The n-Wiener Polynomials of the Cartesian Product of a Complete Graph with some Special Graphs

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Received on:22/4/2008 **Accepted on:11/6/2008**

ABSTRACT

The n-Wiener polynomials of the Cartesian products of a complete graph K_t with another complete graph K_r, a star graph S_r, a complete bipartite graph $K_{r,s}$, a wheel W_r, and a path graph P_r are obtained in this paper. The n-diameters and the n-Wiener indices of $K_t \times K_r$, $K_t \times S_r$, $K_t \times K_{r,s}$, $K_t \times W_r$ and $K_t \times P_r$ are also obtained.

Keywords: n-distance, n-diameter, n-index, n-Wiener polynomial.

متعددات وينر-n لجداءات بيان تام مع بيانات خاصة

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تاريخ الاستلام: 2008/4/22 **تاريخ القبول: 2008/6/11**

الملخص

تضمن هذا البحث إيجاد متعددات وينر-n لجداءات بيان تام K_t مع بيانات خاصة مثل البيان التام K_r , نجمة S_r , وبيان ثنائي التجزئة تام $K_{s,r}$, وعجلة W_r ودرب P_r . كما تضمن إيجاد قطر-n ودليل وينر-n لكل من الجداءات $K_t \times K_r$, $K_t \times S_r$, $K_t \times K_{s,r}$, $K_t \times W_r$, $K_t \times P_r$.

الكلمات المفتاحية: المسافة–n، القطر–n، دليل–n، متعددة حدود وينر–n.

1. Introduction.

We follow the terminology of [5] and [6]. Let v be a vertex of a connected graph G and let S be an (n-1)-subset of vertices of V(G), $n \geq 2$, then the **n-distance** $d_n(v,S)$ is defined as follows[7]

$$d_n(v,S)=\min\{d(v,u):u \in S\}. \qquad ...(1.1)$$

Sometimes, we refer to the n-distance of the pair (v,S) in G by $d_n(v,S \mid G)$. The **n-diameter** $\mathrm{diam}_n G$ of G is defined by

$$\mathrm{diam}_n G=\max\{d_n(v,S): v \in V(G), S \subseteq V(G), |S| =n-1\}. \qquad ...(1.2)$$

It is clear that for all $2 \leq m \leq n \leq p$,

$$\mathrm{diam}_n G \leq \mathrm{diam}_m G \leq \mathrm{diam} G. \qquad ...(1.3)$$

The **n-Wiener index** of G denoted by $W_n(G)$ is defined as

$$W_n(G)= \sum_{(v,S)} d_n(v,S), \qquad ...(1.4)$$

where the summation is taken over all pairs (v,S) for which $v\in V(G)$, $S\subseteq V(G)$ and $|S|=n-1$. The **n-average distance** $\mu_n(G)$ is defined as

$$\mu_n(G)= W_n(G)/p\binom{p-1}{n-1},\ \ 3\le n\le p. \qquad \ldots(1.5)$$

Let v be any vertex of G, then **the n-distance of v** denoted $d_n(v|G)$ or simply $d_n(v)$ is defined as

$$d_n(v)= \sum_{S\subseteq V(G)} d_n(v,S)\ ,\ |S|=n-1. \qquad \ldots(1.6)$$

The Wiener polynomial of G with respect to the n-distance, which is called n-Wiener polynomial and defined as below.

Definition 1.1.[2]. Let $C_n(G,k)$ be the number of pairs (v,S), $|S|=n-1, 3\le n\le p$, such that $d_n(v,S)=k$, for each $0\le k\le\delta_n$**.** Then, **the n-Wiener polynomial** $W_n(G;x)$ is defined by

$$W_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k\ , \qquad \ldots(1.7)$$

in which δ_n is the n-diameter of G .
One may easily see [2] that for $3\le n\le p$, the number of all (v,S) pairs is $p\binom{p}{n-1}$, and [1]

$$\sum_{k=1}^{\delta_n} C_n(G,k)=p\binom{p-1}{n-1}, \qquad C_n(G,0)=p\binom{p-1}{n-2}, \qquad \ldots(1.8)$$

$$C_n(G,1)=p\binom{p-1}{n-1} - \sum_{v\in V(G)}\binom{p-1-\deg_G(v)}{n-1} \qquad \ldots(1.9)$$

Definition 1.2[1] Let v be a vertex of G, and let $C_n(v,G,k)$ be the number of (n-1)-subsets of vertices of G such that

$d_n(v,S|G)=k$, for $n\ge 3$, $0\le k\le\delta_n$.

Then, the **n-Wiener polynomial of vertex v**, denoted by $W_n(v,G;x)$ is defined as

$$W_n(v,G;x)=\sum_{k\ge 0} C_n(v,G,k)x^k. \qquad \ldots(1.10)$$

It is clear that for all $k\ge 0$,

$$\sum_{v\in V(G)} C_n(v,G,k)=C_n(G,k), \qquad \ldots(1.11)$$

and

$$\sum_{v\in V(G)} W_n(v,G,x)=W_n(G;x)\ . \qquad \ldots(1.12)$$

There are many classes of graphs G in which for each $k, 1\le k\le \delta_n$, $C_n(v,G,k)$ is the same for every vertex $v\in V(G)$; such graphs are called [1] **vertex-n-distance regular**. If G is of order p and it is vertex-n-distance regular, then

$$W_n(G;x)=pW_n(v,G;x), \qquad \ldots(1.13)$$

where v is any vertex of G.

The authors of references [2],[3] and [4] obtained the n-Wiener polynomials of some special graphs and some types of composite graphs. In this paper, we obtain n-Wiener polynomials of the Cartesian products $K_t\times K_r$, $K_t\times S_r$, $K_t\times K_{r,s}$, $K_t\times W_r$ and $K_t\times P_r$.

2. The Cartesian Product of a Complete Graph and a Star

Let K_t be a complete graph with $V(K_t)=\{u_1,u_2,\ldots,u_t\}$, and S_r be a star of center v_0 and end vertices $v_1,v_2,\ldots,v_{r-1}$. Each vertex of $K_t\times S_r$ is an ordered pair (u_i,v_j), $1\le i\le t$, $0\le j\le r-1$. Let K_t^j be the clique graph [6] of order t of vertex set $\{(u_i,v_j): i=1,2,\ldots,t,\ 0\le j\le r-1\}$. The graph $K_t\times S_r$ is depicted in Fig. 2.1.

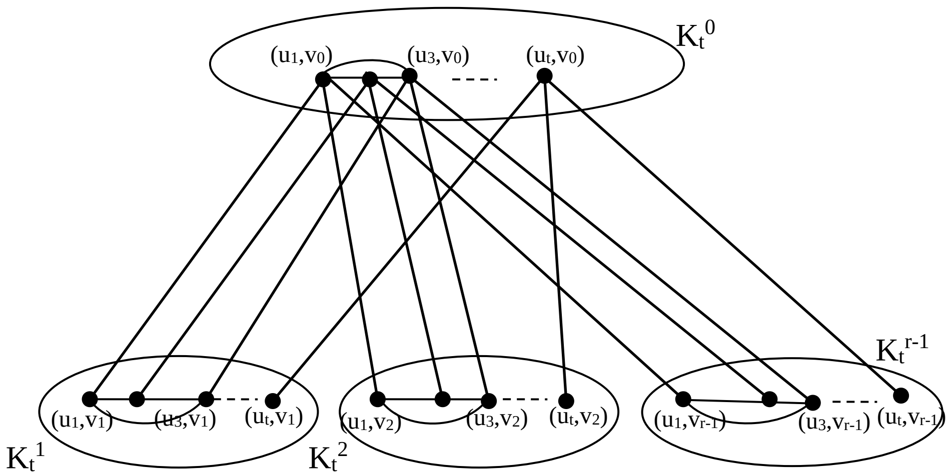

Fig. 2.1. The graph $K_t\times S_r$.

It is clear that $0\le d((u_i,v_j),(u_l,v_m))\le 3$. Thus,

$$\operatorname{diam}_n K_t\times S_r \le \operatorname{diam} K_t\times S_r \le 3.$$

Proposition 2.1. For $t\ge 2$, $r\ge 3$, the n-diameter of $K_t\times S_r$ is given by

$$\operatorname{diam}_n K_t\times S_r=\begin{cases} 3\ , \text{ if } 2\le n\le (t-1)(r-2)+1, \\ 2\ , \text{ if } 1+(t-1)(r-2)<n\le t(r-1), \\ 1, \text{ if } t(r-1)<n\le rt. \end{cases}$$

Proof. The proof is clear from Fig. 2.1. ■

The following theorem gives us the n-Wiener polynomial of $K_t \times S_r$. It is clear that the order of $K_t \times S_r$ is p=rt.

Theorem 2.2. For t≥2, r≥3, 3≤n≤rt,

$$W_n(K_t\times S_r;x)=p\binom{p-1}{n-2}+[p\binom{p-1}{n-1}-t(r-1)\binom{p-t-1}{n-1}-t\binom{p-t-r+1}{n-1}]x$$
$$+t[\binom{p-t-r+1}{n-1}+(r-1)\{\binom{p-t-1}{n-1}-\binom{p-2t-r+2}{n-1}\}]x^2$$
$$+t(r-1)\binom{p-2t-r+2}{n-1}x^3.$$

Proof. It is clear that each vertex of K_t^0 is of degree r+t-2, and each vertex of K_t^j, 1≤j≤r-1, is of degree t. Therefore, by (1.9) we obtained $C_n(K_t\times S_r,1)$ as given in the theorem.

To find $C_n(K_t\times S_r,3)$, we notice that there are (t-1)(r-2) vertices each of distance 3 from each vertex (u,v) of K_t^j, 1≤j≤r-1. Thus,

$$C_n(K_t\times S_r,3)=t(r-1)\binom{(t-1)(r-2)}{n-1}.$$

Finally, by (1.8) and Proposition 2.1, we get

$$C_n(K_t\times S_r,2)=p\binom{p-1}{n-1}-C_n(K_t\times S_r,1)-C_n(K_t\times S_r,3)$$
$$=t[\binom{p-t-r+1}{n-1}+(r-1)\binom{p-t-1}{n-1}-(r-1)\binom{p-2t-r+2}{n-1}].$$

Hence, the proof.■

Corollary 2.3. For t≥2, r≥3, 3≤n≤rt,

$$W_n(K_t\times S_r)=p\binom{p-1}{n-1}+t[(r-1)\binom{p-t-1}{n-1}+\binom{p-t-r+1}{n-1}+(r-1)\binom{p-2t-r+2}{n-1}].\ ■$$

3. The Cartesian Product of Complete Graphs

Let K_t and K_r be disjoint complete graphs, and let $(u_1,v_1),(u_2,v_2)\in V(K_t\times K_r)$, then it is clear that

$$\text{diam } K_t\times K_r=2.$$

Thus,

$$\text{diam}_n\ K_t\times K_r\le 2,\quad 2\le n\le rt.$$

If $u_1 \neq u_2$ and $v_1 \neq v_2$, then (u_1,v_1), (u_2,v_2) are non-adjacent in $K_t \times K_r$; and (u_1, v_1), (u_1,v_2), (u_2,v_2) is a path of length 2. Therefore,

$$d((u_1,v_1),(u_2,v_2))=2.$$

The degree of each vertex (u_1,v_1) is r+t-2. Thus, the number of vertices of distance 2 from (u_1,v_1) is rt-r-t+1. Hence, we have the following result.

Proposition 3.1. For $t,r \geq 2$,

$$\operatorname{diam}_n K_t \times K_r = \begin{cases} 2 & \text{if } 2 \leq n \leq rt-r-t+2, \\ 1 & \text{if } rt-r-t+3 \leq n \leq rt. \end{cases}$$

Now, we find the n-Wiener polynomial of $K_t \times K_r$.

Theorem 3.2. For $r,t \geq 2$, $3 \leq n \leq rt$

$$W_n(K_t \times K_r;x) = rt\binom{rt-1}{n-2} + rt\left[\binom{rt-1}{n-1} - \binom{rt-r-t+1}{n-1}\right]x + rt\binom{rt-r-t+1}{n-1}x^2.$$

Proof. It is clear that $K_t \times K_r$ is vertex-n-distance regular. Thus,

$$C_n(K_t \times K_r,2) = rtC_n((u_1,v_1), K_t \times K_r,2).$$

Since the number of vertices of distance 2 from (u_1,v_1) is rt-r-t+1, and there is no vertex of distance more than 2 from (u_1,v_1), then

$$C_n((u_1,v_1), K_t \times K_r,2) = \binom{rt-r-t+1}{n-1}.$$

The constant term and the coefficient of x follow from (1.8) and (1.9).■

Corollary 3.3. For $r,t \geq 2$, $3 \leq n \leq rt$,

$$W_n(K_t \times K_r) = rt\left[\binom{rt-1}{n-1} + \binom{rt-r-t+1}{n-1}\right].■$$

4. The Cartesian Product of a Complete Graph and a Complete Bipartite Graphs

Let $K_{r,s}$ be a complete bipartite graph of bipartite sets of vertices $V_1=\{v_1,v_2,\ldots,v_r\}$, $V_2=\{w_1,w_2,\ldots,w_s\}$; $r \geq s$, and let

$$V(K_t)=\{u_1,u_2,\ldots,u_t\},$$

then it is clear that in $K_t \times K_{r,s}$

$$d((u_i,v_h),(u_j,v_k))=3 \text{ when } i \neq j,\ h \neq k,$$

because there is a shortest path

$$(u_i,v_h),\ (u_j,v_h),\ (u_j,w),\ (u_j,v_k),\ w \in V_2.$$

Similarly,

$$d((u_i,w_h),(u_j,w_k))=3 \text{ when } i \neq j,\ h \neq k.$$

Moreover,

$$d((u_i,v_h),(u_i,v_k)) = d((u_i,w_h),(u_i,w_k))=2.$$

Therefore,

$$\operatorname{diam} K_t \times K_{r,s} = 3,$$

and so

$$\operatorname{diam}_n K_t \times K_{r,s} \le 3,\ 2 \le n \le p,\ p = t(r+s).$$

For any vertex (u_i,v_h), the number of vertices of distance 3 from (u_i,v_h) in $K_t \times K_{r,s}$ is (t-1)(r-1). Similarly, there are (t-1)(s-1) vertices of distance 3 from (u_i,w_k). Moreover, the degree of each vertex of $K_t \times K_{r,s}$ is either r+t-1 or s+t-1.

Thus, we have the following result.

Proposition 4.1. For t,r,s≥2, r≥s, then the n-diameter of $K_t \times K_{r,s}$ is given

$$\operatorname{diam}_n K_t \times K_{r,s} = \begin{cases} 3, & \text{for } 2 \le n \le tr-t-r+2, \\ 2, & \text{for } tr-t-r+3 \le n \le p-t-s, \\ 1, & \text{for } p-t-s+1 \le n \le p. \end{cases}$$

■

The next theorem determines the n-Wiener polynomial of $K_t \times K_{r,s}$.

Theorem 4.2. For t,r,s≥2, 3≤n≤p, p=t(r+s),

$$W_n(K_t \times K_{r,s};x) = p\binom{p-1}{n-2} + \left[p\binom{p-1}{n-1} - st\binom{p-r-t}{n-1} - rt\binom{p-s-t}{n-1}\right]x$$
$$+\left\{rt\left[\binom{p-s-t}{n-1} - \binom{rt-t-r+1}{n-1}\right] + st\left[\binom{p-r-t}{n-1} - \binom{ts-t-s+1}{n-1}\right]\right\}x^2$$
$$+\left[rt\binom{rt-t-r+1}{n-1} + st\binom{ts-t-s+1}{n-1}\right]x^3.$$

Proof. $C_n(K_t \times K_{r,s},0)$ and $C_n(K_t \times K_{r,s},1)$ are obtained from (1.8) and (1.9). To find the other coefficients, we notice that $C_n((a,b), K_t \times K_{r,s},k)$ is the same for every vertex $(a,b) \in V(K_t) \times V_1$, and $C_n((c,d),K_t \times K_{r,s},k)$ is the same for every vertex $(c,d) \in V(K_t) \times V_2$, for k=2,3. Since the number of vertices of distance 3 from vertex (a,b) is (t-1)(r-1), and the number of vertices of distance 3 from vertex (c,d) is (t-1)(s-1), then we get the coefficient of x^3 as given in the statement of the theorem.

Finally, $C_n(K_t \times K_{r,s},2)$ is obtained using the relation (1.8) and the coefficients already obtained. This completes the proof. ■

Corollary 4.3. For t,r,s≥2, and 3≤n≤p in which p=t(r+s),

$$W_n(K_t \times K_{r,s}) = p\binom{p-1}{n-1} + rt\left[\binom{p-s-t}{n-1} + \binom{rt-t-r+1}{n-1}\right]$$

$$+st[\binom{p-r-t}{n-1}+\binom{ts-t-r+1}{n-1}]\ .$$

Proof. The n-Wiener index is obtained from $W_n(K_t\times K_{r,s};x)$ by taking the derivative with respect to x, and then put x=1, and simplified the expression.■

5. The Cartesian Product of a Complete Graph and a Wheel

Let W_r be a wheel of order $r\geq 4$ and let its center be denoted by v_0 and its other vertices be $v_1,v_2,\ldots,v_{r-1}$. Moreover, let $V(K_t)=\{u_1,u_2,\ldots,u_t\}$. The order of $K_t\times W_r$ is p=rt, and in $K_t\times W_r$

$$\deg(u_i,v_j)=t+2, \text{ for } 1\leq i\leq t,\ 1\leq j\leq r-1,$$
$$\deg(u_i,v_0)=t+r-2.$$

One can easily see that in $K_t\times W_r$

$$d((u_i,v_0),(u_j,v_h))=2, \text{ for } i\neq j,\ h\neq 0,$$
$$d((u_i,v_h),(u_j,v_m))=3, \text{ for } i\neq j,\ h\neq m,\ h,m\neq 0,$$

because (u_i,v_h), (u_j,v_h), (u_j,v_0), (u_j,v_m) is a shortest (u_i,v_h)-(u_j,v_m) when $v_hv_m\notin W_r$.Thus,

$$\text{diam } K_t\times W_r=3, \text{ when } r\geq 5.$$

Thus, for $r\geq 5$, $t\geq 2$,

$$\text{diam}_n K_t\times W_r\leq 3.$$

Since for each vertex (u_i,v_h), $1\leq i\leq t$, $h\neq 0$ there are (t-1)(r-4) vertices of distance 3 from (u_i,v_h), and $\deg(u_i,v_h)=t+2$, then we have the following result.

Proposition 5.1. For $t\geq 2$, $r\geq 5$, the n-diameter of $K_t\times W_r$ is given by

$$\text{diam}_n K_t\times W_r=\begin{cases} 3\ , \text{ for } 2\leq n\leq 1+(t-1)(r-4), \\ 2\ , \text{ for } 2+(t-1)(r-4)\leq n\leq p-t-2, \\ 1\ , \text{ for } p-t-1\leq n\leq p. \end{cases}$$ ■

The following theorem gives us the n-Wiener polynomial of $K_t\times W_r$.

Theorem 5.2. For $t\geq 2$, $r\geq 5$, $3\leq n\leq p$, p=tr

$$W_n(K_t\times W_r;x)= p\binom{p-1}{n-2}+[p\binom{p-1}{n-1}-t(r-1)\binom{p-t-3}{n-1}-t\binom{p-r-t+1}{n-1}]x$$
$$+[t(r-1)\binom{p-t-3}{n-1}+t\binom{p-t-r+1}{n-1}-t(r-1)\binom{p-r-4t+4}{n-1}]x^2$$
$$+t(r-1)\binom{p-r-4t+4}{n-1}x^3.$$

Proof. The coefficients of x^0 and x are obtained using (1.8) and (1.9). To obtain the coefficient of x^3, we notice that for any (u_i,v_0), $1\leq i\leq t$ and every (n-1)-set of vertices S, $d_n((u_i,v_0),S)\le 2$. But for every vertex (u_i,v_j), $1\le i\le t$, $1\le j\le r-1$, there are (t-1)(r-4) vertices each of distance 3 from (u_i,v_j). Therefore, there are $\binom{p-r-4t+4}{n-1}$ sets S, |S|=n-1, such that $d_n((u_i,v_j),S)=3$. Thus,

$$C_n(K_t\times W_r,3)=t(r-1)\binom{p-r-4t+4}{n-1}.$$

We obtain $C_n(K_t\times W_r,2)$ by using (1.8). Hence, the proof. ■

Corollary 3.4.3. For $t\ge 2$, $r\ge 5$ and $3\le n\le rt$,

$$W_n(K_t\times W_r)=p\binom{p-1}{n-2}+t(r-1)\binom{p-t-3}{n-1}+t\binom{p-r-t+1}{n-1}+t(r-1)\binom{p-r-4t+4}{n-1}$$

Proof. The proof follows from Theorem 5.2 and the fact

$$W_n(K_t\times W_r)= W\square_n(K_t\times W_{r;1}).$$ ■

6. The Cartesian Product of a Path and a Complete Graph

Let P_r, $r\ge 2$ be a path graph of order r and

$$P_r: v_1,v_2,\ldots,v_r,$$

and let

$$V(K_t)=\{u_1,u_2,\ldots,u_t\},\ t\ge 3.$$

The Cartesian product $K_t\times P_r$ is shown in Fig. 6.1. The following proposition determines the n-diameter of $K_t\times P_r$.

Proposition 6.1. For $r\ge 2$, $t\ge 3$, $2\le n\le rt$,

$$\text{diam}_n\, K_t\times P_r=r+1-\lceil n/t\rceil.$$

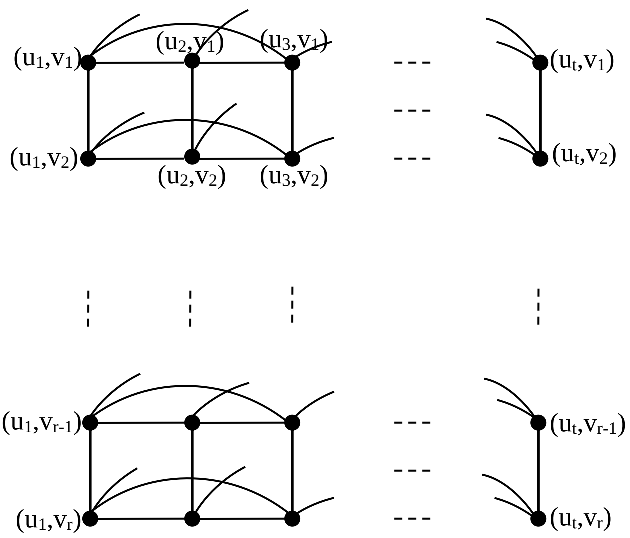

Fig.6.1. The graph $K_t \times P_r$

Proof. From Fig. 6.1, we notice that $d_n((u,v),S)$, $|S|=n-1$ has maximum value when (u,v) is one of the vertices in $A_1 \cup A_r$, where

$$A_i=\{(u_j,v_i): j=1,2,\dots,t\},$$

and S is the (n-1)-set of vertices farthest from (u,v) in $K_t \times P_r$. Thus, we may take the vertex (u_1,v_r), and S consisting of vertices of $A_1,A_2,\dots,A_i$ and some vertices of $A_{i+1}-\{(u_1,v_{i+1})\}$ when

$$it \le n-1 \le t(i+1)-1;$$

and when

$$2 \le n \le t, \text{ then } S \subseteq A_1-\{(u_1,v_1)\}.$$

In the last case,

$$\operatorname{diam}_n K_t \times P_r = r\ ;$$

and in general case of n,

$$\operatorname{diam}_n K_t \times P_r = r-i,\quad it+1 \le n \le (i+1)t.$$

One can easily see that

$$i=\lceil n/t \rceil -1.$$

Hence, in any case of the value of n,

$$\operatorname{diam}_n K_t \times P_r = r+1-\lceil n/t \rceil. \ \blacksquare$$

Now, we obtain the n-Wiener polynomial of $K_t \times P_r$ in the following two theorems.

Theorem 6.2. Let $r=2s$, $s \ge 1$, $t \ge 3$ and $3 \le n \le rt$. Then

$$W_n(K_t \times P_r;x)=\sum_{k=0}^{\delta_n} C_n(K_t \times P_r,k)x^k,$$

where

$$C_n(K_t \times P_r,0)=rt\binom{rt-1}{n-2},$$

$$C_n(K_t \times P_r,1)=rt\binom{rt-1}{n-1}-2t\binom{rt-t-1}{n-1}-t(r-2)\binom{rt-t-2}{n-1},$$

for $2 \le k \le s$

$$C_n(K_t \times P_r,k)=2t[2\sum_{i=1}^{k-1}\{\binom{\alpha+t-ti}{n-1}-\binom{\alpha-it}{n-1}\}+2\{\binom{\alpha+2t-tk-1}{n-1}-\binom{\alpha-tk}{n-1}\}$$
$$+(s-k)\{\binom{\alpha+2t-tk-1}{n-1}-\binom{\alpha-tk-1}{n-1}\}],$$

for $k \ge s+1$

$$C_n(K_t\times P_r,k)=2t\sum_{i=1}^{s}\left\{\binom{\alpha+t-ti}{n-1}-\binom{\alpha-ti}{n-1}\right\},$$

in which

$$\alpha=p-t(k-1)-1.$$

Proof. $C_n(K_t\times P_r,0)$ and $C_n(K_t\times P_r,1)$ are obtained from (1.8) and (1.9). For $2\le k\le \delta_n$ we shall consider three cases for the values of k.

(1) If $2\le k<s$, then for each $1\le i\le s$ the number of vertices of distance k from any vertex, say (u_j,v_i), of A_i is t and the number of vertices of distance more than k from (u_j,v_i) is $p-t(i+k-1)-1=\alpha-ti$ when $1\le i\le k-1$ which gives us

$$a=\sum_{i=1}^{k-1}\sum_{j=1}^{n-1}\binom{t}{j}\binom{\alpha-ti}{n-1-j} \qquad \ldots(6.1)$$

If i=k, then there are 2t-1 vertices of distance k from (u_j,v_i), and there are p-t(2k-1)-1 vertices of distance more than k. This gives us

$$b=\sum_{j=1}^{n-1}\binom{2t-1}{j}\binom{p-2kt+t-1}{n-1-j}$$

$$=\binom{p-2kt+3t-2}{n-1}-\binom{p-2kt+t-1}{n-1}$$

$$=\binom{\alpha+2t-tk-1}{n-1}-\binom{\alpha-kt}{n-1}. \qquad \ldots(6.2)$$

If $k+1\le i\le s$, then there are 2t vertices of distance k from (u_j,v_i) and there are p-t(2k-1)-2 vertices of distance more than k. This gives us

$$c=\sum_{i=k+1}^{s}\sum_{j=1}^{n-1}\binom{2t}{j}\binom{p-2kt+t-2}{n-1-j}$$

$$=(s-k)\left[\binom{p-2kt+3t-2}{n-1}-\binom{p-2kt+t-2}{n-1}\right].$$

$$=(s-k)\left[\binom{\alpha+2t-tk-1}{n-1}-\binom{\alpha-kt-1}{n-1}\right]. \qquad \ldots(6.3)$$

Since r=2s and each A_i consists of t vertices,

$$C_n(K_t\times P_r,k)=2t(a+b+c) \text{ when } 2\le k<s.$$

(2) If k=s, then using the same reasoning as in case (1) we find that (6.1) and (6.2) are true for this case, and (6.3) does not hold. Thus,

$$C_n(K_t\times P_r,k)=2t(a+b) \text{ when } k=s.$$

(3) If $k\ge s+1$, then it is clear that both (6.2) and (6.3) do not hold. Thus,

$C_n(K_t \times P_r,k)==2ta$ when $k \geq s+1$.

Substituting a, b and c, we get the required results. ■

Theorem 6.3. Let $r=2s+1$, $s \geq 1$, $t \geq 3$ and $3 \leq n \leq rt$.
Then

$$W_n(K_t \times P_r;x)=\sum_{k=0}^{\delta_n} C_n(K_t \times P_r,k)x^k,$$

where

$$C_n(K_t \times P_r,0)=rt\binom{rt-1}{n-2},$$

$$C_n(K_t \times P_r,1)=rt\binom{rt-1}{n-1}-2t\binom{rt-t-1}{n-1}-t(r-2)\binom{rt-t-2}{n-1},$$

for $2 \leq k \leq s$

$$C_n(K_t \times P_r,k)=4t[\sum_{i=1}^{k-1}\{\binom{\alpha+t-ti}{n-1}-\binom{\alpha-it}{n-1}\}+\binom{\alpha+2t-tk-1}{n-1}-\binom{\alpha-tk}{n-1}]$$
$$+t(r-2k)\{\binom{\alpha+2t-tk-1}{n-1}-\binom{\alpha-tk-1}{n-1}\},$$

for $k=s+1$,

$$C_n(K_t \times P_r,k)=2t\sum_{i=1}^{s}\{\binom{\alpha+t-ti}{n-1}-\binom{\alpha-ti}{n-1}\}+t\binom{2t-2}{n-1}$$

for $s+1<k \leq \delta_n$,

$$C_n(K_t \times P_r,k)=2t\sum_{i=1}^{s}\{\binom{\alpha+t-ti}{n-1}-\binom{\alpha-ti}{n-1}\},$$

in which

$$\alpha=p-t(k-1)-1.$$

Proof. The proof of $C_n(K_t \times P_r,k)$ for $k \neq s+1$ is similar to that for even r given in Theorem 6.2. For $k=s+1$ we add the number of pairs $((u_j,v_{s+1}),S)$ of n-distance s+1, which equals $\binom{2t-2}{n-1}$ for each $1 \leq j \leq t$. ■

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