if $|x_k - x_{k-1}| < \varepsilon$, then stop.
Step (7): Otherwise, set $\mu_{k+1} = \frac{\mu_k}{10}$ and take $x=x^*$ and set $k=k+1$ and go to Step 3 [3].

5.New Nonlinear Programming <u>without</u> Penalty and Barrier Functions:

Here, we described the concept of filter to nonlinear programming without a Penalty and Barrier Functions such that:

$$\text{Minimize } f(x), \qquad \ldots(14)$$

and

Minimize $h_1(c(x))$, …(15)

Minimize $h_2(c(x))$, …(16)

where

$$h_1(c(x)) := \left\| c^+(x) \right\|_{L_1} := \sum_{j=1}^{m} c_j^+(x), \quad \ldots(17)$$

$$h_2(c(x)) := \left\| c^-(x) \right\|_{L_2} := \sum_{j=1}^{m} c_j^-(x), \quad \ldots(18)$$

is the L_1, L_2 norm of the constraint violations.

Here $c_j^+ = max(0,c_j)$ and $c_j^- = min(0,c_j)$. The problem of satisfiability has now been written as a minimization problem.

A penalty and Barrier functions combine eq.(14); eq.(15) and eq.(16) into a single minimization problem. Instead we prefer to view eq.(14); eq.(15) and eq.(16) as separate aims, akin to a multi-objective optimization problem. However our situation is somewhat different from multi-objective optimization in that it is essential to find a point for which $h_1(c(x))=0$ and $h_2(c(x))=0$ if at all possible. In this sense the minimization of $h_1(c(x))$and $h_2(c(x))$ have priority. Then, we have two new definitions:

Definition 3:

A Pairs $(f_k,h_{1,k})$ and $(f_k,h_{2,k})$ is said to *dominate* another pairs $(f_l,h_{1,l})$ and $(f_l,h_{2,l})$ if and only if both $f_k \le f_{1,l}$ *and* $f_{2,l}$ and $h_k \le h_{1,l}$ *and* $h_{2,l}$.

Definition 4:

A *filter* is a list of pairs $(f_l,h_{1,l})$ and $(f_l,h_{2,l})$ such that no pairs dominates any other. A points $(f_k,h_{1,k})$ and $(f_k,h_{2,k})$ are said to be acceptable for inclusion in the filter if it is not dominated by any point in the filter.

6. New Filter Mixed Exterior-Interior point Algorithm:

Step (1): Find an initial approximation x_0 in the interior of the feasible region for the inequality constraints i.e. $g_i(x_0)<0$.

Step (2): Set $k=1$, and compute $d_k=-H_kg_k$

Step (3): Set $x_{k+1} = x_k + \lambda_k d_k$, where λ is a scalar.

Step (4): Check for convergence i.e. if $|x_k - x_{k-1}| < \varepsilon$, then stop.

Step (5): Otherwise, Check. If $(f_{k+1},h_{1,k+1})$ and $(f_{k+1},h_{2,k+1})$ is acceptable to the filter, then. x_{k+1} is the new point and add $(f_{k+1},h_{1,k+1})$and $(f_{k+1},h_{2,k+1})$ to the filter. and remove any points from the filter that are dominated $(f_{k+1},h_{1,k+1})$ and $(f_{k+1},h_{2,k+1})$.

Step (6): Else set $x_{k+1}=x_k$ and go to step 7.

Step (7): Set $k=k+1$ and go to step 3.

7. Results and Calculation:

Several test functions were tested with different dimensions. Our programs were written in FORTRAN 90.

In order to test the effectiveness of the new algorithm that have been used to Barrier-Penalty function method, the comparative tests involving several well-known test function (see Appendix) have been chosen and solved numerically by utilizing the new and established method. So the new algorithm has been compared with some established algorithm.

In table (1) we have compared a new algorithm with standard Barrier-Penalty algorithm for $1 \leq n \leq 3$ and $1 \leq c_i(x) \leq 7$ using (5) nonlinear test functions.

From table (2) it is clear that, taking the standard Barrier-Penalty algorithm as 100%, and the new algorithm has 61.9%, 48.7% improvements on the standard Barrier-Penalty algorithm in about number of iterations NOI and number of function evaluations NOF respectively.

Table (1)
Comparison between Barrier-Penalty and new algorithms

Test function	Barrier-Penalty algorithm NOI (NOF)	New algorithm NOI (NOF)
1.	7 (61)	4 (22)
2.	8 (2141)	6 (1019)
3.	7 (141)	4 (69)
4.	10 (956)	7 (251)
5.	10 (2205)	5 (1320)
Total	42 (5504)	26 (2681)

Table (2)

	Barrier-Penalty algorithm	New algorithm
NOI	100%	61.9
NOF	100%	48.7

8. Appendix:

Test functions:

1. $min\, f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$ *s.p(7,9)*

s.t.

$x_1 - 2x_2 = -1$

$\frac{-x_1^2}{4} + x_2^2 + 1 \geq 0$

2. $min\, f(x) = x_1 x_2$ *(18,16)*

s.t.

$25 - x_1^2 - x_2^2 = 0$

$x_1 + x_2 \geq 0$

3. $min\, f(x) = x_1^2 + x_2^2$ *s.p.(0.9,2)*

s.t.

$x_1 + 2x_2 = 4$

$x_1^2 + x_2^2 \leq 5$

$x_i \geq 0$

4. $min\, f(x) = (x_1 - 2)^2 + (x_2 - 3)^2$ *(2,7)*

s.t.

$x_1 - 2x_2 = -1$

$-x_1^2 + x_2 \geq 0$

5. $min\, f(x) = x_1 x_4 (x_1 + x_2 + x_3) + x_3$ *s.p(4,3,3,3)*

s.t.

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40$

$x_1 x_2 x_3 \geq 25$

$5 \geq x_i \geq 1$

gnosis $\times$ 100

REFERENCES

[1] Bazaraa, S. and Shetty, C.M. (2003), "Nonlinear Programming: Theory and Algorithms", Newyork, Chichester, Brisbane,Toronto.

[2] Biggs, M.C. (1989), "Halfield Polytechnic Numerical Optimization", The Optima Package, Special Communication.

[3] Chong, K.P.E. and Zak, K.S. (2001), " An Introduction to Optimization", John Wiley and Sons, Inc. New York/ Chichester Weinhein/Brisbane, USA.

[4] Conn, A.R., Gould, N. and Toint, Ph.L., (1994), "Large scale nonlinear Constrained Optimization", A current Survey, Report 94/1 IBM T.J. Watson Research Center, USA.

[5] Fletcher, R. and Leyffer, S., (1997), " Nonlinear Programming Without a Penalty Function", University of Dandee Numerical Analysis Report, NA/171, 22 September, DD1, 4HN, U.K.

[6] Gottfred, B.S. and Weisman, J., (1973), "Introduction to Optimization Theory", Prentice-Hall, Englewood Cliffs, N.J.

[7] Toint, Ph.L. and Necholas, L.M. (1997), "A Note on the Second Order Convergence of Optimization Algorithm by using Barrier Function", A current survey, IBMJJ, Waston Research Center,USA.