Real Dynamics and Bifurcation of the Family $\lambda \frac{\sinh^2(x)}{x^3}$

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ABSTRACT

The purpose of this work is to study the dynamics of the family of nonlinear functions $\mathcal{F}=\{f_\lambda(x)=\lambda\frac{\sinh^2(x)}{x^3},\ x\in\mathbb{R}, x\neq 0, \lambda>0\}$. It is shown that the bifurcation in the dynamics of $f_\lambda(x)$ occurs at the critical parameter value $\lambda_1 \approx 1.2203$.

Keywords: Dynamics, bifurcation, real nonlinear functions.

الدينمية الحقيقية والتشعب للعائلة $\lambda \frac{\sinh^2(x)}{x^3}$

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الملخص

الغرض من هذا البحث هو دراسة دينمية العائلة $\mathcal{F}=\{\lambda f_\lambda(x)=\lambda\frac{\sinh^2(x)}{x^3}, x\in R-\{0\}, >0\}$ من الدوال الحقيقية اللاخطية. سنبرهن على أن التشعب في دينمية العائلة يحدث عندما تأخذ المعلمة القيمة الحرجة $\lambda_1 \approx 1.2203$.

الكلمات المفتاحية: الدينمية، التشعب، الدوال الحقيقية اللاخطية.

1. Introduction

The last fifty years have been seen an explosion of interest in the study on nonlinear dynamical systems. Scientists in all disciplines have come to realize the power and the beauty of the geometric and qualitative techniques developed during this period. More importantly, they have been able to apply these techniques to a number of important nonlinear problems ranging from physics and chemistry to ecology and economics. It is known that many nonlinear systems come from biological, physical and engineering problems[3]. The chaotic behavior of various systems and the complexity in iterates of nonlinear functions, the challenges of their theoretical study, and their wide ranging applications in science and engineering; it has been a popular topic of exploration from mathematicians, physicians and scientists in recent years[4]. Nice introduction to the Quadratic map can be found in [1] and for other types of one parameter families can be found in [5]. In this work we suggest a family of nonlinear functions $\mathcal{F} = \{f_\lambda(x) = \lambda \frac{\sinh^2(x)}{x^3},\ x \in \mathbb{R}, x \neq 0, \lambda > 0\}$ and study the dynamics of this family.

Let f be a non-constant function. Define $f^{(0)}(x) = x, f^{(1)}(x) = f(x), f^{(2)}(x) = f(f(x)), \dots, f^{(n)}(x) = f(f^{n-1}(x)), n \geq 1$ where $f^{(n)}$ denotes the n-th iteration of f. The set $\{f^{(n)}(x): n \in \mathbb{N}\}$ is called the **orbit of x**, and the set of points $\{x: f^{(n)}(x) = x_0\ for\ some\ n \in \mathbb{N}\}$ is called the **backward orbit of x_0**. The point x_0 is called a fixed point of f if $f(x_0) = x_0$ and it is classified as:

(a) $If\ |f'(x_0)| < 1\ then\ x_0\ is\ called\ \textbf{attracting}.$

(b) $If\ |f'(x_0)| = 1\ then\ x_0\ is\ called\ \textbf{rationally indifferent}.$

(c) $If\ |f'(x_0)| > 1\ then\ x_0\ is\ called\ \textbf{repelling}.$

$|f'(x_0)|$ is called the **multiplier of x_0** [2].

Let $\mathcal{F}=\{f_\lambda(x) = \lambda \frac{\sinh^2(x)}{x^3},\ x \in \mathbb{R}, x \neq 0, \lambda > 0\}$ be a family of nonlinear functions with the real parameter λ. In this paper we study the real dynamics of this family of nonlinear functions. We describe the existence and the nature of the fixed points of the function $f_\lambda(x) \in \mathcal{F}$. We find two critical parameter values of f_λ, $\lambda_1 \approx 1.2203$ and $\lambda_2 \approx 0.1252$ where $\lambda_1 = \beta(x_1)$, $\lambda_2 = \beta(x_2)$ and $\beta(x) = \frac{x^4}{\sinh^2(x)}$. Finally we study the bifurcation in the dynamics of $f_\lambda(x) \in \mathcal{F}$. Recall that, bifurcation means a division into two, splitting parts or a changes.

In dynamical systems, the bifurcation is to study the change that maps undergo as parameter changes. These changes often involve the periodic points structure but may involve other changes as well. We show that for $f_\lambda(x) \in \mathcal{F}$, the bifurcation occurs at the critical parameter value $\lambda = \lambda_1$.

2. Dynamics and bifurcation of the functions in the family $\mathcal{F}$:

Let $\mathcal{F}=\{f_\lambda(x) = \lambda \frac{\sinh^2(x)}{x^3}, \lambda > 0\}$ be a one parameter family of transcendental functions. In this section we study the dynamics and bifurcation of the function $f_\lambda(x) \in \mathcal{F}$. First we describe the fixed points of $f_\lambda(x)$ and their nature. Let $f_\lambda(x) = x$. This implies that $\lambda = \frac{x^4}{\sinh^2(x)}$. So the fixed points of $f_\lambda(x)$ are the solutions of $\beta(x)=\lambda$, where

$$\beta(x)=\begin{cases}\frac{x^4}{\sinh^2(x)} & x\neq 0\\ 0 & x=0\end{cases}$$

We give some properties of $\beta(x)$

1. β is continuous in $\mathbb{R}$.
2. $\beta(x)\to 0$ as $x\to\infty$ and $\beta(x)\to 0$ as $x\to-\infty$.
3. $\beta(x)>0\ \forall x\in\mathbb{R}$.
4. $\beta'(x)=(4x^3\sinh(x)-2x^4\cosh(x))/(\sinh^3(x))$. Thus $\beta'(0)=\lim_{x\to 0}\beta'(0)$,i.e. β' is continuous in $\mathbb{R}$.
5. $\beta'(x)$=0 has a unique positive solution $x_1\approx 1.9150$. Note that x_0 is a solution of $\beta'(x)$ iff x_0 is a solution of $2\tanh(x)-x=0$ and by Newton-Raphson method we find x_1 . Also $\beta'(x)=0$ has a unique negative solution at $x=-x_1$.
6. Since $\beta''(x_1)<0$.Thus $\beta(x)$ has exactly one maximum point at $x=x_1$ in $(0,\infty)$. It also follows, by property 2 ,that β strictly increasing in $(0,x_1)$ and strictly decreasing in (x_1,∞) .
7. β is symmetric around the y-axis.

Figure (1) gives the graph of $\beta(x)$.

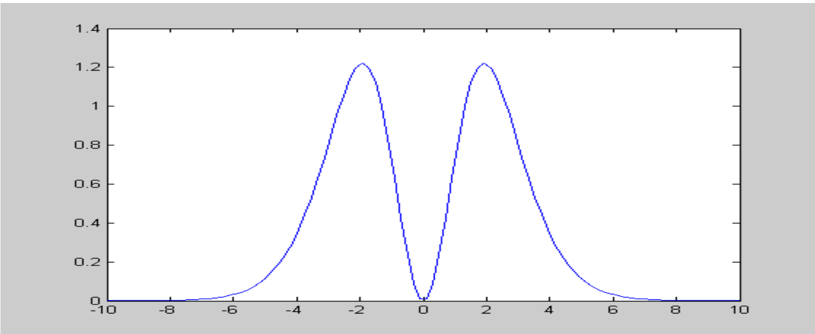

Figure (1)

Let $\lambda_1=\beta(x_1)$.We call it a critical parameter value of $f_\lambda(x)$.

Proposition (2.1) The locations of the fixed points of $f_\lambda\in\mathcal{F}$ are given as follows :

1. For $0<\lambda<\lambda_1$, f_λ has two fixed points one of them in $(0,x_1)$ and the other in(x_1,∞). Further, f_λ has two fixed points living in $(-\infty,-x_1)$and $(-x_1,0)$.

2. For $\lambda=\lambda_1$, f_λ has two fixed points at $x = x_1$ and $x = -x_1$.
3. For $\lambda>\lambda_1$, f_λ has no fixed points.

Proof: **1.** For $0<\lambda<\lambda_1$, by properties 1,2 and 6, the line u=λ intersects the graph of $\beta(x)$ at exactly four points two of them at the right of the y-axis and two at the left of the y-axis (see Fig 1). Using the properties 2,3 and 6, and since $\beta(x_1)=\lambda_1$ one of the solutions of $\beta(x)=\lambda$ lies in the interval $(0,x_1)$.. Also by 6, $\beta(x)$ is decreasing in (x_1,∞) and $\beta(x_1)=\lambda_1$ **,** thus the other solution of $\beta(x) = \lambda$ lies in (x_1,∞). Therefore $f_\lambda(x)$ has one fixed point in $(0,x_1)$ and another one in (x_1,∞). By property 7, $f_\lambda(x)$ has only one fixed point in each of the intervals $(-\infty,-x_1)$ and $(-x_1,0)$.
2. For $\lambda=\lambda_1$**,** since $\beta(x_1)=\lambda_1$, and x_1 is the maximum point for $f_\lambda(x)$ in $(0,\infty)$, thus the line u=λ intersects the graph of $\beta(x)$ at $x = x_1$.Thus $\beta(x) = \lambda$ has only one solution at $x = x_1$.
Hence f_λ has a fixed points at $x = x_1$. Again , by property 7 $x = -x_1$ is the solution of the equation $\beta(x) = \lambda_1$ in the interval $(-\infty,0)$. Thus , $f_\lambda(x)$ has a fixed point, for $\lambda=\lambda_1$, at $x = -x_1$.
3. By property 6, λ_1 is the maximum value of $\beta(x)$. Thus the line u=λ does not intersects the graph of $\beta(x)$ at any point for each $\lambda > \lambda_1$. Hence $\beta(x)=\lambda$ has no solutions for this case. Therefore $f_\lambda(x)$ has no fixed points for each $\lambda > \lambda_1$.

To study the nature of the fixed points of the function $f_\lambda(x)$, put $|f_\lambda'(x)| = 1$. We have two equations, $f'(x) = 1$ and $f'(x) = -1$. The equation $f'(x) = 1$ iff $2tanh(x) - x = 0$. Therefore, it is equivalent to $\beta'(x) = 0$, while the equation $f'(x) = -1$ is equivalent to the equation $tanh(x) - x = 0$, thus we have another critical parameter value $\lambda_2 = \beta(x_2)$, where x_2 is the solution of the equation $tanh(x) - x = 0$. Numerically, $x_2 \approx 0.3616$ and $\lambda_2 \approx .1252$.

The nature of the fixed points of $f_\lambda(x)$ for various values of the parameter λ is described in the following:

Theorem (2.2): Let $f_\lambda(x) = \lambda\frac{\sinh^2(x)}{x^3}$,$x \neq 0, \lambda > 0$. Let x_1 be a solution of $\beta'(x)$=0 and $\lambda_1 = \beta(x_1)$. Then
1. For $0<\lambda<\lambda_1$, the fixed points $a_\lambda \in (0,x_1)$ and $-a_\lambda \in (-x_1,0)$ are attracting and the fixed points $r_\lambda \in (x_1,\infty)$ and $-r_\lambda \in (-\infty,-x_1)$ are repelling.

2. For $\lambda=\lambda_1$, the two fixed points x_1 and $-x_1$ are rationally indifferent.

Proof: The derivative of the function $f_\lambda(x) = \lambda\frac{\sinh^2(x)}{x^3}$ is given by $f_\lambda'(x) = \lambda\frac{2x\sinh(x)\cosh(x)-3\sinh^2(x)}{x^4}$ and the fixed points of the function $f_\lambda(x)$ are the solutions of the equation $\lambda=\frac{x^4}{\sinh^2(x)}$. Thus the multiplier is $|f_\lambda'(x_f)|$ of the fixed point x_f is given by

$$|f_\lambda'(x_f)| = |2x_f\coth(x_f) - 3| \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*).$$

Define $G(x) = \begin{cases} 2x\coth(x) - 3 & x \neq 0 \\ -1 & x = 0 \end{cases}$

The function $G(x)$ is differentiable and

$$G'(x) = \begin{cases} 2\left(\coth(x) - x\,csch^2(x)\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $G'(x) \neq 0$ for $x \neq 0, G'(0) = 0$ *and* $G''(0) > 0$, then the function $G(x)$ has exactly one minimum at $x = 0$ and the minimum value is -1. Since $G'(x)>0$ for $x \in (0,\infty)$ and $G'(x)<0$ for $x\in(-\infty,0)$, then $G(x)$ is increasing from -1 to ∞ as x increases from 0 to ∞ and $G(x)$ is decreasing from ∞ to -1 as x increases from -∞ to 0. From these observations it follows that $|G(x)|$ satisfies :

$$|G(x)| = \begin{cases} < 1 & for\ x \in (-x_1, x_1) \\ = 1 & for\ x = \pm x_1 \\ > 1 & for\ x \in (-\infty, -x_1) \cup (x_1, \infty) \end{cases}$$

See Figure (2).

Therefore, by (*), the multiplier $|f_\lambda'(x_f)|$ satisfies

$$|f_\lambda'(x_f)| = \begin{cases} < 1 & for\ x \in (-x_1, x_1) \ldots\ldots\ldots\ldots (A) \\ = 1 & for\ x = \pm x_1 . \ldots\ldots\ldots\ldots (B) \\ > 1 & for\ x \in (-\infty, -x_1) \cup (x_1, \infty) \ldots\ldots\ldots\ldots (C) \end{cases}$$

Thus **1.** For $0<\lambda<\lambda_1$, by the equation (A) , the fixed points $a_\lambda \in (0, x_1)$ and $-a_\lambda \in (-x_1, 0)$ are attracting and by (*C*) the fixed points $r_\lambda \in (x_1, \infty)$ and $-r_\lambda \in (-\infty, x_1)$ are repelling.

2.For $\lambda=\lambda_1$ we have two fixed points x_1 and $-x_1$. By the equation (B), both of them are rationally indifferent.

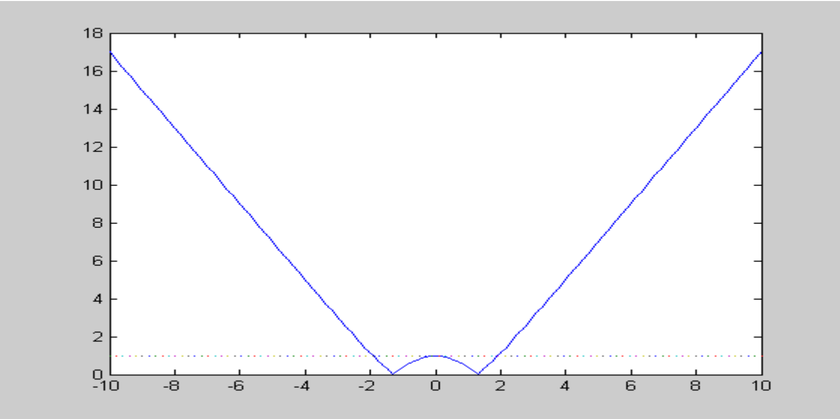

Figure (2): The graph of $|G(x)| = |2x\coth(x) - 3|$

In the following we study the dynamics of $f_\lambda(x), x \in \mathbb{R}, x \neq 0$ and $\lambda > 0$. Also we show that the bifurcation in the dynamics of $f_\lambda(x)$ occurs when the parameter λ crosses the critical parameter value $\lambda_1 = 1.2203$.

Let T_0 be the set of points which are backward orbits of the pole x=0 of the function $f_\lambda(x) = \lambda \frac{\sinh^2(x)}{x^3}$. The dynamics of the functions in our family is studied in the following:

Theorem(2.3) Let $\mathcal{F}=\{f_\lambda(x) = \lambda \frac{\sinh^2(x)}{x^3}, \ x \in \mathbb{R}, x \neq 0, \lambda > 0\}$.

1.For $0<\lambda<\lambda_1$, $f_\lambda^n(x) \to a_\lambda \ as \ n \to \infty \ for \ x \in (\propto, r_\lambda)/T_0$,
$and \ f_\lambda^n(x) \to -a_\lambda \ for \ x \in (-r_\lambda, -\propto)/T_0$. $For \ x \in \{(0, \propto) \cup (r_\lambda, \infty)\}/T_0, f_\lambda^n(x) \to \infty$
and $f_\lambda^n(x) \to -\infty for\{(-\infty, -r_\lambda) \cup (-\propto, 0)\}$ where
$\propto$ *is a positive solution of the equation* $f_\lambda(x) = r_\lambda$, a_λ *is an attracting fixed point of* $f_\lambda(x)$ *and* r_λ *is a repelling fixed point of* $f_\lambda(x)$.

2. $For \ \lambda = \lambda_1, \ f_\lambda^n(x) \to x_1 \ for \ x \in (\mu, x_1) / T_0$ and $f_\lambda^n(x) \to -x_1 \ for \ (-x_1, -\mu)/T_0$.

Moreover, $f_\lambda^n(x) \to \infty$ *for* $x \in \{(0,\mu) \cup (x_1,\infty)\}/T_0$ and $f_\lambda^n(x) \to -\infty$ *for* $x \in \{(-\infty,-x_1) \cup (-\mu,0)\}/T_0$, *where* x_1 *and* $-x_1$ *are the rationally indifferent fixed points of* $f_\lambda(x)$ and

μ *is a positive solution of the equation* $f_\lambda(x) = x_1$.

3. *For* $\lambda > \lambda_1$, $f_\lambda^n(x) \to \infty$ *for* $x \in (0,\infty)/T_0$ and $f_\lambda^n(x) \to -\infty$ *for* $x \in (-\infty,0)/T_0$

Proof: Define $K_\lambda(x) = f_\lambda(x) - x$ *for* $x \in R - \{0\}$. It is clear that $K_\lambda(x)$ is continuous and differentiable on $\mathbb{R}$ -{0}. Further it is easy to see that the solution of $K_\lambda(x) = x$ is exactly the fixed points of $f_\lambda(x)$.

1. For $0<\lambda<\lambda_1$, Theorem (2.2) shows that $f_\lambda(x)$ has two fixed points in the interval (0,∞), $a_\lambda \in (0,x_1)$ *which is attracting and* $r_\lambda \in (x_1,\infty)$ *which is repelling*.

From now on we write $K(x)$ and $f(x)$ instead of $K_\lambda(x)$ and $f_\lambda(x)$.

Since $K'(a_\lambda) < 0$ and $K'(x)$ is continuous in some neighborhood of a_λ. Thus $K'(x) < 0$ in some neighborhood of a_λ. Therefore $K'(x)$ is decreasing in a neighborhood of a_λ. But $K(x)$ is continuous in (0,∞). Thus for sufficiently small $\delta_1, > 0$, $K(x) > 0$ in $(a_\lambda - \delta_1, a_\lambda)$ and $K(x) < 0$ in $(a_\lambda, a_\lambda + \delta_1)$. Further, since $K'(r_\lambda) > 0$ and $K'(x)$ is continuous in some neighborhood of r_λ, then $K'(x) > 0$ in some neighborhood of r_λ. Therefore $K(x)$ is increasing in a neighborhood of r_λ. By the continuity of $K(x)$, for sufficiently small $\delta_2 > 0$, $K(x) < 0$ in $(r_\lambda - \delta_2, r_\lambda)$ and $K(x) > 0$ in $(r_\lambda, r_\lambda + \delta_2)$. But $K(x) \neq 0$ in $(0,\infty)/\{a_\lambda, r_\lambda\}$. Thus $K(x) > 0$ for $x \in (0, a_\lambda) \cup (r_\lambda, \infty)$ and $K(x) < 0$ for $x \in (a_\lambda, r_\lambda)$. Thus

$$K(x) = f(x) - x \begin{cases} > 0 & for\ x \in (0, a_\lambda) \cup (r_\lambda, \infty) \ldots\ldots\ldots\ldots\ldots\ldots (*) \\ < 0 & for\ x \in (a_\lambda, r_\lambda) \ldots\ldots\ldots\ldots\ldots .. (**) \end{cases}$$

See figure (3).

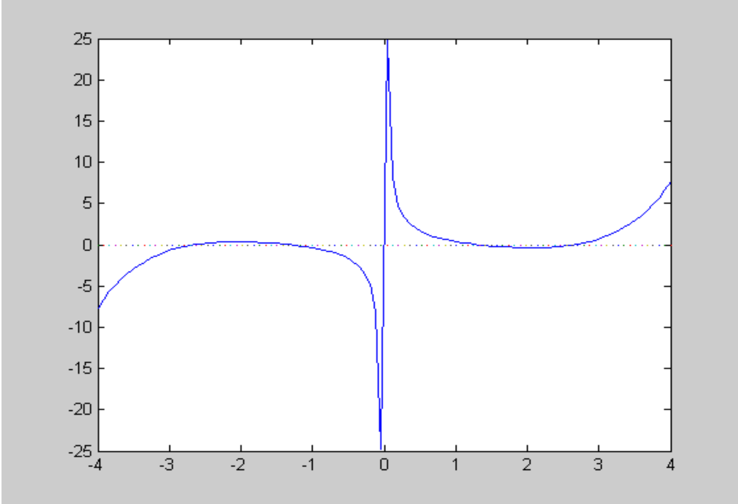

Figure (3): The graph of $K(x) = f(x) - x \quad (0 < \lambda < \lambda_1)$

Now we have two cases:
Case (a). For $x \in (\propto, r_\lambda)/T_0$ and $x \in (-r_\lambda, -\propto)/T_0$
Let $x = x_0$ be the minimum point of $f(x)$ in the interval (a_λ, r_λ) . Then for $x \in (x_0, r_\lambda)$ the function increasing. Further ,by (**) $f(x) < x$.
Thus for $x > x_0 > a_\lambda > 0$ we have $x > f(x) > f^2(x) > \cdots > f^n(x) > \cdots > a_\lambda$, That is the sequence $\{f^n(x)\}$ is decreasing and bounded below by a_λ. Thus $f^n(x) \to a_\lambda$ as $n \to \infty$. for $x \in (x_0, r_\lambda)$. But $f(x)$ is decreasing in $(\propto, x_0)$ and $f(\propto) = r_\lambda$. Thus the function f maps the interval $(\propto, x_0)$ in to (x_0, r_λ). Thus $f_\lambda^n(x) \to a_\lambda \; for \; x \in (\propto, r_\lambda)/T_0$. Since f is an odd function, then $f_\lambda^n(x) \to -a_\lambda$ as $n \to \infty \; for \; x \in (-r_\lambda, -\propto)/T_0$..
Case(b). For $x \in \{(0, \propto) \cup (r_\lambda, \infty) \cup (-\infty, -r_\lambda) \cup (-\propto, 0)\}/T_0$.
By (*) , $f(x) > x$ in each point of the intervals $(0, \propto) \cup (r_\lambda, \infty)$.Since$f(x) > 0$for $x > 0$ and f is strictly increasing in (r_λ, ∞), then $0 < r_\lambda < f(x) < f^2(x) < \cdots < f^n(x) < \cdots$. Thus the sequence $\{f^n(x)\}$ is increasing and not bounded above for each $x \in (r_\lambda, \infty)$. Therefore $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (r_\lambda, \infty)/T_0$. But $f(x)$ is decreasing in $(0, \propto)$ and $f(\propto) = r_\lambda$. Thus $f(x)$ maps $(0, \propto)$ in to (r_λ, ∞). Thus $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in \{(0, \propto) \cup (r_\lambda, \infty)\}/T_0$. Since f is an odd function, then $f_\lambda^n(x) \to -\infty$ as $n \to \infty$ for $x \in \{(-\infty, -r_\lambda) \cup (-\propto, 0)\}/T_0$.

2. For $\lambda = \lambda_1$. By the same arguments used in part **(1)**, we can prove this case.
See figure (4).

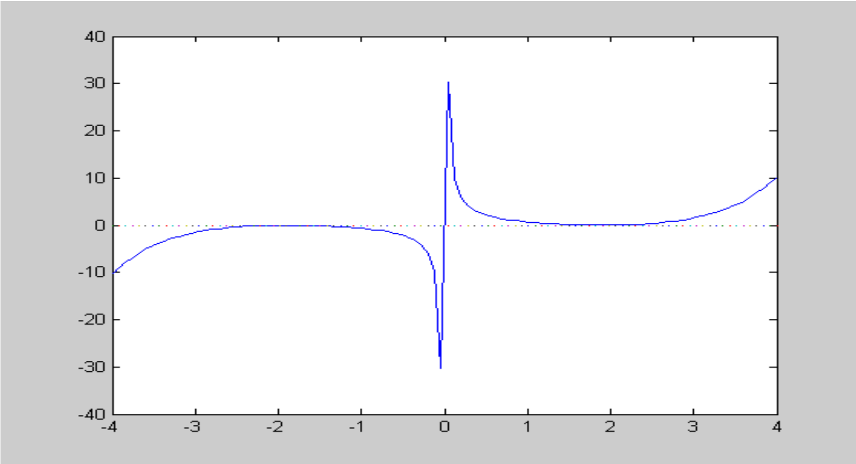

Figure (4): The graph of $K(x) = f(x) - x \quad (\lambda = \lambda_1)$

3. For $\lambda > \lambda_1$, the function $f(x)$ has no fixed points by proposition (2.1). Moreover in this case $f(x) > x$ for $x \in (0, \infty)$ (see Figure 5).
Thus for any $x \in (0,\infty) / T_0$, $0 < x < f(x) < \cdots < f^n(x) < \cdots$.
Therefore, the sequence $\{f^n(x)\}$ is increasing and not bounded above. Thus $f_\lambda^n(x) \to \infty \; for \; x \in (0,\infty) / T_0$. Again since $f(x)$ is an odd function, then $f_\lambda^n(x) \to -\infty \; for \; x \in (-\infty, 0) / T_0$.

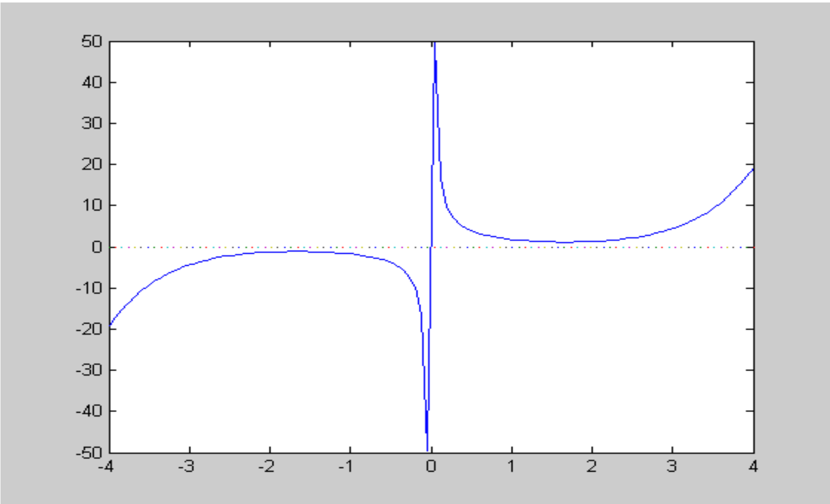

Figure (5): The graph of $K(x) = f(x) - x \quad (\lambda > \lambda_1)$

It follows from Theorem (2.3), that the bifurcation in the dynamics of the function $f_\lambda(x) = \lambda \frac{sinh^2(x)}{x^3}$, $x \in \mathrm{R}, x \neq 0$ occurs at the critical parameter value $\lambda_1 = \frac{x_1^4}{sinh^2(x_1)}$,where $x_{1\approx}1.9150$ is a positive solution of the

equation 2tanh(x)-x =0, and the approximated critical
Value is $\lambda_1 \approx 1.2203$.

Remark:

At the critical parameter value, $\lambda_2 = \beta(x_2)$, where x_2 is the solution of the equation
$tnnh(x) = x$, coming from putting $f'(x) = -1$ ($x_2 \approx 0.3616$ and $\lambda_2 \approx .1252)$. By Theorem
(2.3), at this value of the parameter the dynamics of the function $f_\lambda(x)$ does not change.

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