On Idempotent Elements

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ABSTRACT

In this paper we study idempotent elements, we give some new properties of idempotent elements and provide some exam we also study central idempotent elements and orthogonal idempotent elements and give some new properties of such idempotent.

Finally we study special ring which satisfies the property $x^n = x^{n+1}$ for all x in R and n is a positive integer, we represent such ring in termes of idempotent and nilpotent elements.

Keywords: Rings, idempotent elements, nilpotent elements.

حول العناصر المتحايدة

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الملخص

في هذا البحث درسنا العناصر المتحايدة و أعطينا خواص جديدة لها . و فضلا عن ذلك درسنا العناصر المتحايدة مركزيا و العناصر المتحايدة المتعامدة و أعطينا خواص لهذا النوع من العناصر.

وأخيرا درسنا حلقة خاصة والتي تحقق الخاصية $x^n = x^{n+1}$ ، لكل x فيR وn عدد صحيح موجب ، و أعطينا تمثيلا لهذه الحلقة بدلالة العناصر المتحايدة و المعدومة القوى.

الكلمات المفتاحية: الحلقات، العناصر المتحايدة، المعدومة القوى.

1. Introduction :

Throughout this paper R denotes an associative rings with identity . Recall that:

(1)A ring R is said to be reduced if R contains no non zero nilpotent elements.(2) For any element a of a ring R we define the right annihilator of a in R by, $r(a) = \{x \in R : ax = 0\}$, and likewise the left annihilator of a.in R (3) A ring R is regular provided that for every *x* in R, there exists y in R such that *x* = *xyx* .see[2] (4) An elements e_1, e_2 of a ring R is said to be centeral idempotent elements if $e_1e_2 = e_2e_1$, and orthogonal idempotent elements if $e_1e_2 = e_2e_1 = 0$.

2. Properties of Centeral and Orthogonal Idempotent Elements:

In this section we study centeral and orthogonal idempotent elements and give some basic properties. Also we study special ring which satisfies the relation $x^n = x^{n+1}$, $x \in R$, n is a positive integer .

Proposition2-1: If e_1, e_2 are centeral idempotent elements of R, with $r(e_1 + e_2) = 0$,then $(1-e_1), (1-e_2)$ are orthogonal idempotent elements.

Proof:

Consider $(e_1 + e_2)(1 - e_1 - e_2 + e_1e_2) = \quad e_1 - e_1 - e_1e_2 + e_1e_2 + e_2 - e_2e_1 - e_2 + e_2e_1e_2 = 0$, implies $(1 - e_1 - e_2 + e_1e_2) \in r(e_1 + e_2) = 0$, and $e_1(1-e_2) = (1-e_2)$ Therefore $(1-e_1)(1-e_2) = 0.$ ■

If e_1, e_2 are idempotent elements, then e_1e_2 need not to be idempotent as the following example shows.

Example: let $R(Z_2)$ be the ring of all 2×2 matrices over the ring Z_2 (the ring of integers modulo 2) which are strictly upper triangular.Then the only idempotent matrices of $R(Z_2)$ are:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Now, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not idempotent.

The following result gives the condition for e_1e_2 to be idempotent

Lemma 2-2 : If R is a ring with every idempotent element is centeral, then e_1e_2 is idempotent for every e_1, e_2 are idempotents.

Proof: Trivial.

Theorem 2-3: If e_1, e_2 are centeral idempotent elements of R, and $r(e_1 - e_2) = 0$,then $R = e_1R \oplus e_2R$.

Proof: Let $x \in eR \cap e_2R$ .then $x = e_1r_1$ and $x = e_2r_2$, for some r_1, r_2 in R

Now, $e_1r_1 = e_1e_2r_2$, but $x = e_1r_1$, then $x = e_1e_2r_2$.

Multiplication two sided from left by $(e_1 - e_2)$ we get $(e_1 - e_2)x = 0$, so $x \in r(e_1 - e_2)$ .then $e_1R \cap e_2R \subseteq r(e_1 - e_2)$, but $r(e_1 - e_2) = 0$. So

$$e_1R \cap e_2R = 0. \qquad \ldots (1)$$

Now, consider $(e_1 - e_2)(e_1 + e_2) = (e_1 - e_2)$ implies $(e_1 - e_2)(e_1 + e_2 - 1) = 0$, implies $(e_1 + e_2 - 1) \in r(e_1 - e_2) = 0$. Therefore

$$e_1R + e_2R = R. \qquad \ldots (2)$$

from (1) and (2) we get $R = e_1R \oplus e_2R$. ■

Proposition2-4: If e_1, e_2 are centeral idempotent elements of R , then.

1. $e_1R \cap e_2R = e_1e_2R.$
2. $e_1R \cap e_2R = r(1-e_1) \cap r(1-e_2).$
3. $r(e_1+e_2) = r(e_1) \cap r(e_2)$ if $e_1R \cap e_2R = (0)$.

Proof 1: Let $x \in e_1R \cap e_2R$, then $x = e_1r_1$ and $x = e_2r_2$, for some r_1, r_2 in R . since $e_1x = e_1r_1 = x,$ then $e_1x = e_1e_2r_2$ yields $x = e_1e_2r_2 \in e_1e_2R$, so

$$e_1R \cap e_2R \subseteq e_1e_2R. \quad \ldots(1)$$

Now, let $y \in e_1e_2R$, then $y = e_1e_2r$, for some r in R and this mean $y \in e_1$R . Since e_1, e_2 are centeral idempotent elements , then $y \in e_2R$ implies $y \in e_1R \cap e_2R$ .Therefore

$$e_1e_2R \subseteq e_1R \cap e_2R. \quad \ldots(2)$$

From (1) and (2) we get $e_1R \cap e_2R = e_1e_2R.$.

Proof 2: let $x \in e_1R \cap e_2R$. then $x = e_1r_1$ and $x = e_2r_2$, for some r_1, r_2 in R.

Since $e_1r_1 = e_2r_2 = e_1e_2r_2$, so $x = e_1e_2r_2$.

Multiplication two sided from left by $(1-e_1)$ we get $(1-e_1)x = 0$, and $x \in r(1-e_1)$.

Similarly we get $x \in r(1-e_2)$, hence x $\in r(1-e_1) \cap r(1-e_2)$.

Now , let $y \in r(1-e_1) \cap r(1-e_2)$,then $y \in r(1-e_1)$ *and* $y = e_1y \in e_1R$ also $y \in r(1-e_2)$ and $y = e_2y \in e_2R$, so $y \in e_1R \cap e_2R$ and hence $e_1R \cap e_2R = r(1-e_1) \cap r(1-e_2).$

Proof 3: let $x \in r(e_1) \cap r(e_2)$, then $x \in r(e_1)$ *and* $e_1x = 0, x \in r(e_2)$ *and* $e_2x = 0$

So $(e_1+e_2)x = 0$ and $x \in r(e_1+e_2)$.

Now, let $y \in r(e_1+e_2)$.Then $(e_1+e_2)y = 0$ and $e_1y = -e_2y$ and $e_1y = -e_1e_2y \in e_1R = -e_2e_1y \in e_2R$ (since every idempotent is centeral), then $e_1y \in e_1R \cap e_2R = (0)$ implies $e_1y = 0$ and $y \in r(e_1)$.

Similarly we get $y \in r(e_2)$,then $y \in r(e_1) \cap r(e_2)$.

Hence $r(e_1+e_2) = r(e_1) \cap r(e_2)$. ■

Following [3] a ring R is said to be right semi-regular ring if for every a in R , there exists b in R such that a = ab , and r(a) = r(b).

Proposition 2-5: A ring R is a right semi regular , if and only if , r(a) is generated by an idempotent.
Proof: see [1], theorem (1-1-12).

Theorem 2-6: If R is a right semi-regular ring with every idempotent is centeral .Then for each a in R , there exists e in R such that aR $\cap$ eR=(0).

Proof: let R be a right semi- regular ring, then r(a) = eR , where e is idempotent element ,and let $x \in$ aR $\cap$ eR,then x = ar and x = er' , for some r, r' in R .

Now, $x = er' = e.er' = ex$, since e $\in$ eR= r(a), then ea = ae = 0.

Since x = ar ,then ex =ear= 0 but $ex = er' = x$, so x = 0.

Hence aR $\cap$ eR=(0). ■

Proposition 2-7: If e is centeral idempotent element of R, then for each element x in R there exists y in R such that xye = yxe = e if and only if xe+(1-e) invertibility of ye+(1-e).

Proof: let u = xe+(1-e) and v = ye +(1-e).

Now, uv = (xe +(1-e)).(ye +(1-e)) = $x$eye + xe -xe +ye +1 -e - eye –e+e =1

So, vu =1.

Conversely, let (xe +(1-e))(ye +(1-e)) =1 , implies xye -e = 0 and xye=e.

Similarly we get yxe=e. ■

Theorem 2-8 : If aR = eR .then a = eu , where e is centeral idempotent element and u is unit element of R.

Proof: Let aR= eR,where e is centeral idempotent element of R ,

Now,a = ea = ae.

Also e = ax,for some x in R

Put v =1-e+ex and u = 1-e+a ,we find uv = vu =1

Now,eu = e(1-e+a) = a,Then a = eu = ue. ■

If e_1, e_2 are idempotent element of R, then $(e_1 + e_2)$ need not to be idempotent as the following example shows.

Example: let $R(Z_2)$ be the ring of all 2×2 matrices over the ring Z_2 (the ring of integers modulo 2) which are strictly upper triangular.Then the only idempotent matrices of $R(Z_2)$ are:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Now, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not idempotent.

Lemma 2-9 : If e_1, e_2 are orthogonal idempotent elements of R, then $(e_1 + e_2)$ is idempotent element.

Proof: Trivial.

If e_1, e_2 are idempotent elements of R, then $r(e_1 + e_2) \neq r(e_1) \cap r(e_2)$ in general as the following example shows.

<u>Example:</u> let $R(Z_2)$ be the ring of all 2×2 matrices over the ring Z_2 (the ring of integers modulo 2) which are strictly upper triangular, then the element of $\mathrm{R}(\mathrm{Z}_2)$ are:

I=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, O=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ A=$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,B=$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

C=$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$,D=$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,E=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,F=$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

The only idempotent elements of $R(Z_2)$ are: $\{I, O, A, C, E, F\}$

r(I)=$\{O\}$

r(E)= $\{O, F\}$

r(I)$\cap$r(E)=$\{O\}$ while r(I + E)=r(F) =$\{O, D, E, A\}$

clearly r(I)$\cap$r(E)$\neq$r(I+E)

<u>Proposition 2-10:</u> If e_1, e_2 is orthogonal idempotent elements of R, then $r(e_! + e_2) = r(e_1) \cap r(e_2)$

<u>proof:</u> let $x \in r(e_1 + e_2)$ .Then $(e_1 + e_2)\ x = 0$ and $e_1 x = -e_2 x$ multiplication two sided from left by e_1 we get $e_1 x = 0$ and $x \in r(e_1)$.

Also multiplication two sided from left by e_2 we get $e_2 x = 0$ and $x \in r(e_2)$ so, $x \in r(e_1) \cap r(e_2)$.

Now, let $y \in r(e_1) \cap r(e_2)$, then $y \in r(e_1)$ and $e_1 y = 0$,also $y \in r(e_2)$ and $e_2 y = 0$, implies $(e_1 + e_2) y = 0$ and $y \in r(e_1 + e_2)$,so $r(e_! + e_2) = r(e_1) \cap r(e_2)$. ■

If R is not commutative ring , then $e_1 R + e_2 R \neq (e_1 + e_2) R$ as the following example shows:

Example: let $R(Z_2)$ be the ring of all 2×2 matrices over the ring Z_2 (the ring of integers modulo 2) which are strictly upper triangular, then the element of $R(Z_2)$ are:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

the idempotent matrices are : $\{A, C, D, E, F, G, H, M\}$

ER=$\{A, B, C, E\}$

MR= $\{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, p\}$
Now, ER+MR= $\{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P\}$
(E+M)R= $\{A, G, H, N\}$
Its clearly that ER+MR $\neq$ (E+M)R.

Theorem 2-11: If e_1, e_2 are orthogonal idempotent elements of R, then $e_1 R + e_2 R = (e_1 + e_2)R$

proof: let $x \in (e_1 + e_2)R$, then $x = (e_1 + e_2)$ r,for some r in R ,and this implies $x = e_1 r + e_2 r \in e_1 R + e_2 R$

Now, let $y \in e_1 R + e_2 R$, then $y = e_1 r_1 + e_2 r_2$,for some r_1, r_2 in R
Multiplying two sided from left by $(e_1 + e_2)$ we get $(e_1 + e_2)$ y=y and $y \in (e_1 + e_3)R$ and hence $e_1 R + e_2 R = (e_1 + e_2)R$. ■

Corollary 2-12: If $e_1, e_2, ..., e_r$ are orthogonal idempotent elements of R, then $(e_1 + e_2 + ... + e_{r+1})R = e_1 R + e \ R + ... + e_{r+1} R$

Proof: by induction

1- when n =2 , the equality holds.
2- assume that equality holds when n = r.
3- when n = r+1

$$\left(\sum_{i=1}^{r+1} e_i\right) R = \left(\sum_{i=1}^{r} e_i + e_{r+1}\right) R = \left(\sum_{i=1}^{r} e_i\right) R + e_{r+1} R$$

$$= \sum_{i=1}^{r} e_i R + e_{r+1} R = \sum_{i=1}^{r+1} e_i R. \quad ■$$

Proposition 2-13: If R is a ring and $x^n = x^{n+1}$,for all x in R and n is integer, then :

1. x = e+p, where e is idempotent element and p is nilpotent.
2. Every element of R/N is idempotent .

Proof 1: let $x^n = x^{n+1} = x.x^n$
First we claim that x^n is idempotent element.
Now, $(x^n)^2 = x^n . x^n = x.x.x. \ldots\ldots\ldots\ldots (x.x^n)$
n- times
$= x.x. \ldots\ldots\ldots\ldots\ldots (x.x^n)$
(n-1)-times
$= x^n$

Now, let y $= x - x^n$, then y $= x - x^{n+1}$ and this implies y $= x - x.x^n$,so y = x (1- x^n).

Now, $y^n = (x(1-x^n))^n$ so, $y^n = x^n(1-x^n)^n$ (since every idempotent is centeral) and by [proposition 2-1] , (1-x^n) is also idempotent and by [if e is idempotent element of R, then e can not to be nilpotent] ,(1-x^n) can not be nilpotent. So, $y^n = x^n(1-x^n) = 0$ and y$\in$N, therefore x = x^n+x- $x^n \in$ E +N.

2) let $y = x - x^2$

Now, $x^{n-1}.y = x^x + x^{n+1} = 0$

0 = $x^{n-2}.xy = x^{n-2}x^2 y = x^{n-2}(x-x^2)y = x^{n-2}y^2$

= $x^{n-3}.xy^2 = x^{n-3}x^2\tilde{y}^2 = x^{n-3}(x-x^2)y^2 = \mathrm{x}^{\mathrm{n-3}}\mathrm{y}^3$

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0= y^m this implies $(x-x^2)^m = 0$, then $x - x^2 \in N$.

So, x+N = x^2+N. ■

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