Joint Numerical Range of Matrix Polynomials

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ABSTRACT

Some algebraic properties of the sharp points of the joint numerical range of a matrix polynomials are the main subject of this paper. We also consider isolated points of the joint numerical range of matrix polynomials.

Key words: joint numerical range, matrix polynomial, sharp points**.**

المدى العددي المشترك لمتعددات حدود معاملتها مصفوفات

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الملخص

بعض الخواص الجبرية لنقاط حادة للمدى العددي المشترك لمتعددات حدود معاملتها مصفوفات هى المادة الأساسية لهذا البحث.كذلك درسنا حالة كون النقطة الحادة نقطة شاذة معزولة.

الكلمات المفتاحية: المدى العددي المشترك، متعددة حدود مصفوفة، النقاط الحادة.

1- Introduction:

Let $A \in M_n$ be the algebra of $n \times n$ complex matrices. The classical numerical range of A is the set of a complex numbers $W(A)=\{x^*Ax: x \in C^n, x^*x=1\}$ where C^n vector space (over C) of complex n-vectors [6]. There has been many generalizations and applications of the classical numerical range, see, for example[6]. In the following, we consider a generalization of the classical numerical range. Suppose $p(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$ is a matrix polynomial, where $A_0, A_1, A_2, \ldots, A_m \in M_n$ and λ is a complex variable. Define the joint numerical range of $p(\lambda)$ as $JNR(p(\lambda)) = \{(x^* A_0 x, x^* A_1 x, \ldots, x^* A_m x) : x \in C^n : x^* x = 1\}$ [9].

This generalized joint numerical range has been discussed by [9]. On the other hand The joint numerical range of matrix polynomials, being a continuous image of the unit sphere, is compact and connected but not necessarily convex; see Binding and Li [3]. Its convex hull is denoted by co $\{JNR(p(\lambda))\}$ and it plays an important role in the study of damped vibration systems, with a finite number of degree of freedom [7] and it is useful in various theoretical and applied subjects (see[1,2,3,4 and 5]) and their references. The aim of this paper is to give some algebraic properties of the sharp points of the joint numerical range of matrix polynomials, we also consider an isolated point of the joint numerical range of $p(\lambda)$ .The rest of this paper is organized as follows: In section 2, we present definitions and some basic results which will be used in this paper. In section 3, we prove that if λ_0 is a sharp point of the joint numerical range of the linear pencil $A_1\lambda_1 - B_1, A_2\lambda_2 - B_2, \ldots, A_m\lambda_m - B_m$ then zero is a sharp point of $JNR(A_1\lambda_o - B_1, A_2\lambda_o - B_2, \ldots, A_m\lambda_o - B_m)$.and if λ_0 is an isolated point of $JNR(p(\lambda))$, then $p(\lambda_0) = 0$.

2- Preliminaries:

In this section, we present some definitions and basic results on joint numerical ranges of matrix polynomials.

Definition 2.1 [8]

A point $\lambda_0 \in NR(p(\lambda))$ is called a sharp point of $NR(p(\lambda))$ if for a connected component w_s of $NR(p(\lambda))$ there exists a disk centered at λ_0 and with radius r $S(\lambda_0, r)$, r>0 and angles ϕ_1 *and* ϕ_2 *with* $0 \le \phi_1 < \phi_2 \le 2\pi$ such that

$$Re(e^{i\theta}\lambda_0) = max\{Re\, z : e^{-i\theta} z \in w_s(p(\lambda)) \cap S(\lambda_0, r)\},$$

for all $\theta \in [\phi_1, \phi_2]$.

Definition 2.2 [6]

A matrix $A \in M_n$ is said to be unitary if $A^* A = I$, if in addition $A \in M_n(R)$ then A is said to be real orthogonal.

Theorem 2.3[6]

If $A \in M_n$ the following are equivalent.

a) A is unitary.
b) A is non singular and $A^* = A^{-1}$.
c) $AA^* = I$.
d) A^* Is unitary.

Proposition 2.4 [6]

Suppose that $Q(\lambda, t) = C_{m(t)}\lambda^m + C_{m-1(t)}\lambda^{m-1} + \ldots + C_{1(t)}\lambda + C_0$ is a polynomial matrix in λ where the coefficient $C_{j(t)}$ depend continuously on the parameter $0 \le t < \varepsilon$ *and* $C_{m(t)} \ne 0$ *for every* $o \le t < \varepsilon$ then m roots $\lambda_{j(t)}$; j=1,2,…,m of the equation Q(λ,t)=0 are continuous functions in $t \in [0, \varepsilon)$.

Definition 2.5 [6]

An n-by-n Hermitiat matrix A is said to be positive definite if $x^* Ax > 0 \quad for\ all\ non-zero\ x \in C^n$.

Definition 2.6 [6]

A matrix B∈M_n is said to be positive semi-definite if x*Bx ≥ 0 for all x∈C^n

Definition 2.7 [6]

The matrix adjoint A* of A∈M_n(C) is define by A*=A^{-T} where A^- is the component-wise conjugate, and A^T is the transpose of A.

Definition 2.8 [6]

The matrix A∈M_n(C) is said to be Hermition if A=A* ,it is skew-Hermition if A=-A* and for any A∈M_n(C) can be written A=(A+A*)/2 +(A-A*)/2=H(A)+S(A) where H(A) =(A+A*)/2 the Hermition part of A, and S(A)=(A-A*)/2 is the skew-Hermition part of A.

Definition 2.9 [6]

Let p_0 be an element of anon-empty set A, we say that p_0 is an isolated point of A if $\exists N_r(p_0)\ such\ that\ N_r(p_0) \cap A = \{p_0\}$

3- Properties of Sharp points

In the following, we will restrict ourselves to the definition of sharp points. The next theorem gives a connection of these points with respect to the origin as a joint numerical range of matrix polynomials.

Theorem 3.1

Suppose that x_o is a unit vector such that $0 = x_o^* A_1 x, \ldots, 0 = x_o^* A_n x$ belongs to the joint numerical range of $A_1, A_2, \ldots A_n \in M_n$ if $x^* A_1 x, \ldots, x^* A_n x$ have non-negative real parts for all x of the neighborhood $S(x_o, \varepsilon) = \{x \in c^n : \|x - x_o\|_2 < \varepsilon\}$ then $A_1 + A_1^*, \ldots, A_n + A_n^*$ are positive semi-definite.

Proof:

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of each of the Hermitian matrices $A_1 + A_1^*, \ldots, A_n + A_n^*$. We see that $x^*(A_1 + A_1^*)x = y_1^* D y_1, \ldots, x^*(A_n + A_n^*)x = y_n^* D y_n$ $where\ D = diag(\lambda_1, \lambda_2, \ldots \lambda_n),\ y = u^* x$ and u is a unitary matrix. For

$$y_o = u^* x_o = [y_1, y_2, \ldots, y_n]^T \quad we\ have\ y_o^* D y_o = \lambda_1 |y_1|^2 + \ldots + \lambda_n |y_n|^2 = 0 \qquad \cdots (*),$$

since, $\|y - y_o\|_2 = \|u^*\|_2 \|x - x_o\|_2 = \|x - x_o\|_2$, $and \quad Re(x^* A_1 x) \geq 0, \ldots, Re(x^* A_n x) \geq 0$ for all $x \in S(x_o, \varepsilon)$, there exists a neighborhood $S(y_o, \varepsilon)$ such that $y^* D y \geq 0\ for\ any\ y \in S(y_o, \varepsilon)$. Now in (*) assume that $\lambda_k < 0$ consider the vector $y_\delta = y_o + \delta e_k = [y_1 ... y_k + \delta ... y_n]^T$ where $\delta \in C$ such that $0 < |\delta| < \varepsilon$ and $|y_k + \delta| > |y_k|$. Then for a vector y_δ of the neighborhood $S(y_o, \varepsilon)$ we have $y_\delta^* D y_\delta = \lambda_1 |y_1|^2 + ... + \lambda_k |y_k + \delta|^2 + ... + \lambda_n |y_n|^2 = \lambda_k (|y_k + \delta|^2 + |y_k|^2) < 0$ which is contradiction. Therefore $\lambda_k \geq 0$ for k=1,2,…n and $D \geq 0$. Hence the matrices $A_1 + A_1^*, ..., A_n + A_n^*$ are positive semi-definite.

Theorem 3.2

Suppose x_o is a unit vector such that $0 = x_o^* A_1 x, ...0 = x_o^* A_n x$ belongs to the joint numerical range of $A_1, A_2, ...A_n \in M_n$ .If $x^* A_1 x = 0, ..., x^* A_n x = 0$ for all $x \in S(x_o, \varepsilon)$ then $A_1 = A_2 = ... = A_n = 0.$

Proof:

For the matrices $A_1, A_2, ..., A_n$ we consider the Hermitian matrices,

H(A1)= $\frac{1}{2}$ (A1+A2*),H(A2)= $\frac{1}{2}$ (A2+A2*),…,H(An)= $\frac{1}{2}$ (An+An*) and

S(A1)= $\frac{1}{2}$ (A1*-A1),S(A2)= $\frac{1}{2}$ (A2*-A2),…,S(An)= $\frac{1}{2}$ (An*-An) then from the hypotheses that $x_o^* A_1 x_o = 0, ..., x_o^* A_n x_o = 0$ *and* $x^* A_1 x = 0, ..., x^* A_n x = 0$ for any $x \in S(x_o, \varepsilon)$ it is clear that $x_o^* H(A_1) x_o = 0, ..., x_o^* H(A_n) x_o = 0$ *and* $x^* H(A_1) x = 0, ..., x^* H(A_n) x = 0$ for each $x \in s(x_o, \varepsilon)$ and also $x_o^* S(A_1) x_o = 0, ..., x_o^* S(A_n) x_o = 0$ *and* $x^* S(A_1) x = 0, ..., x^* S(A_n) x = 0$ for each $x \in S(x_o, \varepsilon)$ thus by Theorem (3.1) we have $H(A_1), H(A_2), ..., H(A_n)$ *and* $S(A_1), S(A_2), ..., S(A_n)$ are both positive semi-definite and negative semi-definit and this implies that $H(A_1) = S(A_1) = 0, ..., H(A_n) = S(A_n) = 0,$ then $A_1 = H(A_1) + iS(A_1) = 0, \ldots, A_n = H(A_n) + iS(A_n) = 0$.

Theorem 3.3

Let $A_1, A_2, ...A_n \in M_n$ and x_o be a unit vector such that $x_0^* A_1 x = x_0^* A_2 x = ... = x_0^* A_n x = 0$ belongs to the joint numerical range of $A_1, A_2, ..., A_n$, then zero is a sharp point of joint numerical range of A1,A2,…,An if and only if there exists $\varepsilon > 0,\ \phi_1$ *and* ϕ_2 *such that* $\phi_1 \leq arg(x^* A_1 x) \leq \phi_2, \phi_1 \leq arg(x^* A_2 x) \leq \phi_2 ...,$ $\phi_1 \leq arg(x^* A_n x) \leq \phi_2$ *with* $\phi_2 - \phi_1 < \pi$ for all $x \in S(x_o, \varepsilon)$.

Proof:

Suppose there exists $\varepsilon > 0$, φ_1 and φ_2 such that $\varphi_1 \leq \arg(x^* A_1 x) \leq \varphi_2$, $\varphi_1 \leq arg(x^* A_2 x) \leq \varphi_2, \ldots, \varphi_1 \leq arg(x^* A_n x) \leq \varphi_2$. If $w_1 = \frac{\pi}{2} - \varphi_1$ and $w_2 = \frac{3\pi}{2} - \varphi_2$

then $0<w_2-w_1<\pi$,and for the matrices $e^{iw}A_1+e^{-iw}A_1^*$, $e^{iw}A_2+e^{-iw}A_2^*$, $\ldots, e^{iw}A_n-e^{-iw}A_n^*$

We have $x_0^*(e^{iw}A_1+e^{-iw}A^*{}_1)x_0=0$, $x_0^*(e^{iw}A_2+e^{-iw}A^*{}_2)x_0=0$, $\ldots$, $x_0^*(e^{iw}A_n+e^{-iw}A^*{}_n)x_0=0$, and $\mathrm{Re}(x^*e^{iw}A_1x)=\frac{1}{2}x^*(e^{iw}A_1+e^{-iw}A_1^*)x\le 0$, $\mathrm{Re}(x^*e^{iw}A_2x)=\frac{1}{2}x^*(e^{iw}A_2+e^{-iw}A_2^*)x\le 0,\ldots,$

$\mathrm{Re}(x^*e^{iw}A_nx)=\frac{1}{2}x^*(e^{iw}A_n+e^{-iw}A_n^*)x\le 0$,

for any $w\in[w_1,w_2]$ and for all $x\in S(x_0,\varepsilon)$.Therefore by theorem (3.1) the matrices $e^{iw}A_1+e^{-iw}A_1^*$, $e^{iw}A_2+e^{-iw}A_2^*,\ldots,e^{iw}A_n+e^{-iw}A_n^*$ are non-negative semi definite and $\max\{x^*(e^{iw}A_1+e^{-iw}A_1^*)x:\|x\|=1\}=0$, $\max\{x^*(e^{iw}A_2+e^{-iw}A_2^*)x:\|x\|=1\}=0,\ldots,$ $\max\{x^*(e^{iw}A_n+e^{-iw}A_n^*)x:\|x\|=1\}=0$, for any $w\in[w_1,w_2]$ thus max{Rez: $z\in e^{iw}\,JNR(A_1,A_2,...,A_n)\}=0$,this means that the origin is a sharp point of the joint numerical range of $A_1,A_2,\ldots,A_n$. Conversely, assume that zero is a sharp point of the joint numerical range of $A_1,A_2,\ldots,A_n$, then by the definition of sharp point there exists w_1 and w_2 belongs to $[0,2\pi]$, such that for each w belongs to $[w_1,w_2]$,

max{Rez: $z\in JNR(e^{iw}(A_1,A_2,...,A_n))\}=0$, where $w_1=\frac{\pi}{2}-\varphi_1$ and $w_2=\frac{3\pi}{2}-\varphi_2$, so this implies that $\max\{x^*(e^{iw}A_1+e^{-iw}A_1^*)x:\|x\|=1\}=0$, $\ldots$, $\max\{x^*(e^{iw}A_n+e^{-iw}A_n^*)x:\|x\|=1\}=0$,because $x^*(e^{iw}A_1+e^{-iw}A_1^*)x=$ $\mathrm{Re}(x^*e^{iw}A_1x),\ldots,$ $x^*(e^{iw}A_n+e^{-iw}A_n^*)x=\mathrm{Re}(x^*e^{iw}A_nx)$, we obtain that $arg(x^*A_1x), arg(x^*A_2x),\ldots,\arg(x^*A_nx)$ belongs to $[\varphi_1,\varphi_2]$ where $\varphi_2-\varphi_1=(\frac{3\pi}{2}-w_2)-(\frac{\pi}{2}-w_1)<\pi.$

Theorem 3.4

Let λ_o be a sharp point of the joint numerical range of the linear pencil $A_1\lambda_1-B_1,A_2\lambda_2-B_2,...,A_m\lambda_m-B_m$.Then zero is a sharp point of $JNR(A_1\lambda_o-B_1,A_2\lambda_o-B_2,...,A_m\lambda_o-B_m)$.

Proof:

By the equality $JNR(A_1(\lambda_1+\lambda_o)-B_1,A_2(\lambda_2+\lambda_o)-B_2,...,A_m(\lambda_m+\lambda_o)-B_m)=$ $JNR(A_1\lambda_1-B_1,A_2\lambda_2-B_2,...,A_m\lambda_m-B_m)-\lambda_o$, since λ_o is a sharp point of $JNR(A_1\lambda_1-B_1,A_2\lambda_2-B_2,...,A_m\lambda_m-B_m)$. This implies that zero is a sharp point of $JNR(A_1\lambda_1+(A_1\lambda_o-B_1),A_2\lambda_2+(A_2\lambda_o-B_2),...,A_m\lambda_m+(A_m\lambda_o-B_m))$, thus there exists a vector x_o such that $x_o^*(A_1\lambda_o \quad B_1)x_o=0,...,x_o^*(A_m\lambda_o \quad B_m)x_o=0$ and $x_o^*A_1x_o=k_1\neq 0,...,x_o^*A_mx_o=k_m\neq 0$. It is not possible to have $x_o^*(A_1\lambda_o-B_1)x_o=x_o^*A_1x_o=0, x_o^*(A_2\lambda_o-B_2)x_o=x_o^*A_2x_o=0,...,x_o^*(A_m\lambda_o-B_m)x_o=x_o^*A_mx_o=0$ because $JNR(A_1\lambda_1+A_1\lambda_o-B_1,A_2\lambda_2+A_2\lambda_o-B_2,...,A_m\lambda_m+A_m\lambda_o-B_m)$ $=C^n$. Since zero is a sharp point of $JNR(A_1\lambda_1+A_1\lambda_o\text{-}B_1,A_2\lambda_2+A_2\lambda_o\text{-}B_2,$ $...,A_m\lambda_m+A_m\lambda_o\text{-}B_m)$, there exists $r_1,r_2,...,r_m>0$ such that for any complex number

$$\gamma_{x_1}=\frac{-x^*(A_1\lambda_o-\ \ B_1)x}{x^*A_1x}\in S(0,r_1)\cap[\lambda_1<A_1x,x>+\lambda_0<A_1x,x>-B_1]$$

$$\gamma_{x_2}=\frac{-x^*(A_2\lambda_o-\ \ B_2)x}{x^*A_2x}\in S(0,r_2)\cap[\lambda_2<A_2x,x>+\lambda_0<A_2x,x>-B_2]\ .$$

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$$\gamma_{x_m}=\frac{-x^*(A_m\lambda_o-\ \ B_m)x}{x^*A_mx}\in S(0,r_m)\cap[\lambda_{\mathrm{m}}<A_mx,x>+\lambda_0<A_mx,x>-B_m]\ .$$

We have
$$\phi_1\leq arg(\frac{-x^*(A_1\lambda_o-B_1)x}{x^*A_1x})\leq\phi_2$$
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$$\phi_1\leq arg(\frac{-x^*(A_m\lambda_o-B_m)x}{x^*A_mx})\leq\phi_2$$

with each $\phi_2-\phi_1<\pi$.Moreover, by the continuity of the functions $F_{11}(x)=x^*A_1x, F_{12}(x)=x^*A_2x,...,F_{1m}(x)=x^*A_mx$ and $F_{21}(x)=x^*(A_1\lambda_o-B_1)x,$ $..., F_{2m}(x)=x^*(A_m\lambda_o-B_m)x$ for any $\varepsilon>0$ there exists a neighborhood $S(x_o,\delta)$ such that for any $x\in S(x_o,\delta)$ $x^*A_1x,x^*A_2x,...,x^*A_mx$ *belong to* $S(k_1,\varepsilon),...,S(k_m,\varepsilon)$ respectively and $\gamma_{x_1},\gamma_{x_2},...,\gamma_{x_m}$ belong to $S_1(0,r_1),S_2(0,r_2),\ldots,S_m(0,r_m)$ respectively.

thus by equation arg(x*A1x)+arg(γ_{x_1})=arg(x*($A_1\lambda_0$-B_1)x), arg(x*A_2x)+ arg(γ_{x_2})=arg(x*($A_2\lambda_0$-B_2)x),…,arg(x*A_mx)+arg($\boldsymbol{\gamma_{x_m}}$)=arg(x*($A_m\boldsymbol{\lambda_0}$-$B_m$)x), we have that each of $\arg(x^*(A_1\lambda_o-B_1)x),...,\arg(x^*(A_m\lambda_o-B_m)x)$ belongs to the $[\theta_1,\theta_2]$ for any $x\in S(x_o,\delta)$ and for suitable θ_1,θ_2 *with* $\theta_2-\theta_1<\pi$. And then by Theorem (3.3) it is clear that zero is a sharp point of the $JNR(A_1\lambda_o - B_1, A_2\lambda_o - B_2, ..., A_m\lambda_o - B_m)$.

Degenerate cases of sharp points are the isolated points, and we have the following statement:

Theorem 3.5

Let $p(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$ be an nxn matrix polynomial such that zero does not belong to the joint numerical range $JNR(A_m)$ of the leading coefficient matrix. If λ_o is an isolated point of the joint numerical range $JNR(p(\lambda_o))$ *then* $p(\lambda_o) = 0$.

Proof:

Assume $\lambda_o = 0$ then there exists a unit vector such that $<A_j x, x> = 0,\ j = 0,1,2,...,m$. This means that $<A_0 x, x> = 0$, now if $<A_0 y, y> = 0$ for any unit vector y then the matrix A_o satisfies $A_o = 0$. We assume that there exists a unit vector y with $<A_j y, y> \neq 0;\ j = 0,1,2,...,m$ i.e. $<A_o y, y> \neq 0$. We consider a polynomial

$Q(\lambda, t) = <A_m(\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y), (\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y)> \lambda^m + ... + <A_1(\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y),$
$(\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y)> \lambda + <A_o(\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y), (\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y)>$.This satisfies $<A_m(\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y), (\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y) > \neq 0$, because zero does not belong to the joint numerical range of the leading coefficient A_m. Now it is sufficient to prove that there exists a sequence (t_n) for which t_n tends to zero as n tends to infinity and $Q(\lambda_n, t_n) = 0$,at first we fix 0<t<1 the condition $Q(0,t) = <A_o(\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y), (\cos\frac{t\pi}{2} x + \sin\frac{t\pi}{2} y)> = 0$, is equal to $<A_o(x + \tan(\frac{t\pi}{2})y, x + \tan(\frac{t\pi}{2})y> = 0$ on the other hand we consider the function $<A_o(x + sy), x + sy> = s$, where we have $<A_o y, y> \neq 0$ by the choice of the unit vector y. $<A_o x, y> + <A_o y, x> \neq 0$ then we have $<A_o(x + sy), x + sy> \neq 0$
for sufficiently small s>0 and if $<A_o x, y> + <A_o y, x> \neq 0$ then we have $<A_o(x + sy), x + sy> = s^2 <A_o y, x> \neq 0$.Thus we conclude that $Q(0,t) \neq 0$ for a sufficiently small t>0 so that $<A_o(x + \tan(\frac{t\pi}{2})y, x + \tan(\frac{t\pi}{2})y> \neq 0$. Hence the equation Q(λ,t)=0 in λ has m roots by the fundamental theorem of algebra. The roots of the algebraic equation depends continuously on the coefficient, hence P(0)= A_0=0

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