Using Predictor-Corrector Methods for Numerical Solution of System of Non-linear Volterra Integral Equations of Second Kind

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ABSTRACT

The aim of this paper is solving system of non-linear Volterra integral equations of the second kind (**NSVIEK2**) numerically using Predictor-Corrector methods (**P-CM**). Two multistep methods (Adams-Bashforth, Adams-Moulton). Convergence and stability of the methods are proved and some examples are presented to illustrate the methods. Programs are written in matlab program version 7.0.

Keywords: Adams-Bashforth method, Adams-Moulton method, Runge-Kutta method, system of non-linear Volterra integral equation.

استخدام طرائق التكهن–المصحح للحل العددي لمنظومة معادلات فولتيرا التكاملية غير الخطية من النوع الثاني

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الملخص

الهدف من هذا البحثِ، هو حل منظومة معادلات فولتيرا التكاملية غير الخطية من النوع الثاني عدديا مستخدما طرائق التكهن–المصحح، حيث تم استخدام طريقتين من طرق متعددة الخطوات (طريقة أدم–باشفورث و طريقة أدم–مولتن). تم مناقشة التقارب والاستقرارية لمثالين وبالاعتماد على أخطاء التربيعيات الصغرى, استخدمت برنامج (matlab version 7.0) لكتابة البرامج الخاصة بهذه الطريقة.

الكلمات المفتاحية: طريقة آدمز–باشفورث، طريقة آدمز–مولتون، طريقة رانج–كوتا، نظام معادلة فولتيرا التكاملية غير الخطية.

1. Introduction

A predictor-corrector method (**P-CM**) is the combination of an explicit and implicit technique. **(**Delves and Mohamed, [3]), **(**Delves and Walsh, [4]), (Hall and Watt [5])**.**

(Ahmed, [1]**)** Solved system of non-linear Volterra integral equations of the second kind using computational methods, (Babolian and Biazar, [2]) used Adomian decomposition method to find the solution of a system of non-linear Volterra integral equations of the second kind, (Jumaa, [6]) find approximate solutions for a system of non-linear Volterra integral equations using B-Spline function, (Linz, [8]) Solve Volterra integral equations of the second kind using two (block-by-block) method, (Maleknejad and Shahrezaee, [9]) solve a system of Volterra integral equation numerically using Runge-Kutta method, (Waswas, [10]), used modified decomposition method to treatment non-linear integral equations and system of non-linear integral equations analytically, (Laurene**,** [7]) in this book derive the formula of Runge-Kutta method of order (three, four, five), Adams method (Moulton, Bashforth), and Adams Predictor-Corrector method and use this method to find numerical solution of ordinary and partial differential equation .

In this paper, the Adams Predictor-Corrector method is applied for the first time to find the numerical solution for a (**NSVIEK2**), which is defined by Jumaa, [6]:

$$F(x) = G(x) + \int_0^x K(x,t,F(t))dt\ , \qquad ...(1)$$

where

$$F(x) = (f_1(x),...,f_m(x))^T\ ,\ F(t) = (f_1(t),...,f_m(t)),$$

$$G(x) = (g_1(x),...,g_m(x))^T\ ,$$

$$K(x,t,F(t)) = (k_1(x,t,F(t),...,k_m(x,t,F(t)))^T\ ,$$

In this paper, the method is based on the explicit fourth-order Adams Bashforth method as Predictor and the implicit fourth-order Adams-Moulton method as Corrector, with the starting values from the fourth-order Runge-Kutta method (Laurene, [7]).

2. Adams Method: (Delves and Walsh, [4]), (Hall. and Watt, [5])

The general multistep method for approximating the solution to the initial-value problem:

$$u' = f(t,u), \qquad a \le t \le b, \qquad u(a) = \alpha$$

can be written in the form:

$$w_0 = \alpha,\quad w_1 = \alpha_1,\quad w_2 = \alpha_2,\quad ...\quad , w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + ... + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, ..., w_{i+1-m}) \qquad ...(2)$$

where

$$F(t_i, h, w_{i+1}, w_i, ..., w_{i+1-m}) = h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + ... + b_0 f(t_{i+1-m}, w_{i+1-m})] \qquad ...(3)$$

for each $i = m-1, m, ..., N-1$ *where* $a_0, a_1, ..., a_{m+1}$ *and* $b_0, b_1 ..., b_{m+1}$ are constants and, as usual, $h = (b-a)/N$ *and* $t_i = a + ih$.

when $b_m = 0$, the method is called explicit. when $b_m \neq 0$, the method is called implicit.

The explicit Adams methods, known as Adams-Bashforth methods. The implicit Adams methods, known as Adams-Moultons method.

2.1 Explicit Fourth-Order Adams-Bashforth Method (A-BM):

(Delves and Walsh, [4]), (Hall and Watt, [5])

put m=4 in equation (2), we get:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + hF(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}, w_{i-3}) \qquad ...(4)$$

where $F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}, w_{i-3})$ is the backward difference polynomial through $(t_i, f(t_i, u(t_i)))$, $(t_{i-1}, f(t_{i-1}, u(t_{i-1})))$, $(t_{i-2}, f(t_{i-2}, u(t_{i-2})))$, $(t_{i-3}, f(t_{i-3}, u(t_{i-3})))$ that given by:

$$F(t_i, h, w_i, w_{i-1}, w_{i-2}, w_{i-3}) = h[f(t_i, w_i) + \frac{1}{2}\nabla f(t_i, w_i) + \frac{5}{12}\nabla^2 f(t_i, w_i) + \frac{3}{8}\nabla^3 f(t_i, w_i)]$$

$$F(t_i, h, w_i, w_{i-1}, w_{i-2}, w_{i-3}) = h\{f(t_i, w_i) + \frac{1}{2}[f(t_i, w_i) - f(t_{i-1}, w_{i-1})] + \frac{5}{12}[f(t_i, w_i) - 2f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] + \frac{3}{8}[f(t_i, w_i) - 3f(t_{i-1}, w_{i-1}) + 3f(t_{i-2}, w_{i-2}) - f(t_{i-3}, w_{i-3})]\}$$

$$F(t_i, h, w_i, w_{i-1}, w_{i-2}, w_{i-3}) = h\{(1 + \frac{1}{2} + \frac{5}{12} + \frac{3}{8}) f(t_i, w_i) + (-\frac{1}{2} - \frac{10}{12} - \frac{9}{8}) f(t_{i-1}, w_{i-1}) + (\frac{5}{12} + \frac{9}{8}) f(t_{i-2}, w_{i-2}) + (-\frac{3}{8}) f(t_{i-3}, w_{i-3})\}$$

$$F(t_i, h, w_i, w_{i-1}, w_{i-2}, w_{i-3}) = \frac{h}{24}[55 f(t_i, w_i) - 59 f(t_{i-1}, w_{i-1}) + 37 f(t_{i-2}, w_{i-2}) - 9 f(t_{i-3}, w_{i-3})] \qquad ...(5)$$

so $m = 4, b_m = 0, b_{m-1} = \frac{55}{24}, b_{m-2} = -\frac{59}{24}, b_{m-3} = \frac{37}{24}, b_{m-4} = -\frac{9}{24}$

i.e

$$w_{i+1} = w_i + \frac{h}{24}[55 f(t_i, w_i) - 59 f(t_{i-1}, w_{i-1}) + 37 f(t_{i-2}, w_{i-2}) - 9 f(t_{i-3}, w_{i-3})] \quad ...(6)$$

for each $i = 3, 4, ..., N-1$, the local truncation error for the predictor is

$\tau_{i+1}(h) = \frac{251}{720} u^{(5)}(\varepsilon_i) h^5$, for some $\varepsilon_i \in (t_{i-3}, t_{i+1})$

2.2 Implicit fourth-Order Adams-Moulton Method (A-MM):

(Delves and Walsh, [4]),(Hall and Watt, [5])

put m=3 in equation (2), we get:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + hF(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}) \quad ...(7)$$

Where $F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2})$ is the forward difference polynomial through $(t_i, f(t_i, u(t_i)))$, $(t_{i-1}, f(t_{i-1}, u(t_{i-1})))$, $(t_{i-2}, f(t_{i-2}, u(t_{i-2})))$, that given by:

$$F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}) = h[f(t_i, w_i) + \frac{1}{2}\Delta f(t_i, w_i) - \frac{1}{12}\Delta^2 f(t_i, w_i) + \frac{1}{24}\Delta^3 f(t_i, w_i)]$$

$$F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}) = h\{f(t_i, w_i) + \frac{1}{2}[f(t_{i+1}, w_{i+1}) - f(t_i, w_i)] - \frac{1}{12}[f(t_{i+1}, w_{i+1}) - 2f(t_i, w_i) + f(t_{i-1}, w_{i-1})] + \frac{1}{24}[-f(t_{i+1}, w_{i+1}) + 3f(t_i, w_i) - 3f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})]\}$$

$$F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}) = h\{(\frac{1}{2} - \frac{1}{12} - \frac{1}{24})f(t_{i+1}, w_{i+1}) + (1 - \frac{1}{2} + \frac{2}{12} + \frac{3}{24})f(t_i, w_i) + (-\frac{1}{12} - \frac{3}{24})f(t_{i-1}, w_{i-1}) + (\frac{1}{24})f(t_{i-2}, w_{i-2})\}$$

then

$$F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}) = \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] \quad ...(8)$$

so $m = 3, b_m = \frac{9}{24}, b_{m-1} = \frac{19}{24}, b_{m-2} = -\frac{5}{24}, b_{m-3} = \frac{1}{24}$

i.e

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] \quad ...(9)$$

for each $i = 2,3,4,\ldots,N-1$, the local truncation error for the corrector is

$\tau_{i+1}(h) = -\frac{19}{720}u^{(5)}(\varepsilon_i)h^5$, for some $\varepsilon_i \in (t_{i-2}, t_{i+1})$

2.3 Adams Predictor-Corrector Method (P-CM): (Laurene, [7])

The (P-CM) combines the fourth-order Adams Bashforth method as Predictor and the implicit fourth-order Adams-Moulton method as Corrector:

w_0 is given; w_1, w_2, w_3 are found from a Runge-Kutta method.

Then, for $i = 3,4,\ldots,N-1$

$$w^*_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] \quad ...(10)$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w^*_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] \quad ...(11)$$

3. Solution of NSVIEK2 using (P-CM):

Consider the ith equation of (1):

$$f_i(x) = g_i(x) + \int_a^x k_i(x, t, f_1(t), f_2(t), ..., f_m(t))dt \,, \qquad ...(12)$$

The form of the explicit fourth-order Adams Bashforth equation (10) can be written as:

$$w_{i,j+1} = w_{i,j} + \frac{h}{24}[55k_i(x_{j+1}, t_j, f_{1j}, f_{2,j}, ..., f_{m,j}) - 59k_i(x_{j+1}, t_{j-1}, f_{1,j-1}, f_{2,j-1}, ..., f_{m,j-1}) + 37k_i(x_{j+1}, t_{j-2}, f_{1,j-2}, f_{2,j-2}, ..., f_{m,j-2}) - 9k_i(x_{j+1}, t_{j-3}, f_{1,j-3}, f_{2,j-3}, ..., f_{m,j-3})] \qquad ...(13)$$

$$f_{i,j+1} = g_{i,j+1} + w_{i,j+1} \qquad ...(14)$$

where $f_{i,j+1} = f_i(x_{j+1}), \quad g_{i,j+1} = g_i(x_{j+1}), \quad w_{i,j+1} = w_i(x_{j+1})$

The form of the implicit fourth-order Adams Moulton equation (11) can be written as:

$$w_{i,j+1} = w_{i,j} + \frac{h}{24}[9k_i(x_{j+1}, t_{j+1}, f_{1,j+1}, f_{2,j+1}, ..., f_{m,j+1}) + 19k_i(x_{j+1}, t_j, f_{1,j}, f_{2,j}, ..., f_{m,j}) - 5k_i(x_{j+1}, t_{j-1}, f_{1,j-1}, f_{2,j-1}, ..., f_{m,j-1}) + k_i(x_{j+1}, t_{j-2}, f_{1,j-2}, f_{2,j-2}, ..., f_{m,j-2})] \qquad ...(15)$$

$$f_{i,j+1} = g_{i,j+1} + w_{i,j+1} \qquad ...(16)$$

where $f_{i,j+1} = f_i(x_{j+1}), \quad g_{i,j+1} = g_i(x_{j+1}), \quad w_{i,j+1} = w_i(x_{j+1})$

4. Stability

Definition 1: **(**Delves and Walsh, [4])

Let $\lambda_1, \lambda_2, ..., \lambda_m$ denote the (not necessarily distinct) roots of the characteristic equation

$$\lambda^m - a_{m-1}\lambda^{m-1} - .... - a_1\lambda^1 - a_0 = 0 \qquad ...(17)$$

associated with the multistep difference method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad ... \quad, w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + ... + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, ..., w_{i+1-m})$$

If $|\lambda_i| \le 1$ for each $i = 1,2,3,...,m$ and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 1: (William and Richard, [11])

A multistep method of the form

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad ... \quad, w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + ... + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, ..., w_{i+1-m})$$

is stable if and only if it satisfies root condition; Moreover, if the difference method is consistent with the differential equation, then the method is stable if and only if it is convergent.

4.1 Stability of (P-CM)

We have seen that in the equation (13),

$a_0 = 0, a_1 = 0, a_2 = 0, and\ a_3 = 1$

Then the characteristic equation for (13) is, consequently.

$\lambda^4 - \lambda^3 = \lambda^3(\lambda - 1) = 0$

which has roots $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 0, and\ \lambda_4 = 0$.

Hence, by definition 1 equation (13) satisfies the root condition
Then, by theorem 1 it is stable.
Also, we have seen that in the equation (15),

$a_0 = 0, a_1 = 0, and\ a_2 = 1$

Then, the characteristic equation for (15) is, consequently.

$\lambda^3 - \lambda^2 = \lambda^2(\lambda - 1) = 0$

which has roots $\lambda_1 = 1,\ \lambda_2 = 0,\ and\ \lambda_3 = 0$.

Hence, by definition 1 equation (15) satisfies the root condition
Then, by theorem 1 it is stable.

Algorithm of Runge-Kutta Method:

Step1: fix $f_{i,0} = g_i(0)$, $i = 1,2,..m$

Step2: Letting $h = \dfrac{b-a}{n}$, $n \in N$.

Step3: Letting $j = 1,2,..n+1$

Step4: Find $u_{i,j}, L_{i,j}^{(1)}, L_{i,j}^{(2)}, L_{i,j}^{(3)}, L_{i,j}^{(4)}, f_{i,j}$ *For* $i = 1,2,..m$ using the following

$$u_{i,j} = f_{i,j-1} - g_i((j-1)h)$$

$$L_{i,j}^{(1)} = f_{i,j-1}$$

$$L_{i,j}^{(2)} = u_{i,j} + g_i((j-1)h + h/2) + \frac{h}{2}[k_i(jh, (j-1)h + h/2, L_{i,j}^{(1)})$$

$$L_{i,j}^{(3)} = u_{i,j} + g_i((j-1)h + h/2) + \frac{h}{2}[k_i(jh, (j-1)h + h/2, L_{i,j}^{(2)})$$

$$L_{i,j}^{(4)} = u_{i,j} + g_i((j-1)h + h) + h[k_i(jh, (j-1)h + h, L_{i,j}^{(3)})$$

$$f_{i,j} = u_{i,j} + g_i(jh) + \frac{h}{6}[k_i(jh, (j-1)h, L_{i,j}^{(1)}) + 2k_i(jh, (j-1)h + h/2, L_{i,j}^{(2)})$$

$$+ 2k_i(jh, (j-1)h + h/2, L_{i,j}^{(3)}) + k_i(jh, jh, L_{i,j}^{(4)})$$

Algorithm of (P-CM):

Step (1): Fix $f_{i0} = g_i(a)$, $i = 1, 2, \ldots, m$.

Step (2): Letting $h = \dfrac{b-a}{n}$, $n \in N$.

$x_i = a + ih, \quad t_i = a + ih$.

Step (3): Calculate $g_i(x)$, for $i = 1, 2, \ldots, m$.

Step (4): Using the algorithm of Rung-Kutta method, to find the unknown's $f_{i,1}$, $f_{i,2}$, and $f_{i,3}$.

Step (5): For $j = 3,4,5...,N$

$$w_{i,j+1}^{(0)} = w_{i,j} + \frac{h}{24}[55k_i(x_{j+1}, t_j, f_{1j}, f_{2,j},..., f_{m,j}) - 59k_i(x_{j+1}, t_{j-1}, f_{1,j-1}, f_{2,j-1},..., f_{m,j-1}) + 37k_i(x_{j+1}, t_{j-2}, f_{1,j-2}, f_{2,j-2},..., f_{m,j-2}) - 9k_i(x_{j+1}, t_{j-3}, f_{1,j-3}, f_{2,j-3},..., f_{m,j-3})] \quad ...(18)$$

$$f_{i,j+1}^{(0)} = g_{i,j+1} + w_{i,j+1}^{(0)} \quad ...(19)$$

$$w_{i,j+1} = w_{i,j} + \frac{h}{24}[9k_i(x_{j+1}, t_{j+1}, f_{1,j+1}^{(0)}, f_{2,j+1}^{(0)},..., f_{m,j+1}^{(0)}) + 19k_i(x_{j+1}, t_j, f_{1,j}, f_{2,j},..., f_{m,j}) - 5k_i(x_{j+1}, t_{j-1}, f_{1,j-1}, f_{2,j-1},..., f_{m,j-1}) + k_i(x_{j+1}, t_{j-2}, f_{1,j-2}, f_{2,j-2},..., f_{m,j-2})] \quad ...(20)$$

$$f_{i,j+1} = g_{i,j+1} + w_{i,j+1} \quad ...(21)$$

Step(6): L.S.E= $(exact(x) - f_{i,j})$^2 $\quad i = 1,...,m, \quad j = 1,..N$

5. Illustrative Examples

In this section, two examples are presented for demonstrating the method and a comparison among the solutions obtained by this method against the exact solution which has been made depending on the least square errors (L.S.E).

Example 1: (Waswas, [10])

Solve a system of non-linear VIEK2's:

$$f_1(x) = \sec(x) - x + \int_0^x (f_1^2(t) - f_2^2(t))dt$$

$$f_2(x) = 3\tan(x) - x - \int_0^x (f_1^2(t) + f_2^2(t))dt$$

The exact solution of this system is:

$f_1(x) = \sec(x)$ and $f_2(x) = \tan(x)$

After solving this system by Predictor-Corrector method with h=0.1 in equations (13)-(16) for **A-BM** and **A-MM** and equations (18)-(21) for **P-CM**, we obtain the following numerical solution.

Table (1) comparison between the exact solution $\sec(x)$ and the numerical solution $f_1(x)$ of Example 1 taking h=0.1.

x	Exact	A-BM	A-MM	P-CM
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	1.0050209184	1.0050206529	1.0050206529	1.0050206529
0.2	1.0203388449	1.0203381445	1.0203381445	1.0203381445

0.3	1.0467516015	1.0467500295	1.0467500295	1.0467500295
0.4	1.0857044284	1.0857025412	1.0852627697	1.0856976449
0.5	1.1394939273	1.1394615248	1.1381423415	1.1394784299
0.6	1.2116283145	1.2115250364	1.2085446655	1.2115971159
0.7	1.3074592597	1.3071839148	1.3011583728	1.3073969157
0.8	1.4353241997	1.4346596963	1.4229915908	1.4351938722
0.9	1.6087258105	1.6071582434	1.5846849365	1.6084310654
1.0	1.8508157177	1.8470647612	1.8026544923	1.8500734142
L.S.E		1.70560e-005	3.10079e-003	6.60024e-007
R.T		0.1100second	0.1560second	0.1400second

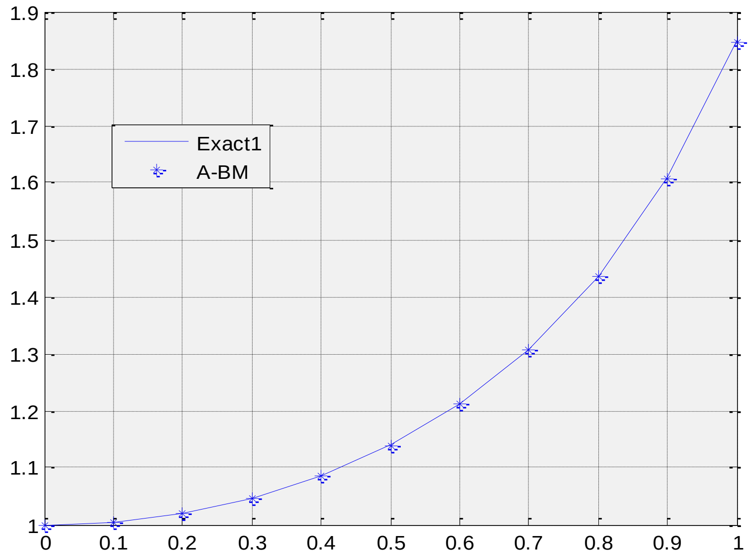
Table (2) shows the comparison between the exact solution $\tan(x)$ and the numerical solution $f_2(x)$ of Example 1 taking h=0.1

x	Exact	A-BM	A-MM	P-CM
0.0	0	0	0	0
0.1	0.1003346721	0.1003342422	0.1003342421	0.1003342422
0.2	0.2027100355	0.2027090714	0.2027090714	0.2027090714
0.3	0.3093362496	0.3093344666	0.3093344666	0.3093344665
0.4	0.4227932187	0.4229478548	0.4223727949	0.4227722795
0.5	0.5463024898	0.5466490912	0.5452558909	0.5462537940
0.6	0.6841368083	0.6847911760	0.6821812399	0.6840434458
0.7	0.8422883805	0.8434304213	0.8390598963	0.8421161023
0.8	1.0296385571	1.0316635320	1.0246489149	1.0293138426
0.9	1.2601582175	1.2638895190	1.2527272100	1.2595096789
1.0	1.5574077246	1.5647973445	1.5465474314	1.5559902860
L.S.E		7.45061e-005	2.13582e-004	2.57638e-006
R.T		0.1100second	0.1560second	0.1400second

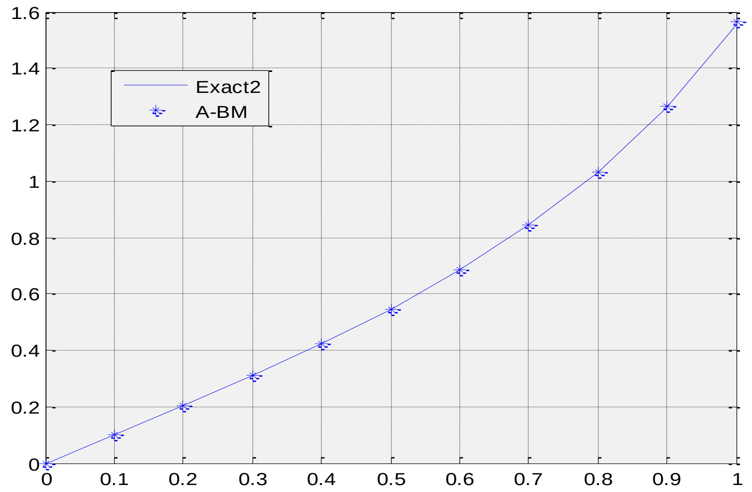
Table (3) shows the least square errors for $f_1(x)$ and $f_2(x)$ with different values of h for Example 1.

Numerical solution of	**method**	**P-CM**		
	h	0.05	0.02	0.01
$f_1(x)$	L.S.E	6.29169e-010	6.73993e-013	1.95962e-014
	R.T	0.2030second	0.3290 second	0.5940 second
$f_2(x)$	L.S.E	1.98911e-008	3.01182e-011	2.24865e-013
	R.T	0.2030 second	0.3290 second	0.5940 second

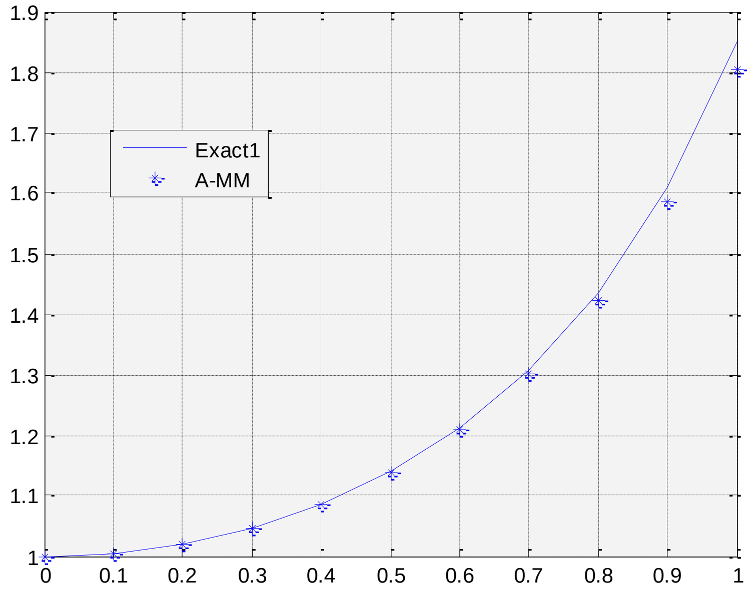
Figure(1) shows the comparison between the exact solution $\sec(x)$ and the numerical solution $f_1(x)$ using **A-BM** of Example 1 taking h=0.1.



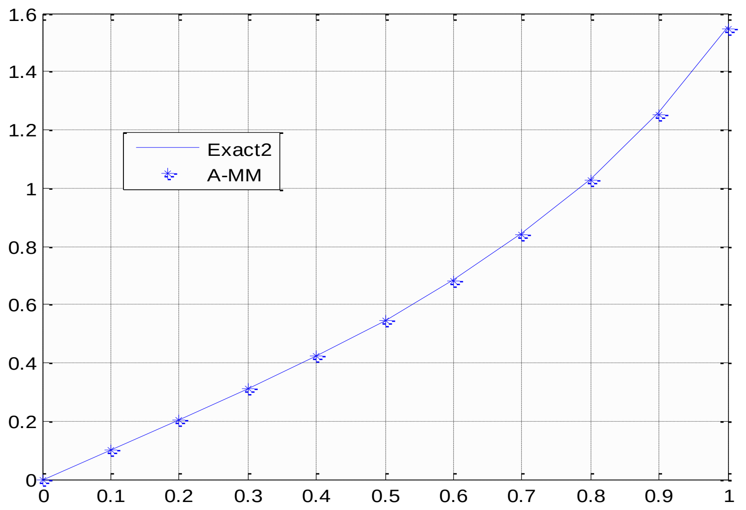
Figure(2) shows the comparison between the exact solution $\tan(x)$ and the numerical solution $f_2(x)$ using **A-BM** of Example 1 taking h=0.1



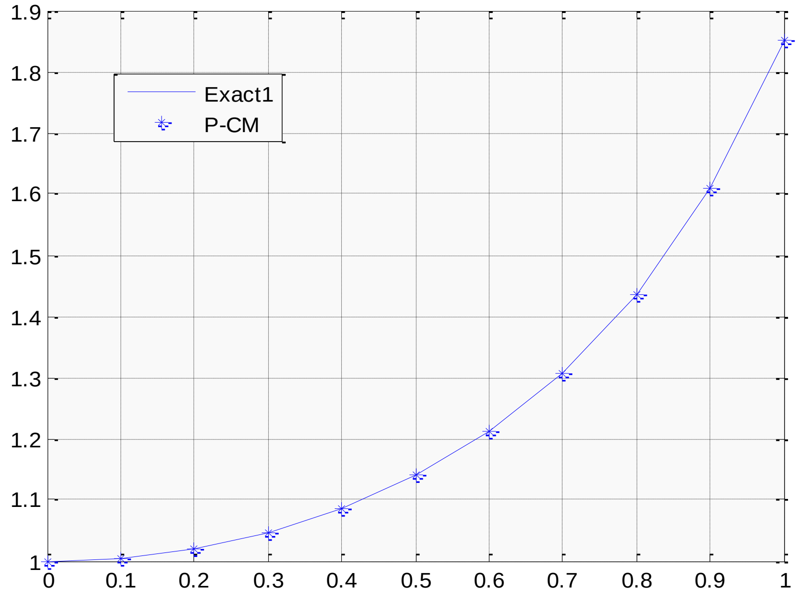
Figure(3) shows the comparison between the exact solution $\sec(x)$ and the numerical solution $f_1(x)$ using **A-MM** of Example 1 taking h=0.1.



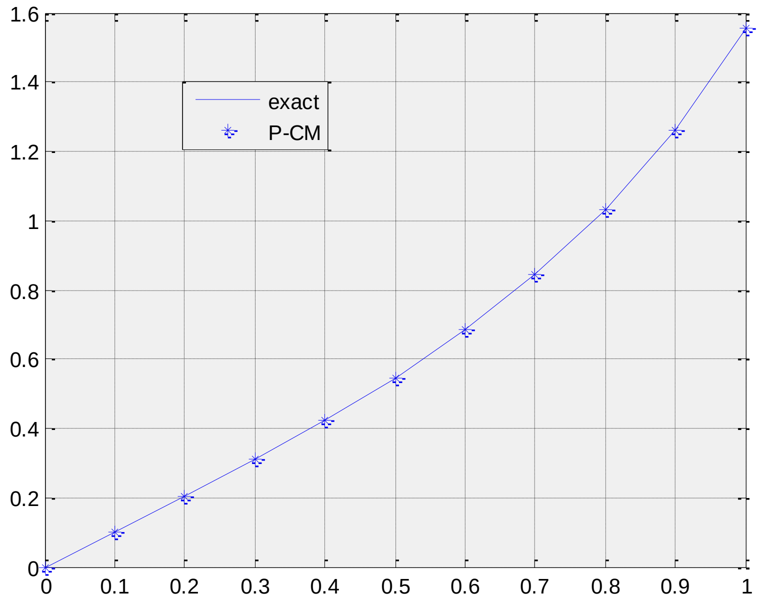
Figure(4) shows the comparison between the exact solution $\tan(x)$ and the numerical solution $f_2(x)$ using **A-MM** of Example 1 taking h=0.1



Figure(5) shows the comparison between the exact solution $\sec(x)$ and the numerical solution $f_1(x)$ using **P-CM** of Example 1 taking h=0.1.



Figure(6) shows the comparison between the exact solution $\tan(x)$ and the numerical solution $f_2(x)$ using **P-CM** of Example 1 taking h=0.1



Example 2: (Jumaa, [6])

Solve a system of non-linear VIEK2's:

$$f_1(x) = x(1-\frac{1}{2}x) + \int_0^x \ln\left| f_2(t) \right| \, dt$$

$$f_2(x) = -(xe^x + 1) + \int_0^x t e^{f_1(t)} dt$$

The exact solution of this system is:

$f_1(x) = x$ and $f_2(x) = -e^x$

After solving this system by Predictor-Corrector method with h=0.1 in equations (13)-(16) for **A-BM** and **A-MM** and equations (18)-(21) for **P-CM**, we obtain the following numerical solution.

Table (4) comparison between the exact solution x and the numerical solution $f_1(x)$ of Example 2 taking h=0.1.

x	Exact	A-BM	A-MM	P-CM
0.0	0	0	0	0
0.1	0.1000000000	0.0999999309	0.0999999309	0.0999999309
0.2	0.2000000000	0.1999998578	0.1999998578	0.1999998578
0.3	0.3000000000	0.2999997793	0.2999997793	0.2999997793
0.4	0.4000000000	0.3999998194	0.4004392373	0.4000002609
0.5	0.5000000000	0.5000025721	0.5009111633	0.5000006337
0.6	0.6000000000	0.6000049457	0.6014126140	0.6000009292
0.7	0.7000000000	0.7000087268	0.7019404466	0.7000011459
0.8	0.8000000000	0.8000134286	0.8024905501	0.8000012879
0.9	0.9000000000	0.9000188603	0.9030580543	0.9000013590
1.0	1.0000000000	1.0000249835	1.0036373570	1.0000013629
L.S.E		1.26755e-009	3.55689e-005	8.08306e-012
R.T		0.1090 second	0.1250 second	0.1250 second

Table (5) shows the comparison between the exact solution $-e^x$ and the numerical solution $f_2(x)$ of Example 2 taking h=0.1

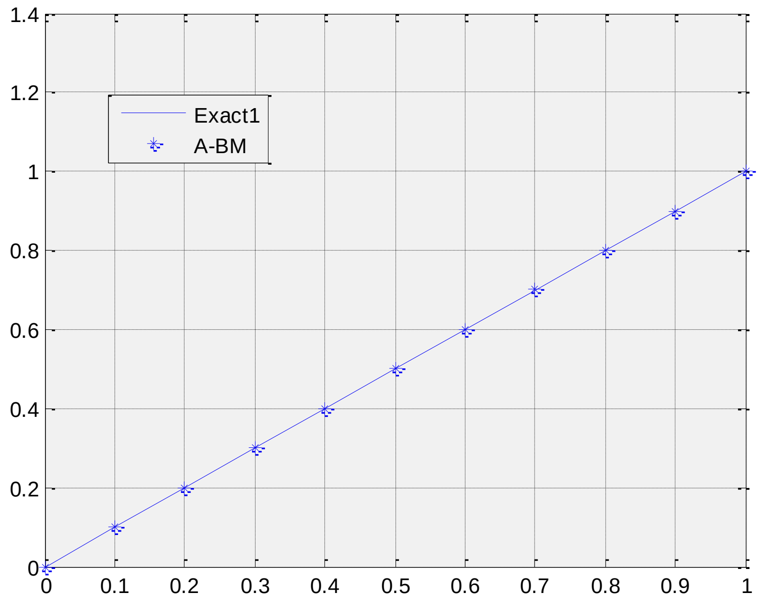
x	Exact	A-BM	A-MM	P-CM
0.0	-1.000000000	-1.000000000	-1.000000000	-1.000000000
0.1	-1.105170918	-1.105171026	-1.105171026	-1.105171026
0.2	-1.221402758	-1.221403013	-1.221403013	-1.221403013
0.3	-1.349858808	-1.349859257	-1.349859257	-1.349859257
0.4	-1.491824698	-1.491842965	-1.491813842	-1.491823668
0.5	-1.648721271	-1.648759691	-1.648659805	-1.648718533
0.6	-1.822118800	-1.822179503	-1.821943458	-1.822114094
0.7	-2.013752707	-2.013838457	-2.013364768	-2.013745732
0.8	-2.225540928	-2.225653949	-2.224796393	-2.225531344
0.9	-2.459603111	-2.459745773	-2.458300725	-2.459590535

1.0	-2.718281828	-2.718456185	-2.716149269	-2.718265841
L.S.E		7.63744e-008	6.98349e-006	5.85238e-010
R.T		0.1090 second	0.1250 second	0.1250 second

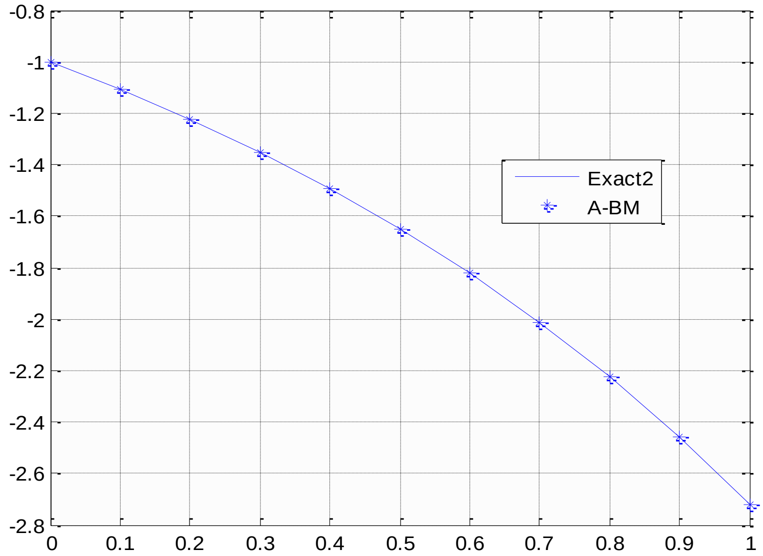
Table (6) shows the least square errors for $f_1(x)$ and $f_2(x)$ with different values of h for Example 2.

Numerical solution of	**method**	**P-CM**		
	h	0.05	0.02	0.01
$f_1(x)$	L.S.E	6.13653e-015	1.54294e-016	2.18979e-018
	R.T	0.1720 second	0.3120 second	0.5000 second
$f_2(x)$	L.S.E	6.06538e-012	1.10743e-014	8.89313e-017
	R.T	0.1720 second	0.3120 second	0.5000 second

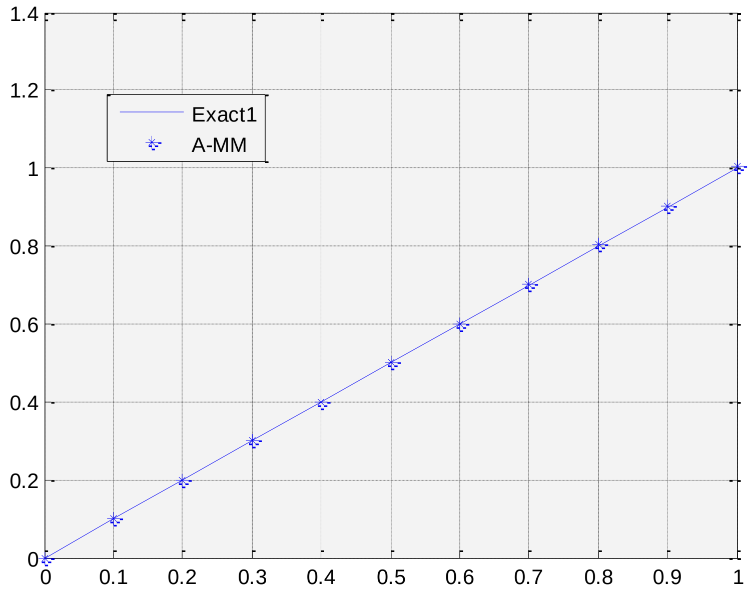
Figure(7) comparison between the exact solution x and the numerical solution $f_1(x)$ using **A-BM** of Example 2 taking h=0.1.



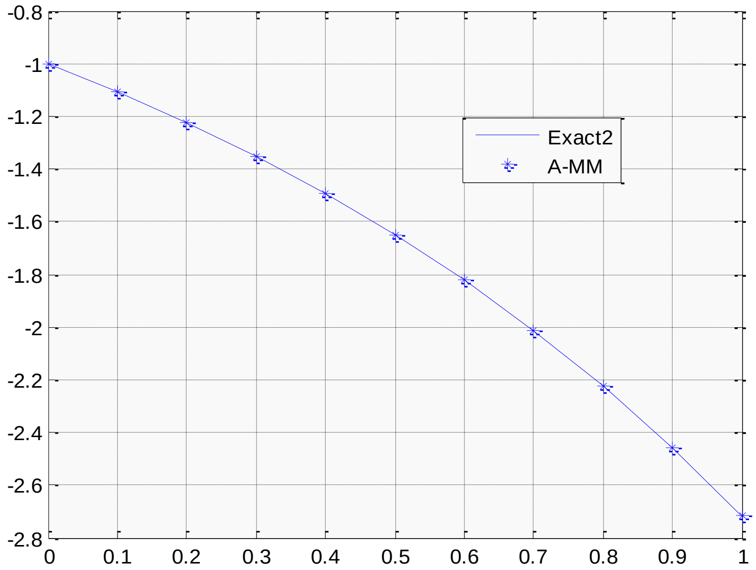
Figure(8) shows the comparison between the exact solution $-e^{x}$ and the numerical solution $f_2(x)$ using **A-BM** of Example 2 taking *h*=0.1



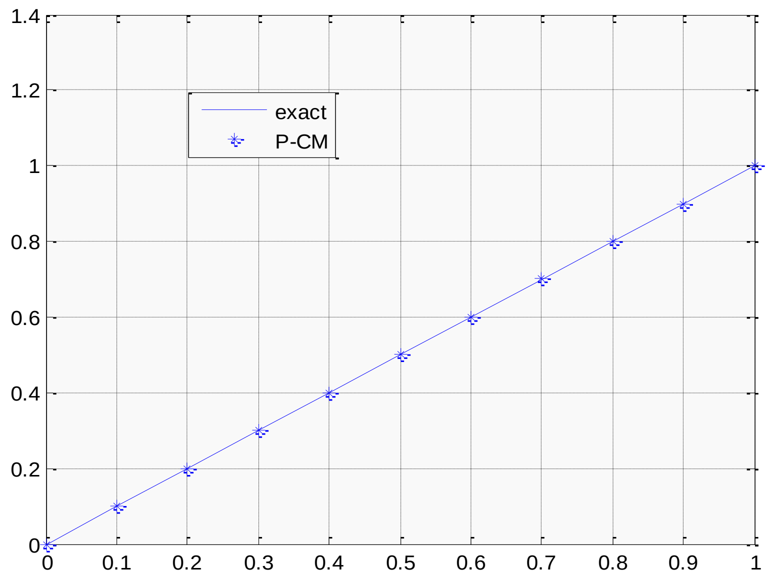
Figure(9) comparison between the exact solution x and the numerical solution $f_1(x)$ using **A-MM** of Example 2 taking *h*=0.1.



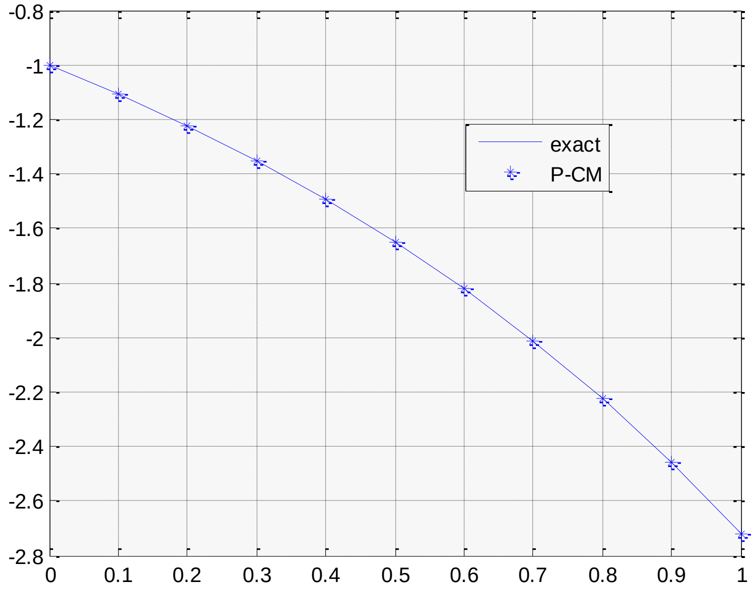
Figure(10) shows the comparison between the exact solution $-e^x$ and the numerical solution $f_2(x)$ using **A-MM** of Example 2 taking h=0.1



Figure(11) comparison between the exact solution x and the numerical solution $f_1(x)$ using **P-CM** of Example 2 taking h=0.1.



Figure(12) shows the comparison between the exact solution $-e^x$ and the numerical solution $f_2(x)$ using **P-CM** of Example 2 taking h=0.1



6. Conclusions

According to the numerical results which obtaining from the illustrative examples we concludes the following:

1. The explicit fourth-order Adams-Bashforth method gave better results than the implicit fourth-order Adams-Moulton method.
2. If we use the explicit fourth-order Adams Bashforth method as Predictor and the implicit fourth-order Adams-Moulton method as Corrector, then the method gave better results than explicit fourth-order Adams-Bashforth method and than the implicit fourth-order Adams-Moulton method.
3. The A-BM, A-MM, and P-CM methods are stable by section (4-1).
4. In P-CM, the error will be decreasing if we chose small values for h (step size) and it is the faster.

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