A Generalized Curvature of a Generalized Envelope

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ABSTRACT

In this paper we study one of the applications of a generalized curvature [3] on the generalized envelope of a family of lines given in [7], [8], using some concepts of nonstandard analysis given by **Robinson**, **A**. [5] and axiomatized by **Nelson**, E..

Keywords: infinitesimals, monad, envelope, generalized curvature

حول الانحناء المعمم للغلاف المعمم

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الملخص

الهدف من هذا البحث هو دراسة بعض تطبيقات الانحناء المعمم [3] على الغلاف المعمم لعائلة من المستقيمات معطاة [7]،[8] باستخدام بعض مفاهيم التحليل غير القياسي الذي أوجده Robinson, A. [5] ووضعه Nelson, E. بأسلوب منطقي.

الكلمات المفتاحية: ما لانهاية من الصغر، هالة، غلاف، انحناء معمم.

1- Introduction:

The following definitions and notations are needed throughout this paper.

Every concept concerning sets or elements defined in classical mathematics is called **standard** [4].

Any set or formula which does not involve new predicates "standard, infinitesimals, limited, unlimited…etc" is called **internal,** otherwise it is called **external** [2], [4].

A real number x is called **unlimited** if and only if $|x| > r$ for all positive standard real numbers, otherwise it is called **limited** [2].

A real number x is called **infinitesimal** if and only if $|x| < r$ for all positive standard real numbers r [2].

Two real numbers x and y are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \cong y$ [2], [6].

If x is a limited number in $\mathbf{R}$, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x or **shadow** of x denoted by $st(x)$ or ${}^0x$ [2], [4].

If x is a real limited number, then the set of all numbers, which are infinitely close to x, is called the **monad** of x and denoted by $m(x)$ [2], [3].

A curve ν is called **envelope** of a family of curves $\{\gamma_\alpha\}$ depending on a parameter α , if at each of its points, it is tangent to at least one curve of the family $\{\gamma_\alpha\}$, and if each of its segments is tangent to an infinite set of these curves [1].

The **projective homogenous plane** over $\mathbf{R}$, denoted by $\mathbf{P}^2_{\mathbf{R}}$ is the set:

$\mathbf{P}^2_{\mathbf{R}} = \mathbf{R}^2 \cup$ {one point at ∞ for each equivalence classes of parallel lines },we denoted it by (***PHP***) [1].

The **projective homogeneous coordinates** of a point $p(x,y) \in \mathbf{R}^2$ are $(x\alpha, y\alpha, \alpha)$, where α is any nonzero number, we denote it by (***PHC***). In this sense the projective homogeneous coordinates of any point is not unique. [1]

By a **parameterized differentiable curve**, we mean a differentiable map $\gamma : \mathbf{I} \to \mathbf{R}^3$ of an open interval $\mathbf{I}$ =(***a***,***b***) of the real line $\mathbf{R}$ into $\mathbf{R}^3$ such that: $\gamma(t)=(x(t), y(t), z(t)) = x(t)e_1 + y(t)e_2 + z(t)e_3$, and ***x, y,*** and ***z*** are differentiable at ***t*** ; it is also called **spherical curve** [2].

Definition 1.1 [7]

Let $A=\gamma(t)$ be a standard point on the curve γ, then the following cases occur for the point ***A*** with the existence of the order of derivatives of γ :

1- If $\gamma' \neq 0, \gamma'' \neq 0$ and $\gamma' \cdot \gamma'' \neq 0$ then the point is called **biregular** point.

2- If $\gamma' \neq 0$ then the point is called **regular** point.

3- If $\gamma' \neq 0$ and $\gamma' \cdot \gamma'' \neq 0$ then the point is called **only regular** point, and we say that the point is only regular point of order ***p-1*** if $\gamma' \neq 0$ and $\gamma' = \gamma'' = \cdots = \gamma^{(p-1)} = 0$, but $\gamma' \cdot \gamma^{(p)} \neq 0$. In this case we say that ***p*** is the order of the first vector derivative not **collinear** with γ'

4- If $\gamma' = 0$ then the point is called **singular** point. In general if $\gamma' = \gamma'' = \cdots = \gamma^{(p-1)} = 0$ but $\gamma^{(p)} \neq 0$,then the point is called **singular point of order p**.

Theorem 1.2 [7]

Let γ be a standard curve of order C^n and **A** be a standard singular point of order *p-1* on γ**;** and let *B* and *C* be two points infinitely close to the point *A***, t**hen the generalized curvature of γ at the point denoted by K_G and given by

$$K_G = \frac{(p!)^{\frac{q}{p}}\left|x^{(p)}y^{(q)} - x^{(q)}y^{(p)}\right|}{q!\left(x^{(p)2} + y^{(p)2}\right)^{\frac{q+p}{2p}}} = \frac{(p!)^{\frac{q}{p}}\left|\gamma^{(p)} \times \gamma^{(q)}\right|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}},$$

where q is the order of the first vector derivative of γ not collinear with $\gamma^{(p)}$ **.**

Theorem 1.3 [7]

If $p_k(t)= r_k(t)= q_k(t)=0$ for $1 \le k \le n$ (n standard) and $p_n(t)$, $r_n(t)$, $q_n(t)$ are not all zeros, then the ***PHC*** points of $\gamma(t)$ are of the form ($p_n(t)$, $r_n(t)$, $q_n(t)$) which does not depend on e . Thus, we get the generalized nonclassical form of the envelope curve $\gamma(t)$ as follows:

$$(x(t),y(t)) = \left(\frac{X_e(t)}{Z_e(t)}, \frac{Y_e(t)}{Z_e(t)}\right)$$

$$= \left(\frac{v^{(n)}(t)w(t) - w^{(n)}(t)v(t)}{u^{(n)}(t)v(t) - v^{(n)}(t)u(t)}, \frac{w^{(n)}(t)u(t) - u^{(n)}(t)w(t)}{u^{(n)}(t)v(t) - v^{(n)}(t)u(t)}\right)$$

2- A Generalized Curvature of the Envelope of a Family of Lines

Throughout this section, we give a curvature formula for the envelope of a family of lines $L_t : u(t)x + v(t)y + w(t)z = 0$ represented by the components u, v, and w .

It is clear that every two infinitely closed points (points in the same monad) on the envelope curve of a family of lines determine two infinitely close lines in that monad.

That is, $\forall$ $A(t_o), B(t_o) \in \gamma(t_o)$, where $B(t_o+\alpha) \in m(A(t_o))$ there exists a line $L_{t_o+a} \in \{L_t\}$ such that $L_{t_o+a} > L_{t_o}$ in $m(A(t_o))$, where $m(A(t_o))$ denotes the monad of the point ***A***, where α is an infinitesimal number.

For finding curvature formula of the envelope of a family of lines, we follow the following algorithm.

1. Find the envelope curve using **Theorem 1.3** according to the case under consideration.
2. Find the singularity and collinearity order of the envelope curve.
3. Consider three infinitely closed points $A(t_o)$, $B(t_o+\alpha)$ and $C(t_o+\beta)$ on the envelope curve $g(t)$ such that

$A(t_o) \in L_{t_o}$, $B(t_o+\alpha) \in L_{t_o+a}$ and $C(t_o+\beta) \in L_{t_o+b}$

4. Apply the generalized curvature formula given in **Theorem 1.2** at the points $A(t_o)$, $B(t_o+\alpha)$ and $C(t_o+\beta)$.

   Where α and β are infinitesimal numbers.

The following theorems will give a new formula of the generalized curvature of the envelope of a family of lines.

Theorem 2.1

Let $A= (t_o)$ be a regular point of the envelope curve γ of the family $L_t : u(t)X + v(t)Y + w(t)Z =0$ in ***PHC***, then the generalized curvature K_G of the envelope curve at a point A is given by

$$\frac{\left|\left(r'(t)q''(t)-r''(t)q'(t)\right)^2+\left(p''(t)q'(t)-p'(t)q''(t)\right)^2+\left(p'(t)r''(t)-p''(t)r'(t)\right)^2\right|^{\frac{1}{2}}}{2\left|p'(t)^2+q'(t)^2+r'(t)^2\right|^{\frac{3}{2}}}, \quad \text{...(2.1.1)}$$

where $p(t)$, $r(t)$ and $q(t)$ are as given in **Theorem 1.3** for $n=1$

Proof:

Let $A= \gamma(t_o)$ be a standard point on the envelope of the curve γ, and $B = \gamma(t_o+\alpha)$, $C = \gamma(t_o+\beta)$ be two points infinitely close to A. Let L_t , L_{t+a} and L_{t+b} be three lines of the family $\{L_t\}$ having A ,B and C as contact points with the envelope curve, respectively.

Then;

L_t : $u(t)X + v(t)Y + w(t)Z =0$,

L_{t+a} : $u(t+\varepsilon)X + v(t+\varepsilon)Y + w(t+\varepsilon)Z =0$,

L_{t+b} : $u(t+\varepsilon)X + v(t+\varepsilon)Y + w(t+\varepsilon)Z =0$.

Since, the point A is regular, then **Theorem 1.3** for $n=1$ is satisfied, and therefore $\gamma(t)= (p_1(t), r_1(t), q_1(t))$

Using the spherical case of the generalized curvature given in **Theorem 1.2** for a curve $\gamma = \left(x(t),y(t),z(t)\right)$, we get

$$K_G = \frac{\left|(y'z''-y''z')^2+(x''z'-x'z'')^2+(x'y''-x''y')^2\right|^{\frac{1}{2}}}{2\left|x'^2+y'^2+z'^2\right|^{\frac{3}{2}}} \quad \text{… (2.1.2)}$$

Now replacing each of x, y and z by $p_1(t)$, r_1(t) and $q_1(t)$, respectively, we get the required result. ■

Theorem 2.2

Let $A = \gamma(t_o)$ be a singular point of the envelope curve γ of order $n-1$, and let m be the order of the first nonzero derivative which is not collinear with $\gamma^{(n)}(t)$, that is, $\gamma'(t) = \gamma''(t) = \cdots = \gamma^{(n-1)}(t) = 0$, $\gamma^{(n)}(t) \neq 0$, and
$\gamma'(t).\gamma''(t) = \gamma'(t)\square.\gamma'''(t) = \cdots = \square\,\gamma'(t)\square.\gamma^{(m-1)}(t) = \cdots = \gamma^{(n-1)}\square.(t)\,\gamma^{(m-1)}(t) = 0$,
$\gamma^{(n)}(t).\gamma^{(m)}(t) \neq 0$

Then, the generalized curvature K_G of the envelope curve γ at the points of the monad of A is given by

$$\frac{(n!)^{\frac{m}{n}}\left|\left(r^{(n)}q^{(m)}-r^{(m)}q^{(n)}\right)^2+\left(p^{(m)}q^{(n)}-p^{(n)}q^{(m)}\right)^2+\left(p^{(n)}r^{(m)}-p^{(m)}r^{(n)}\right)^2\right|^{\frac{1}{2}}}{m!\left|p^{(n)2}+q^{(n)2}+r^{(n)2}\right|^{\frac{m+n}{2n}}} \quad \ldots(2.2.1)$$

Moreover, the Cartesian coordinate of the generalized curvature K_G of the envelope curve γ at the points of the monad of A is given by

$$\frac{(n!)^{\frac{m}{n}}\left|\left(\frac{p(t)}{q(t)}\right)^{(n)}\left(\frac{r(t)}{q(t)}\right)^{(m)}-\left(\frac{p(t)}{q(t)}\right)^{(m)}\left(\frac{r(t)}{q(t)}\right)^{(n)}\right|}{m!\left|\left(\frac{p(t)}{q(t)}\right)^{(n)2}+\left(\frac{r(t)}{q(t)}\right)^{(n)2}\right|^{\frac{m+n}{2n}}} \quad \ldots(2.2.2)$$

where n and m are positive integer numbers.

Proof:

First, applying the spherical case of the generalized curvature given in **Theorem 1.2** at $x = p_1(t)$, $y = r_1(t)$ and $z = q_1(t)$, we get the generalize curvature formula **(2.2.1)**. Since the point $(p_1(t), r_1(t), q_1(t))$ in ***PHC*** is equivalent to the point $(p_1(t)/q_1(t), r_1(t)/q_1(t), 1)$, so again, applying the spherical case of generalized curvature, we get

$$K_G = \frac{(n!)^{\frac{m}{n}}\left|\left(y^{(n)}z^{(m)}-y^{(m)}z^{(n)}\right)^2+\left(z^{(n)}x^{(m)}-z^{(m)}x^{(n)}\right)^2+\left(x^{(n)}y^{(m)}-x^{(m)}y^{(n)}\right)^2\right|^{\frac{1}{2}}}{m!\left|x^{(n)2}+y^{(n)2}+z^{(n)2}\right|^{\frac{m+n}{2n}}} \ldots(2.2.3)$$

Thus, putting $x = p_1(t)/q_1(t)$, $y = r_1(t)/q_1(t)$ and $z = 1$, in **(2.2.3),** we obtain the formula **(2.2.2)**. ■

Corollary 2.3

Let $A = \gamma(t_o)$ be a singular point of the envelope curve γ satisfying the hypothesis of **Theorem 2.2**.

Moreover, let the coefficient vector $(u(t), v(t), w(t))$ of the envelope curve has a singularity of order $n-1$, then the generalized curvature K_G of the envelope curve γ at points in the monad of A is given by

$$\frac{(n!)^{\frac{m}{n}}\left|(r_n(t)q_m(t)-r_m(t)q_n(t))^2+(p_m(t)q_n(t)-p_n(t)q_m(t))^2+(p_n(t)r_m(t)-p_m(t)r_n(t))^2\right|^{\frac{1}{2}}}{m!\,\left|p_n^2(t)+q_n^2(t)+r_n^2(t)\right|^{\frac{m+n}{2n}}} \quad \ldots(2.3.1)$$

and the cartesian coordinate curvature $K_G(t)$ of the envelope curve γ at A is given by

$$\frac{(n!)^{\frac{m}{n}}\left|\left(\frac{p_n(t)}{q_n(t)}\right)\cdot\left(\frac{r_m(t)}{q_m(t)}\right)-\left(\frac{p_m(t)}{q_m(t)}\right)\cdot\left(\frac{r_n(t)}{q_n(t)}\right)\right|}{m!\,\left|\left(\frac{p_n(t)}{q_n(t)}\right)^2+\left(\frac{r_n(t)}{q_n(t)}\right)^2\right|^{\frac{m+n}{2n}}} \quad \ldots(2.3.2)$$

Proof:

By **Theorem 2.2** we have

$$K_G(t)=\frac{(n!)^{\frac{m}{n}}\left|\left(r^{(n)}q^{(m)}-r^{(m)}q^{(n)}\right)^2+\left(p^{(m)}q^{(n)}-p^{(n)}q^{(m)}\right)^2+\left(p^{(n)}r^{(m)}-p^{(m)}r^{(n)}\right)^2\right|^{\frac{1}{2}}}{m!\,\left|p^{(n)^2}+q^{(n)^2}+r^{(n)^2}\right|^{\frac{m+n}{2n}}}$$

Since the coefficient vector ($\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$, $\boldsymbol{w}(t)$) of the envelope curve has a singularity of order $\boldsymbol{n-1}$,so we get

$u'(t)=v'(t)=w'(t)=\cdots=u^{(n-1)}(t)=v^{(n-1)}(t)=w^{(n-1)}(t)=0$,

and $\quad (u^{(n)}(t),v^{(n)}(t),w^{(n)}(t))\neq 0$

Therefore,

$$\left.\begin{aligned} p^{(n)}(t) &= v^{(n)}(t)w(t)-w^{(n)}(t)v(t)=p_n(t)\\ r^{(n)}(t) &= w^{(n)}(t)u(t)-u^{(n)}(t)w(t)=r_n(t)\\ q^{(n)}(t) &= u^{(n)}(t)v(t)-v^{(n)}(t)u(t)=q_n(t)\end{aligned}\right\} \quad \ldots(2.3.3)$$

Hence, the result of the first part is proved.

To prove the second part put $x=\mathbf{p_n(t)/q_n(t)}$,$y=\mathbf{r_n(t)/q_n(t)}$ and $z=\mathbf{1}$ and then apply the spherical curvature formula **(2.2.3)** to obtain the formula **(2.3.2)**. ■

Corollary 2.4

Let $A=\gamma(t_o)$ be a singular point of the envelope curve γ satisfying the hypothesis of **Theorem 2.2**. Let $\gamma(t)=(p(t), r(t), q(t))$ be such that $q(t)$ has a nonzero constant value, then the generalized curvature K_G of the envelope curve γ at points of the monad of A is given by

$$\frac{(n!)^{\frac{m}{n}}\left|\left(p^{(n)}(t)r^{(m)}(t)-p^{(m)}(t)r^{(n)}(t)\right)^2\right|}{m!\,\left|p^2(t)+r^2(t)\right|^{\frac{m+n}{2n}}}\cdot q^{\frac{m-n}{n}} \quad \ldots(2.4.1)$$

Proof:
Without loss of generality we use the cartesian coordinate form **(2.2.2)** of **Theorem 2.2** to obtain

$$K_G(t)=\frac{(n!)^{\frac{m}{n}}\left|\left(\frac{p(t)}{q(t)}\right)^{(n)}\cdot\left(\frac{r(t)}{q(t)}\right)^{(m)}-\left(\frac{p(t)}{q(t)}\right)^{(m)}\cdot\left(\frac{r(t)}{q(t)}\right)^{(n)}\right|}{m!\left|\left(\frac{p(t)}{q(t)}\right)^{(n)^2}+\left(\frac{r(t)}{q(t)}\right)^{(n)^2}\right|^{\frac{m+n}{2n}}} \quad \ldots (2.4.2)$$

Since the value of $q(t)$ is constant, we get

$$K_G(t)=\frac{(n!)^{\frac{m}{n}}\left(\frac{1}{q}\right)^2\left|p(t)^{(n)}r(t)^{(m)}-p(t)^{(m)}r(t)^{(n)}\right|}{m!\left(\frac{1}{q}\right)^{\frac{m+n}{n}}\left|p(t)^{(n)^2}+r(t)^{(n)^2}\right|^{\frac{m+n}{2n}}}$$

$$=\frac{(n!)^{\frac{m}{n}}\left|\left(p^{(n)}(t)r^{(m)}(t)-p^{(m)}(t)r^{(n)}(t)\right)^2\right|}{m!\left|p^2(t)+r^2(t)\right|^{\frac{m+n}{2n}}}\cdot q^{\frac{m-n}{n}} \quad . \ \blacksquare$$

Remark 2.5

If $q(t)=0$ then, by using either equation **(2.2.1)** or the equation **(2.3.1)**, we can find a spherical generalized curvature K_G , but it does not represent a real curvature of the envelope curve . We shall call such value of curvature **Ideal Curvature** of a curve γ at points of the monad of $A=\gamma\ (t_o)$.

Example 2.6

Consider the family of lines $2x - 3ty + t^3 = 0$

By applying the algorithm given at the beginning of this section, we get

$u = 2$	$u' = 0$	$u'' = 0$	$u''' = 0$
$v = -3t$	$v' = -3$	$v'' = 0$	$v''' = 0$
$w = 2t^3$	$w' = 6t^2$	$w'' = 12t$	$w''' = 12$

Now we determine the singularity and collinearity

$\gamma(0)=(0,0)$ $\gamma'(0)=(0,0)$ $\gamma''(0)=(0,2)$ $\gamma'''(0)=(12,0)$

Thus γ has a first singularity order (that is $n=2$) and the order of collinearity is equal to 3. The envelope curve $\gamma(t)$ is given by

$(X\varepsilon\ (t), Y\varepsilon\ (t), Z\varepsilon(t))$

$=(v'(t)w(t)-w'(t)v(t), w'(t)u(t)-u'(t)w(t), u'(t)v(t)-v'(t)u(t))$

$= (6t^3, 6t^2, 12)$

Since the value of **q(t)** is constant, so using **Corollary 2.4**, we get,

$$K_G = \frac{(2!)^{\frac{3}{2}} \left| \left(p^{(2)}(t) r^{(3)}(t) - p^{(3)}(t) r^{(2)}(t) \right)^2 \right|}{3! \; \left| p^2(t) + r^2(t) \right|^{\frac{3+2}{2\times 2}}} \cdot q^{\frac{3-2}{2}} = \frac{1}{\sqrt{6}} \cdot \sqrt{12} = \sqrt{2}$$

Note that if we use the cartesian coordinate, we find that $\gamma(t)$ is equal to

$$(x(t), y(t)) = \left(\frac{X_e(t)}{Z_e(t)}, \frac{Y_e(t)}{Z_e(t)} \right) = \left(\frac{v'(t)w(t) - w'(t)v(t)}{u'(t)v(t) - v'(t)u(t)}, \frac{w'(t)u(t) - u'(t)w(t)}{u'(t)v(t) - v'(t)u(t)} \right)$$

$$= (1/2)(t^3, t^2)$$

Here γ also has a first singularity order (that is $n=2$) and the order of collinearity is equal to 3. Thus by using the usual two dimensional forms of the generalized curvature, we get, (see **Figure 2.3**)

$$K_G = \frac{(2!)^{\frac{3}{2}} \left| \left(x^{(2)}(t) y^{(3)}(t) - x^{(3)}(t) y^{(2)}(t) \right)^2 \right|}{3! \; \left| \left(x^{(2)}(t) \right)^2 + \left(y^{(2)}(t) \right)^2 \right|^{\frac{3+2}{2\times 2}}} = \left. \frac{(2!)^{\frac{3}{2}} \left| 3t \cdot 0 - 3 \cdot 1 \right|}{3! \; \left| (3t)^2 + (1)^2 \right|^{\frac{3+2}{2\times 2}}} \right|_{t=0} = \sqrt{2}$$

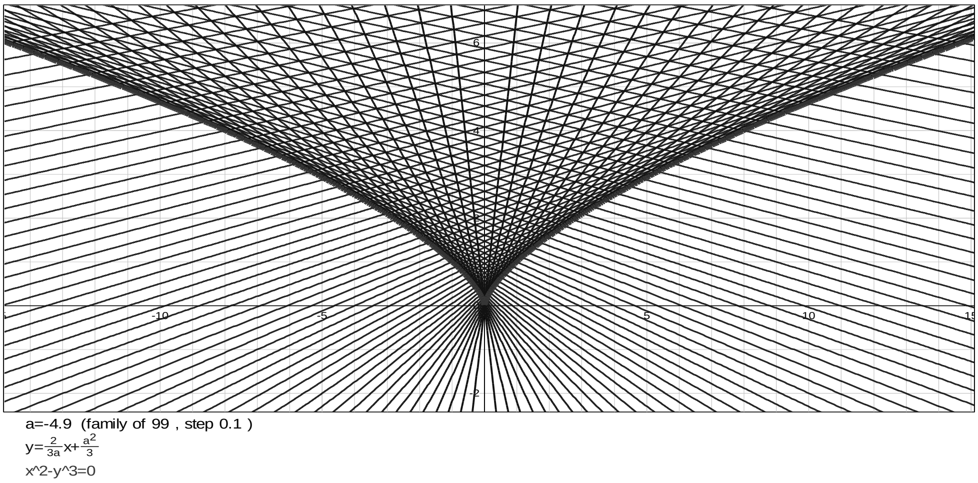

Figure 2.3

Remark: The graph of the equation of the above example is plotted with specific software **Omnigraph V3.1b-2005**.

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