The n-Wiener Polynomials of Straight Hexagonal Chains and $K_t \times C_r$

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ABSTRACT

The n-Wiener polynomials of straight hexagonal chains and the Cartesian product of a complete graph K_r and a cycle C_r are obtained in this paper. The n-diameter and the n-Wiener index of each such graphs are also determined.

Keywords: n-distance, n-diameter, n-Wiener index, n-Wiener Polynomials, hexagonal chains.

متعددة حدود وينر-n لسلاسل سداسية مستقيمة وللبيان $K_t \times C_r$

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الملخص

تضمن هذا البحث ايجاد متعددة حدود وينر-n لبيانات بشكل سلسلة سداسية مستقيمة وكذلك للبيان الناتج من الجداء الديكارتي لبيان تام K_t مع بيان دارة C_r. كما تضمن البحث ايجاد القطر-n ودليل وينر-n لكل من هذه الانواع من البيانات.

الكلمات المفتاحية: المسافة−n، القطر−n، دليل وينر−n، متعددة حدود وينر−n، سلاسل سداسية.

1.Introduction.

We follow the terminology of [5,6]. Let v be a vertex of a connected graph G and let S be an (n-1)-subset of vertices of V(G), n≥2, then the **n-distance** $d_n(v,S)$ is defined as follows[7]

$$d_n(v,S)=\min\{d(v,u):u\in S\}. \quad ...(1.1)$$

Sometimes, we refer to the n-distance of the pair (v,S) in G by $d_n(v,S\,|\,G)$. The **n-diameter** $diam_n G$ of G is defined by

$$diam_n G=\max\{d_n(v,S):\ v\in V(G),\ S\subseteq V(G), |S|=n-1\}. \quad ...(1.2)$$

It is clear that for all $2\le m\le n\le p$,

$$diam_n G\le diam_m G\le diam G. \quad ...(1.3)$$

The **n-Wiener index** of G denoted by $W_n(G)$ is defined as

$$W_n(G)=\sum_{(v,S)} d_n(v,S), \quad ...(1.4)$$

where the summation is taken over all pairs (v,S) for which $v\in V(G)$, $S\subseteq V(G)$ and $|S|=n-1$. The **n-average distance** $\mu_n(G)$ is defined as

$$\mu_n(G)= W_n(G)/p\binom{p-1}{n-1},\ \ 3\le n\le p. \qquad ...(1.5)$$

Let v be any vertex of G, then **the n-distance of v** denoted $d_n(v|G)$ or simply $d_n(v)$ is defined as

$$d_n(v)= \sum_{S\subseteq V(G)} d_n(v,S)\ \ ,|S|=n-1. \qquad ...(1.6)$$

The Wiener polynomial of G with respect to the n-distance, which is called n-Wiener polynomial and defined as below.

Definition 1.1.[2]. Let $C_n(G,k)$ be the number of pairs (v,S), $|S|=n-1, 3\le n\le p$, such that $d_n(v,S)=k$, for each $0\le k\le\delta_n$**.** Then, **the n-Wiener polynomial** $W_n(G;x)$ is defined by

$$W_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k\ , \qquad ...(1.7)$$

in which δ_n is the n-diameter of G .

One may easily see [2] that for $3\le n\le p$, the number of all (v,S) pairs is $p\binom{p}{n-1}$, and

$$\sum_{k=1}^{\delta_n} C_n(G,k)=p\binom{p-1}{n-1}, \qquad C_n(G,0)=p\binom{p-1}{n-2}, \qquad ...(1.8)$$

$$C_n(G,1)=p\binom{p-1}{n-1} - \sum_{v\in V(G)}\binom{p-1-\deg_G(v)}{n-1}. \qquad ...(1.9)$$

Definition 1.2[1] Let v be a vertex of a G, and let $C_n(v,G,k)$ be the number of (n-1)-subsets of vertices of G such that

$d_n(v,S|G)=k$, for $n\ge 3$, $0\le k\le\delta_n$.

Then, the **n-Wiener polynomial of vertex v**, denoted by $W_n(v,G;x)$ is defined as

$$W_n(v,G;x)=\sum_{k\ge 0} C_n(v,G,k)x^k. \qquad ...(1.10)$$

It is clear that for all $k\ge 0$,

$$\sum_{v\in V(G)} C_n(v,G,k)=C_n(G,k), \qquad ...(1.11)$$

and

$$\sum_{v\in V(G)} W_n(v,G,x)=W_n(G;x)\ . \qquad ...(1.12)$$

There are many classes of graphs G in which for each k, $1 \le k \le \delta_n$, $C_n(v,G,k)$ is the same for every vertex $v \in V(G)$; such graphs are called [1] **vertex-n-distance regular**. If G is of order p and it is vertex-n-distance regular, then

$$W_n(G;x)=pW_n(v,G;x), \qquad ...(1.13)$$

where v is any vertex of G.

The authors of papers [2,3,4] obtained the n-Wiener polynomials and n-Wiener index for some special graphs and of some kind of compound graphs. In this paper, we obtain n-Wiener polynomials for straight hexagonal chains and for the Cartesian product $K_t \times C_r$.

2. The Cartesian Product of a Cycle and a Complete Graph

Let C_r be a cycle of order $r \ge 3$ and vertices $v_1, v_2, \ldots, v_r, v_1$, and let K_t be a complete graph of vertex set $V(K_t)=\{u_1, u_2, \ldots, u_t\}$.

It is clear that $K_t \times C_r$ is regular of degree t+1, and it is vertex-n-distance regular. Thus, for every vertex (u_i,v_j) of $K_t \times C_r$ and each k

$$C_n((u_i,v_j), K_t \times C_r, k)$$

has the same value for $2 \le n \le tr$. Therefore,

$$\operatorname{diam}_n K_t \times C_r = \max\{d_n(u_1,v_1) : S \subseteq V(K_t \times C_r), |S|=n-1\}.$$

The n-diameter of $K_t \times C_r$ is determined in the next proposition.

Proposition 2.1. For $r=2s$, $s \ge 2$, $t \ge 3$,

$$\operatorname{diam}_n K_t \times C_r = s+1-\lfloor (n+t-1)/2t \rfloor, \text{ when } 2 \le n \le tr.$$

Proof. Let $A_i=\{(u_j,v_i): j=1,2,\ldots,t\}$, $1 \le i \le r$. The induced subgraph $\langle A_i \rangle$ is denoted by $K_t^{(i)}$ and called the i^{th} copy of K_t. It is clear that

$$\operatorname{diam} K_t \times C_r = s+1,$$

therefore, for $2 \le n \le tr$

$$\operatorname{diam}_n K_t \times C_r \le s+1.$$

Now if $2 \le n \le t$, then we take $S \subseteq A_{s+1}-\{(u_1,v_{s+1})\}$, and we find that

$$d_n((u_1,v_1),S)=s+1,$$

as it is clear from $K_t \times C_r$ shown in Fig.2.1. Thus,

$$\operatorname{diam}_n K_t \times C_r = s+1, \text{ when } 2 \le n \le t.$$

Now, assume that $t+1 \le n \le tr$.

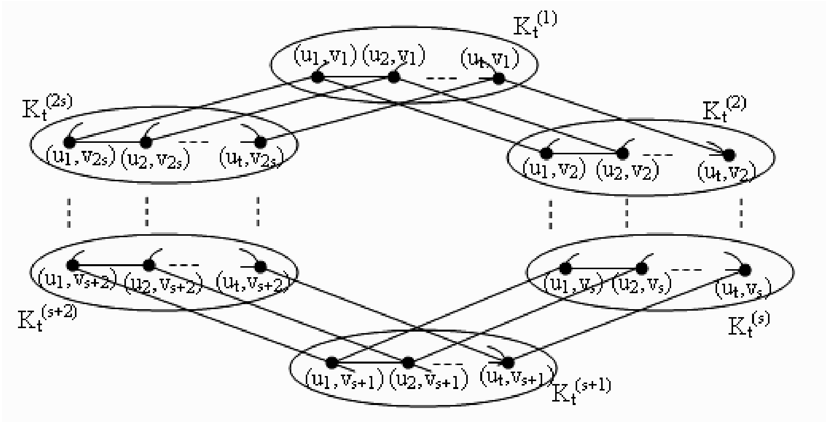

Fig.2.1. The graph $K_t \times C_{2s}$

If S□ is an (n-1)-set of vertices such that $d_n((u_1,v_1),S$□$)$ is maximum, then S□ must contain A_{s+1}, and the other n-t-1 vertices are taken from the set
$[(A_s \cup A_{s+2}) \cup (A_{s-1} \cup A_{s+3}) \cup \ldots \cup (A_{s+1-j} \cup A_{s+j+1})]-\{(u_1,v_{s+1-j}),(u_1,v_{s+j+1})\}$
such that

$$t+2t(j-1) \leq n-1 \leq 3t-2+2t(j-1)$$

Solving for j, we get

$$(n-t+1)/2t \leq j \leq (n+t-1)/2t.$$

Since j is an integer, we have

$$j=\lfloor (n+t-1)/2t \rfloor.$$

From Fig.2.1, one can easily see that

$$d_n((u_1,v_1),S\square)=s+1-j.$$

Hence the proof is completed. ■

Proposition 2.2. For $r=2s+1$, $s \geq 1$, $t \geq 3$,

$$diam_n\, K_t \times C_r = s+1-\lfloor n/2t \rfloor, \text{ for } 2 \leq n \leq tr.$$

Proof. Consider the graph $K_t \times C_{2s+1}$ shown in Fig.2.2 and use the notations used in the proof of Proposition 2.1.

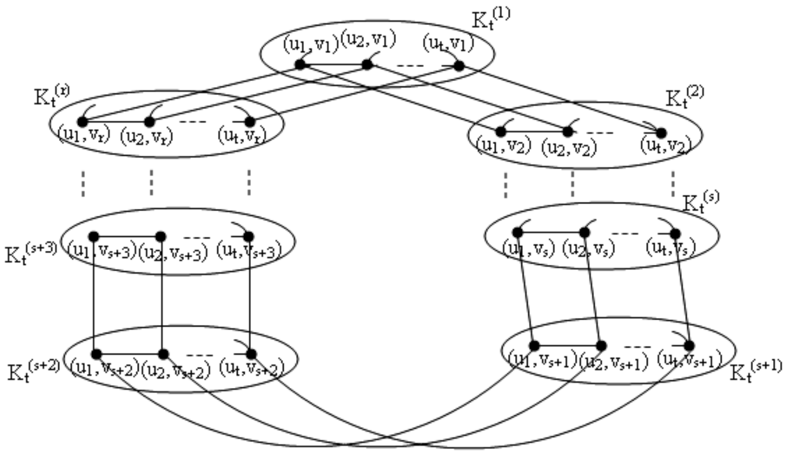

Fig.2.2. The graph $K_t \times C_{2s+1}$.

Let S be an (n-1)-set of vertices such that $d_n((u_1,v_1),S)$ is maximum. If $2 \le n \le 2t-1$, then $S \subseteq A_{s+1} \cup A_{s+2} - \{(u_1,v_{s+1}),(u_1,v_{s+2})\}$,and

$$d_n((u_1,v_1),S)=s+1=diam_n\, K_t \times C_r,$$

as given in the proposition.

If $2t \le n \le rt$, then S must consist of vertices from

$[(A_{s+1} \cup A_{s+2}) \cup (A_s \cup A_{s+3}) \cup \ldots \cup (A_{s+1-j} \cup A_{s+j+2})]-\{(u_1,v_{s+1-j}),(u_1,v_{s+j+2})\}$ such that

$$2tj-1 \le n-1 \le 2t(j+1)-2 \text{ (See Fig.2.2).}$$

Solving for j, we get

$$(n/2t)-((2t-1)/2t) \le j \le n/2t.$$

Since j is positive integer, then $j=\lfloor n/2t \rfloor$. It is clear from the figure that

$$d_n((u_1,v_1),S)=s+1-j.$$

Therefore,

$$diam_n\, K_t \times C_{2s+1}=s+1-j=s+1-\lfloor n/2t \rfloor. \quad \blacksquare$$

We determine the n-Wiener polynomial of $K_t \times C_r$ in the following theorems.

Theorem 2.3. For $t \ge 2$, $3 \le n \le rt$, $r=2s+1$, we have

$$W_n(K_t \times C_r;x)=\sum_{k=0}^{\delta_n} C_n(K_t \times C_r,k)x^k,$$

in which

$$C_n(K_t \times C_r,0)=rt\binom{rt-1}{n-2}$$

$$C_n(K_t\times C_r,1)=rt\left[\binom{rt-1}{n-1}-\binom{rt-t-2}{n-1}\right],$$

and for $2\le k\le s$,

$$C_n(K_t\times C_r,k)=rt\left[\binom{\alpha+2t}{n-1}-\binom{\alpha}{n-1}\right],$$

where

$$\alpha=2t(s-k+1)-2,$$

$$C_n(K_t\times C_r,s+1)=rt\binom{2t-2}{n-1}, \text{ where } 3\le n\le 2t-1;$$

and δ_n is the n-diameter of $K_t\times C_r$.

Proof. $C_n(K_t\times C_r,0)$ and $C_n(K_t\times C_r,1)$ follow from (1.8) and (1.9).
Since $K_t\times C_r$ is vertex-n-distance regular, we have for $2\le k\le\delta_n$,

$$C_n(K_t\times C_r,k)=rtC_n((u_1,v_1),K_t\times C_r,k) \text{ (See Fig.2.2).}$$

For $2\le k\le s$ there are 2t vertices each of distance k from vertex (u_1,v_1), and there are $2t(s-k+1)-2(=\alpha)$ vertices each of distance more than k from (u_1,v_1). Thus,

$$C_n((u_1,v_1),K_t\times C_r,k)=\sum_{j=1}^{n-1}\binom{2t}{j}\binom{\alpha}{n-1-j},\ 2\le k\le s.$$

If $3\le n\le 2t-1$, then $\delta_n=s+1$. In this case, there are exactly 2t-2 vertices each of distance s+1 from vertex (u_1,v_1), and there is no vertex of distance more than s+1.
Therefore,

$$C_n((u_1,v_1),K_t\times C_r,s+1)=\binom{2t-2}{n-1}.\ \blacksquare$$

Theorem 2.4. For $t\ge 2$, $3\le n\le rt$, $r=2s\ge 4$, we have

$$W_n(K_t\times C_r;x)=\sum_{k=0}^{\delta_n} C_n(K_t\times C_r,k)x^k,$$

where δ_n is the n-diameter, and

$$C_n(K_t\times C_r,0)=rt\binom{rt-1}{n-2}$$

$$C_n(K_t\times C_r,1)=rt\left[\binom{rt-1}{n-1}-\binom{rt-t-2}{n-1}\right],$$

and for $2\le k\le s-1$,

$$C_n(K_t\times C_r,k)=rt\left[\binom{\beta+2t}{n-1}-\binom{\beta}{n-1}\right],\ \beta=t(r-2k+1)-2,$$

$$C_n(K_t\times C_r,s)=rt\left[\binom{3t-2}{n-1}-\binom{t-1}{n-1}\right],$$

$$C_n(K_t\times C_r,s+1)=rt\binom{t-1}{n-1},\ \text{where } \delta_n=s+1.$$

Proof. $C_n(K_t\times C_r,0)$ and $C_n(K_t\times C_r,1)$ follow from (1.8) and (1.9). For $2\le k\le s-1$, we notice that there are 2t vertices each of distance k from (u_1,v_1), and there are (2t(s-k)+t-2) vertices each of distance more than k from vertex (u_1,v_1). Therefore, for $2\le k\le s-1$,

$$C_n((u_1,v_1),K_t\times C_r,k)=\sum_{j=1}^{n-1}\binom{2t}{j}\binom{\beta}{n-1-j}$$

$$=\binom{\beta+2t}{n-1}-\binom{\beta}{n-1}\ \text{(See Fig. 2.1).}$$

For $3\le n\le 3t-1$, then $diam_n\, K_t\times C_r\ge s$, and for k=s, there are (2t-1) vertices each of distance s from (u_1,v_1), and there are (t-1) vertices each of distance more than s from (u_1,v_1). Therefore,

$$C_n((u_1,v_1),K_t\times C_r,s)=\sum_{j=1}^{n-1}\binom{2t-1}{j}\binom{t-1}{n-1-j}$$

$$=\binom{3t-2}{n-1}-\binom{t-1}{n-1}.$$

For $3\le n\le t$, then $diam_n K_t\times C_r$ =s+1 by Proposition 2.1, and there are exactly (t-1) vertices each of distance s+1 from vertex (u_1,v_1), and there is no vertex of distance more than s+1 from (u_1,v_1). Thus,

$$C_n((u_1,v_1),K_t\times C_r,s+1)=\binom{t-1}{n-1}.$$

Since $K_t\times C_r$ is vertex-n-distance regular, then the proof of the theorem is completed. ■

3. Straight Hexagonal Chains

A straight hexagonal chain is a graph ς_t consisting of t hexagons $H_1,H_2,\ldots,H_t$ such that H_i and H_{i+1}, $1\le i\le t-1$, have one edge in common as shown in Fig.3.1.

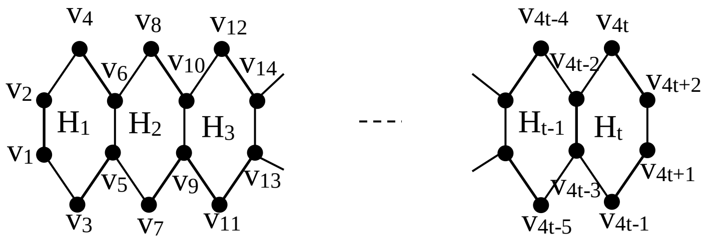

Fig.3.1. The graph ς_t.

It is clear that $p(\varsigma_t)=4t+2$, $q(\varsigma_t)=5t+1$.

Let $n\geq 2$, and consider the vertex v_1. The n-diameter of ς_t is the n-distance of (v_1,S) such that S is an (n-1)-set consisting of vertices farthest from v_1. To find S, we notice that

$$d(v_1,v_{4t+2})=2t+1,\ d(v_1,v_{4t+1})=2t,$$
$$d(v_1,v_{4t})=2t\ ,\ d(v_1,v_{4t-1})=2t-1,$$
$$d(v_1,v_{4t-2})=2t-1\ ,\ d(v_1,v_{4t-3})=2t-2$$

in general

$$d(v_1,v_i)=\lfloor i/2 \rfloor,\ \text{for } i=1,2,3,\dots,4t+2.$$

Therefore, if $d_n(v_1,S)$ is maximum, then S consists of the first n-1 vertices from the sequence:

$$v_{4t+2},v_{4t+1},v_{4t},v_{4t-1},\dots,v_5,v_4,v_3,v_2.$$

Thus, the vertex of S nearest to v_1 is v_{4t+4-n}.

If n is even, then

$$d(v_1,v_{4t+4-n})=2(t+1)-(n/2),$$

and when n is odd,

$$d(v_1,v_{4t+4-n})=2(t+1)-(n+1)/2.$$

Therefore,

$$d(v_1,v_{4t+4-n})=2(t+1)-\lceil n/2 \rceil,$$

which completes the proof of the following proposition.

Proposition 3.1. For $t\geq 1$, $2\leq n\leq 4t+2$,

$$\mathrm{diam}_n\ \varsigma_t=2(t+1)-\lceil n/2 \rceil .\ \blacksquare$$

To find the n- Wiener polynomial, $n\geq 3$, for ς_t we redraw ς_t as in Fig.3. 2 with new labels for its vertices.

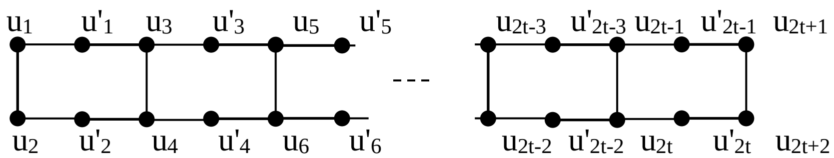

Fig. 3.2-the graph ς_t

From Fig.3.2 we notice that ς_t is $K_2\times P_{2t+1}$ with the edges $\{u'_iu'_{i+1}:i=1,3,5,\dots,2t-1\}$ removed.

Theorem 3. 2. For $t \geq 3$, $3 \leq n \leq 4t+2$, we have

$$W_n(\varsigma_t;x)=\sum_{k=0}^{\delta_n} C_n(\varsigma_t,k)x^k,$$

where δ_n is the n-diameter, and

$$C_n(\varsigma_t,0)=p\binom{p-1}{n-2},\ p=4t+2,$$

$$C_n(\varsigma_t,1)=p\binom{p-1}{n-1}-(2t+4)\binom{p-3}{n-1}-(2t-2)\binom{p-4}{n-1},$$

$$C_n(\varsigma_t,2)=2\{(t+2)\binom{p-3}{n-1}+(t-1)\binom{p-4}{n-1}-2\binom{p-5}{n-1}-2\binom{p-6}{n-1}-(t-2)\binom{p-7}{n-1}$$
$$-(t-1)\binom{p-8}{n-1}\},$$

$$C_n(\varsigma_t,3)=2\{2\binom{p-5}{n-1}+2\binom{p-6}{n-1}+(t-4)\binom{p-7}{n-1}+(t-1)\binom{p-8}{n-1}-2\binom{p-9}{n-1}-$$
$$2\binom{p-11}{n-1}-(2t-5)\binom{p-12}{n-1}\},$$

for $4 \leq k \leq \delta_n$,

$$C_n(\varsigma_t,k)=C_n(K_2 \times P_{2t+1},k),$$

in which $C_n(K_2 \times P_{2t+1},k)$ is given in Theorem 3.5.3. Ref[1].

Proof. $C_n(\varsigma_t,0)$ and $C_n(\varsigma_t,1)$ are obtained from (1.8) and (1.9.

To find $C_n(\varsigma_t,2)$ we notice that for $u \in \{u_1,u_2,u_{2t+1},u_{2t+2}\}$ there are exactly 2 vertices of distance 2 from u, and there are (p-5) vertices of distance more than 2 from u. For this case, the number of pairs (u,S) such that $d_n(u,S)=2$ is

$$4\sum_{j=1}^{n-1}\binom{2}{j}\binom{p-5}{n-1-j}=4[\binom{p-3}{n-1}-\binom{p-5}{n-1}]. \qquad ...(3.1)$$

If $u \in \{u\square_1,u\square_2,u\square_{2t-1},u\square_{2t}\}$, then there are exactly 3 vertices of distance 2 from u, and there are (p-6) vertices of distance more than 2 from u. For these vertices, the number of pairs (u,S) such that $d_n(u,S)=2$ is

$$4\sum_{j=1}^{n-1}\binom{3}{j}\binom{p-6}{n-1-j}=4[\binom{p-3}{n-1}-\binom{p-6}{n-1}]. \qquad ...(3.2)$$

If $u \in \{u_3,u_4,\ldots,u_{2t-1},u_{2t}\}$, then there are 4 vertices of distance 2 from u and there are (p-8) vertices of distance more than 4 from u. For these vertices u, the number of pairs (u,S) such that $d_n(u,S)=2$ is

$$2(t-1)\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-8}{n-1-j}=2(t-1)\left[\binom{p-4}{n-1}-\binom{p-8}{n-1}\right]. \qquad ...(3.3)$$

Finally, if $u\in\{u'_3,u'_4,u'_5,\ldots,u'_{2t-3}, u'_{2t-2}\}$, then there are 4 vertices of distance 2 from u, and there are (p-7) vertices of distance more than 2 from u. Therefore, the number of pairs (u,S) such that $d_n(u,S)=2$ for these vertices u is

$$2(t-2)\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-7}{n-1-j}=2(t-2)\left[\binom{p-3}{n-1}-\binom{p-7}{n-1}\right]. \qquad ...(3.4)$$

Summing the numbers in (3.1)-(3.4), we obtain the value of $C_n(\varsigma_t,2)$ as given in the theorem.

To find $C_n(\varsigma_t,3)$, we use the same method. If $u\in\{u_1,u_2,u_{2t+1},u_{2t+2}\}$, then the number of (u,S) pairs of n-distance 3 is

$$4\sum_{j=1}^{n-1}\binom{2}{j}\binom{p-7}{n-1-j}=4\left[\binom{p-5}{n-1}-\binom{p-7}{n-1}\right]. \qquad ...(3.5)$$

If $u\in\{u'_1,u'_2,u'_{2t-1},u'_{2t}\}$, then the number of (u,S) pairs is

$$4\sum_{j=1}^{n-1}\binom{3}{j}\binom{p-9}{n-1-j}=4\left[\binom{p-6}{n-1}-\binom{p-9}{n-1}\right]. \qquad ...(3.6)$$

If $u\in\{u_3,u_4,u_{2t-1},u_{2t}\}$, then the number of (u,S) pairs is

$$4\sum_{j=1}^{n-1}\binom{3}{j}\binom{p-11}{n-1-j}=4\left[\binom{p-8}{n-1}-\binom{p-11}{n-1}\right]. \qquad ...(3.7)$$

If $u\in\{u'_3,u'_4,\ldots,u'_{2t-3},u'_{2t-2}\}$, then the number of (u,S) pairs of n-distance 3 is

$$2(t-2)\sum_{j=1}^{n-1}\binom{5}{j}\binom{p-12}{n-1-j}=2(t-2)\left[\binom{p-7}{n-1}-\binom{p-12}{n-1}\right]. \qquad ...(3.8)$$

If $u\in\{u_5,u_6,\ldots,u_{2t-3},u_{2t-2}\}$, then the number of (u,S) pairs of n-distance 3 is

$$2(t-3)\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-12}{n-1-j}=2(t-3)\left[\binom{p-8}{n-1}-\binom{p-12}{n-1}\right]. \qquad ...(3.9)$$

Summing the numbers in (3.5)-(3.9), we get $C_n(\varsigma_t,3)$ as given in the statement of the theorem.

From Fig.3.2, we notice that for $4\le k\le\delta_n$, if $d(u,v)=k$ in the graph ς_t then it is also k in $K_2\times P_{2t+1}$, and conversely. Thus, if $d_n(u,S)=k$ in ς_t then it is also k in $K_2\times P_{2t+1}$, and conversely. Therefore,

$$C_n(\varsigma_t,k)=C_n(K_2\times P_{2t+1},k), \text{ for } 4\le k\le\delta_n. \ \blacksquare$$

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