# Construction of Complete (k,n)-arcs from Conics in PG(2,13)

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## ABSTRACT

This research contains the construction of complete (k,n)-arcs using geometric method and putting a condition that each constructed arc must begin with reference unit four points ,then constructing all conics that each one contains the fundamental four points ,and then make union operation between each two conics and compute all the points need to be deleted (or added) to get complete (k,n)-arcs each time for $2 \le n \le 12$ . This research is a construction procedure not a classification method .

***keywords:*** *Complete arc, (k,n)-arc, conic ,PG(2,13)* .

بناء الاقواس – (k,n) التامة من المخروطيات في PG(2,13)

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**الملخص**

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تضمن هذا البحث بناء الاقواس -(k,n) التامة باستخدام الطريقة الهندسية مبتدئين ومشترطين ان يبتديء البناء بالنقاط الاربع الاساسية ومن ثم تكوين كافة المخروطيات Conics التي تضم كل النقاط الاساسية الاربع ، يلي ذلك اجراء عملية الاتحاد بين كل اثنين منهم واحتساب النقاط المطلوب حذفها(أو إضافتها) في كل حالة وذلك للوصول الى الاقواس-(k,n) التامة ولقيم $2 \le n \le 12$ . إن هذا البحث هو وسيلة للبناء وليس طريقة للتصنيف.

***الكلمات المفتاحية : الاقواس التامة،*** الاقواس -(k,n) ، الاقواس المخروطية *PG(2,13)* .

## 1. Introduction

Finding the exact value of k where (k,n)-arc exists, is a big problem, researchers try to fix an n to get run on k and others try to fix k also fix n to check whether the (k,n)-arc exist or not, see for example, [6],[2]and[5], some researchers try to use new algorithms to get exact results, but does not meet the general goal, because these algorithms are trying to check and build little number of the huge number of these (k,n)-arcs, see for example [3] ,[11] and [12].

Researching in this problem; driving man to think in different ways ,the famous direction is classifying the (k,n)-arcs into classes according to n, and trying to study the groups, spectrum (the number of i-secant lines $\tau_i, i = 0, \dots, n$ in the plane [4]), point types, and other properties analytically or computationally according to the way that each class had derived ,see for example, [13]. Another way for fighting this problem is to split the problem into small pieces and study each piece lonely, this idea is needed when the plane is large and the splitting is beneficial, so constructing the piece is so important like the study on this piece, may this piece be a cubic as in [13] or to be a conic as in [9] and[10].

This research is to construct the (k,n)-arcs that are arising from conics and each one of them containing the fundamental arc.

## 2. Preliminaries

**2.1 Definition** [7]

Let V be an (n+1)-dimensional vector space over the field K with origin 0, then consider the equivalence relation on the points of V\{0} whose equivalence classes are the one-dimensional subspaces of V with the origin deleted ;that is ,if $x, y \in V \setminus \{0\}$ and for some basis x=$(x_0, \dots, x_n)$,y=$(y_0, \dots, y_n)$ ,x is equivalent to y if ,for some t in the field K without the origin 0, we have $y_i = tx_i$ for all i. Then, the set of equivalence classes is the n-dimensional projective space over K and is denoted by PG(n,K) or, if K=GF(q), by PG(n,q).

In this paper set n=2,K=GF(13), and we get the finite projective plane PG(2,13).

**2.2 Definition** [7]

A (k,n)-arc A in PG(2,q) is a set of k points such that some line of the plane meets A in n points, but such that no line meets A in more than n points ,where $n \geq 2$.A (k,n)-arc is complete if there is no (k+1,n)-arc containing it.

**2.3 Definition** [7]

Let K$[x_0, \dots, x_n]$ be the polynomial ring in $x_0, \dots, x_n$ variables over the field K, if f in K$[x_0, \dots, x_n]$ be homogeneous; it is called a form. Let x=$(x_0, \dots, x_n)$,then f(x)=f$(x_0, \dots, x_n)$. A subset F of PG(n,K) is a variety(over K) if there exists forms $f_1, f_2, \dots, f_r$ in K[x] such that

$F = \{p(a) \in PG(n,K) \mid f_1(a) = f_2(a) = \cdots = f_r(a) = 0\}$=V$(f_1, f_2, \dots, f_r)$.

Here, the points $p$(a) are points of an F .

**2.4 Definition** [7]

A point set of a conic is a (q + l)-arc of PG(2, q) .In PG(2, q), q odd, every conic is complete as (q + l)-arc. A (q + l)-arc of PG(2, q), q even, is always incomplete. It can be extended in a unique way to a (q + 2)-arc by a point r which is called the nucleus of this (q + l)-arc [7].

**2.5 Theorem 1 ( B. Segre's famous theorem)**[7]

Every (q+ 1)-arc of PG(2, q), q odd, is a conic .

**2.6 Theorem 2** [8]

In PG(2,q) with q≥4, there is a unique conic through a 5-arc .

**2.7 Definition** [7]

An i-secant for (k,n)-arc is a line in PG(2,q) such that it contains exactly i points from this (k,n)-arc.

**2.8 Theorem 3** [7]

In PG(2, q), q ≡-1mod 4, there exists a complete k-arc with k = (q + 5)/2.

**2.9 Theorem 4** [7]

In PG(2,q), q even and q ≥ 8, there exists a complete k-arc with k = (q + 4)/2.

**2.10 Definition** [1]

Let K be a (k,n)-arc ,if the order of the plane is q ,then $k \leq 1 + (q+1)(n-1) = qn - q + n$ with equality if and only if every line intersects the arc in 0 or n points. Arcs realizing the upper bound are called maximal arcs.

**2.11 Definition** [12]

Points of index zero are points which belong to PG(2,q) such that it does not belong to (k,n)-arc nor to any n-secant for this (k,n)-arc .

**3. Construction of PG(2,13)**[7],[10]

The plane PG(2,q) contains $q^2 + q + 1$ points , $q^2 + q + 1$ lines , q+1 points lying on every line and q+1 lines passing through every point. So, in PG(2,13) we will get 183 points and 183 lines and 14 points on every line and 14 lines through every point.

**3.1 Analytic View**

Any point of the plane PG(2,q) has the form of triple $(x_0, x_1, x_2)$where $x_0, x_1$ $and$ $x_2$ are elements of GF(q) with the exception of a triple consisting of three zero elements, two triples represent the same point if there exists λ in GF(q)\{0},such that
$(y_0, y_1, y_2)$= λ$(x_0, x_1, x_2)$.

Similarly, any line of the plane has the form of a triple $[x_0, x_1, x_2]$, where $x_0, x_1$ $and$ $x_2$ are in GF(q) with the exception of a triple consisting three zero elements. Two lines $[x_0, x_1, x_2]$ and $[y_0, y_1, y_2]$represent the same line if there exists λ in GF(q)\{0} such that
$[y_0, y_1, y_2] = \lambda[x_0, x_1, x_2]$.
The point $(x_0, x_1, x_2)$ is incident with the line $[y_0, y_1, y_2]$ if $x_0y_0 + x_1y_1 + x_2y_2 \equiv 0 \ mod \ q$.

**3.2 Construction of Lines and Points for PG(2,13)**

**Firstly:** Let us construct the points of PG(2,13) using lexical manner(A related industry lexical, linguistic style Dictionary i.e. reordering elements in ascending or descending order for each coordinate to any projective point) as in the following way :

Without the loss of generality, we can take $p_1 = (1,0,0)$ be the first point ,then we can use all the plane's rest points as follows :
$p_k = (i, 1, 0)$ ,i=0,1,…,12; (k=i+2). Then, let $p_k = (i, j, 1)$, i=0,1,…,12 for each j=0,1,…,12.(Here k runs from 15 up to 183).

This will collect all the points of PG(2,13) in lexical form. Now write each point in an independent row

**Secondly:** For simplicity, think of coordinates of each point as the coordinates for the line you want to construct (this implies that we think of the point coordinates $(x_0, x_1, x_2)$as the line $[x_0, x_1, x_2]$ and the index k of the point is the same index k of the line) . Now, you can choose the position of the points (their index) to write it beside the coordinate of the line according to the relation:
The point $(x_0, x_1, x_2)$ is incident with the line $[y_0, y_1, y_2]$ if $x_0y_0 + x_1y_1 + x_2y_2 \equiv 0 \ mod \ 13$.
So, we will get the following PG(2,13) plane :

First line :
Line index : k=1 , Line coordinates=[1,0,0],
Line Points={2,15,28,41,54,67,80,93,106,119,132,145,158 and 171}
Second line :
Line index : k=2 ,Line coordinates=[ 0,1,0],
Line Points={1,15,16,17,18,19,20,21,22,23,24,25,26 and 27},
Third line :
Line index : k=3 ,Line coordinates=[1,1,0],
Line Points={14,15,40,52,64,76,88,100,112,124,136,148,160 and 172}
and so on …(all points and lines are in the Appendix )
Fifteenth line:
Line index : k=15 ,Line coordinates=[0,0,1],
Line Points={1,2,3,4,5,6,7,8,9,10,11,12,13 and 14}
For example, note that (just for check) if you take the coordinates for the fifteenth line, you will take the vector$[y_0, y_1, y_2]$=[0,0,1], now, choose the points that their coordinates $(x_0, x_1, x_2)$are perpendicular to [0,0,1] ,clearly $(x_0, x_1, x_2)$=(1,0,0) is one of them which has index k=1, so as $(x_0, x_1, x_2)$=(0,1,0) which has index k=2, also $(x_0, x_1, x_2)$=(1,1,0) has k=3 , $(x_0, x_1, x_2)$=(2,1,0) of index 4,and so on … . Last line will be :
Line index: k=183, Line coordinates=[12,12,1],
Line Points ={14,16,28,53,65,77,89,101,113,125,137,149,161 and 173}

## 4. The Fundamental Arc

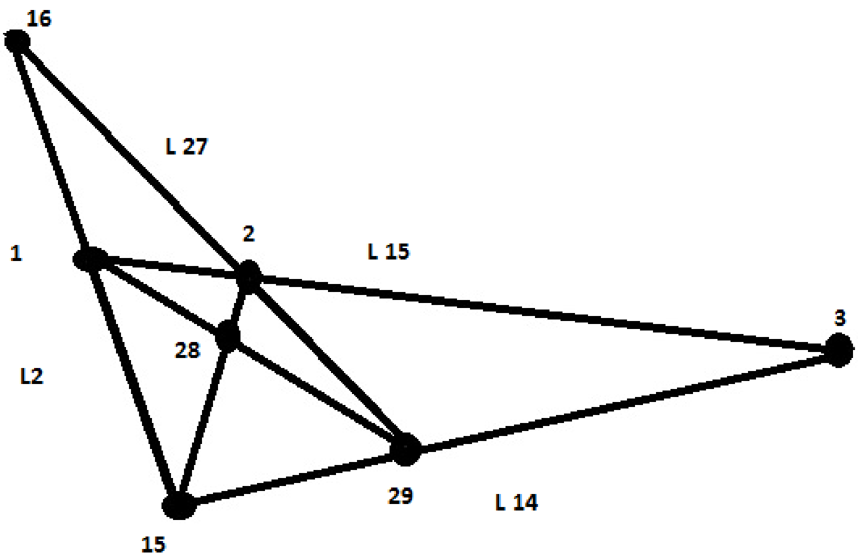
The fundamental theorem of projective geometry states that in any non-trivial projective space, there are four points, no three of them are collinear. Without the loss of generality, we can take an $s_1 = (1,0,0), s_2 = (0,1,0), s_3 = (0,0,1)$ $and$ $s_4 = (1,1,1)$ because in finite projective geometry the relation x= λx ∀ x ∈ $\mathrm{GF(q)}\backslash\{0\}$ is achieved .It is clear that the coordinates of $s_1, s_2, s_3$ and $s_4$ are the simplest coordinates that can be taken. From now and on, we call the arc that contains from the simplest four points $\{s_1, s_2, s_3$ and $s_4\}$ by the fundamental arc. In our PG(2,13) fundamental arc is the points {1,2,15 and 29}, clearly it is a (4,2)-arc and it is not complete.

### 4.1 Geometrical Representation of fundamental arc [10]

According to the fundamental theorem in projective geometry there are four points (namely fundamental arc here) no three of them are collinear, so there is a line connecting each two points of them, this implies that there are six common lines as follows:

L15=$\overleftrightarrow{(1\ ,2)}$={1,2,3,4,5,6,7,8,9,10,11,12,13 and 14};
L2=$\overleftrightarrow{(1\ ,15)}$={1,15,16,17,18,19, 20,21,22,23,24,25,26 and 27};
L171=$\overleftrightarrow{(1\ ,29)}$={1,28,29,30,31,32,33,34,35,36,37,38,39 and 40};
L1=$\overleftrightarrow{(2\ ,15)}$={2,15,28,41,54,67,80,93,106,119,132,145,158 and 171};
L27=$\overleftrightarrow{(2\ ,29)}$={2,16,29,42,55,68,81,94,107,120,133,146,159 and 172};
L14=$\overleftrightarrow{(15\ ,29)}$={3,15,29,43,57,71,85,99,113,127,141,155,169 and 183};

Now, if we plot these lines according to fundamental arc we will get the following figure:




**Figure(1) presents the six lines of the fundamental arc with its diagonal points**

We see from figure (1) that the diagonal points are {3,16 and 28}which arise from the intersection of each two lines from the six lines as follows:
$L14 \cap L15 = \{3\}$ , $L2 \cap L27 = \{16\}$ , $L1 \cap L171 = \{28\}$ .

## 5. Construction of Complete (k,2)-arcs

In this section, we build all the conics that are containing the fundamental arc. So, take the variety V(2) in general manner as done in[10]
$V(x_0, x_1, x_2) = a_1x_0^2 + a_2x_1^2 + a_3x_2^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_2 = 0$ ,
$a_i \in GF(13)$ ... (1)
Substitute fundamental arc point coordinates in (1) we get
$V(s_1)=V(1,0,0)=0 \Rightarrow a_1=0$
$V(s_2)=V(0,1,0)=0 \Rightarrow a_2=0$
$V(s_3)=V(0,0,1)=0 \Rightarrow a_3=0$
$V(s_4)=V(1,1,1)=0 \Rightarrow a_4+a_5+a_6=0$
So,(1) becomes
$a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_2 = 0$ , ... (2)
Now, if $a_4=0 \Rightarrow x_2(a_5x_0+a_6x_1)=0 \Rightarrow$ it is degenerated (It can be expressed as a product of the two factors) $\Rightarrow a_4$ must not be zero. Similarly, $a_5 \neq 0$ and $a_6 \neq 0$. Dividing eq.(2) by $a_4$ we get
$x_0x_1 + \alpha x_0x_2 + \beta x_1x_2 = 0,$ ... (3)
$\alpha = \frac{a_5}{a_4}, \beta = \frac{a_6}{a_4}$ , Now substitute $s_4$ we get
$1 + \alpha + \beta = 0 \Rightarrow \beta = -(1 + \alpha)$ , Now $\alpha \neq 0$ & $\alpha \neq 12$ because if not then, it will be degenerated (For contradicting, let $\alpha = 0$ this implies $= -1$ , now substitute this in (3) we get $x_0x_1 - x_1x_2 = 0 \Rightarrow x_1(x_0 - x_2) = 0$ this means (3) degenerate ), so it must be $\alpha = 1,2, \dots ,11$ .
Now, we can run our work on different values of $\alpha$ to get the conics that contains the fundamental arc,So

1- If $\alpha$ =1, then eq.(3) becomes the conic $C_1 = x_0x_1 + x_0x_2 + 11x_1x_2 = 0$ have the points {1,2,15,29,51,62,79,86,104,111,128,139,148 and 162}when check the number of points of index zero, we see it zero. So, $C_1$ is complete (14,2)-arc.

2- If $\alpha$ =2, then eq.(3) becomes the conic $C_2 = x_0x_1 + 2x_0x_2 + 10x_1x_2 = 0$ have the points {1,2,15,29,49,61,69,84,105,117,124,138,154 and 181} when check the number of points of index zero we see it zero. So, $C_2$ is complete (14,2)-arc.
3- If $\alpha$ =3, then eq.(3) becomes the conic $C_3 = x_0x_1 + 3x_0x_2 + 9x_1x_2 = 0$ have the points {1,2,15,29,53,56,73,89,100,114,129,135,163 and 182} when check the number of points of index zero we see it zero. So, $C_3$ is complete (14,2)-arc.
4- If $\alpha$ =4, then eq.(3) becomes the conic $C_4 = x_0x_1 + 4x_0x_2 + 8x_1x_2 = 0$ have the points {1,2,15,29,47,58,76,90,96,108,131,156,166 and 178} when check the number of points of index zero we see it zero. So, $C_4$ is complete (14,2)-arc.
5- If $\alpha$ =5, then eq.(3) becomes the conic $C_5 = x_0x_1 + 5x_0x_2 + 7x_1x_2 = 0$ have the points {1,2,15,29,52,66,74,83,101,116,134,149,167 and 176} when check the number of points of index zero we see it zero. So, $C_5$ is complete (14,2)-arc.
6- If $\alpha$ =6, then eq.(3) becomes the conic $C_6 = x_0x_1 + 6x_0x_2 + 6x_1x_2 = 0$ have the points {1,2,15,29,46,65,75,82,103,123,144,151,161 and 180} when check the number of points of index zero we see it zero. So, $C_6$ is complete (14,2)-arc.
7- If $\alpha$ =7, then eq.(3) becomes the conic $C_7 = x_0x_1 + 7x_0x_2 + 5x_1x_2 = 0$ have the points {1,2,15,29,50,59,77,92,110,125,143,152,160 and 174} when check the number of points of index zero we see it zero. So, $C_7$ is complete (14,2)-arc.
8- If $\alpha$ =8, then eq.(3) becomes the conic $C_8 = x_0x_1 + 8x_0x_2 + 4x_1x_2 = 0$ have the points {1,2,15,29,48,60,70,95,118,130,136,150,168 and 179}when check the number of points of index zero we see it zero. So, $C_8$ is complete (14,2)-arc.
9- If $\alpha$ =9, then eq.(3) becomes the conic $C_9 = x_0x_1 + 9x_0x_2 + 3x_1x_2 = 0$ have the points {1,2,15,29,44,63,91,97,112,126,137,153,170 and 173}when check the number of points of index zero we see it zero. So, $C_9$ is complete (14,2)-arc.
10- If $\alpha$ =10, then eq.(3) becomes the conic $C_{10} = x_0x_1 + 10x_0x_2 + 2x_1x_2 = 0$ have the points {1,2,15,29,45,72,88,102,109,121,142,157,165 and 177} when check the number of points of index zero we see it zero. So, $C_{10}$ is complete (14,2)-arc.
11- If $\alpha = 11$, then eq.(3) becomes the conic $C_{11} = x_0x_1 + 11x_0x_2 + x_1x_2 = 0$ have the points {1,2,15,29,64,78,87,98,115,122,140,147,164 and 175}when check the number of points of index zero we see it zero. So, $C_{11}$ is complete (14,2)-arc.

## 6. Construction of complete (k,n)-arcs for n=3,…,11

First we must put an important condition to do this construction, we strongly recommended that must each complete (k,n)-arc contains the fundamental arc with all of its four points and not eliminate any point from it for any reason or process in constructing the complete (k,n)-arcs. This paper distinguishes with this condition because the derivation of the conics (in section 5) depends on the coordinates of the fundamental arc points. So, we will do the same way as [9], but we will be far away from his mistakes of eliminating some points from fundamental arc (he eliminate the point 17 to form the $F_2$ and eliminate the point 9 to get G, while these two points are two from the four points of fundamental arc in PG(2,7)).

Man sometimes does some mistakes(for example in[9] ,the point 5 does not belong to the points of zero index for $D_1$,but he writes it as a point of zero index),for this reason, I write a MATLAB® script and run it (after check it on PG(2,5), PG(2,7) and PG(2,11)) to get the correct results on PG(2,13),I program it to find all the conics and all the different combinations of the different conics to get the (k,3)-arcs, so the number of all combinations are $C_2^{11} = \frac{11!}{2!(11-2)!} = 55$ cases ,but if we continue evaluating for (k,4)-arcs we will see that $C_2^{55} = \frac{55!}{2!(55-2)!}$ =1485 cases, for (k,5)-arcs and upper, this number will be increased into bigger and bigger numbers. Because of these huge numbers, this research is made for constructing ,not for classifying, so we can take determined numbers for each (k,n)-arcs, so I see to take firstly the whole 55 cases and, then take (55+1)/2 different cases and, then take 55 cases and so on alternatively .

According to last paragraph, the numbers of (k,3)-arcs,(k,5)-arcs, (k,7)-arcs, (k,9)-arcs and (k,11)-arcs are 55 different cases for each one of them and the numbers of (k,4)-arcs, (k,6)-arcs, (k,8)-arcs, (k,10)-arcs and (k,12)-arcs are 28 different cases for each one of them.

The program can compute and save other things like the details for each (k,n)-arc and which points are to be added and which points must be eliminated (here, it avoids any point from fundamental arc), save the history of each (k,n)-arc came from, save the spectrum for each complete(k,n)-arc and save the complete(k,n)-arc itself, furthermore all this different information is stored in just one variable of class 'structure'.

In the following subsections (6.2 up to 6.10), we will write just one example for simplicity, except in section 6.1, we will write two examples to show how the process works ,but if anyone likes to see all the results , he(or she) can contact with the researcher.

### 6.1 Construction of Complete (k,3)-arcs

Using the union of each two different conics from section 5 as follows:
Let $D_t = C_i \cup C_j$ where $i < j, \forall i, j = 1,2, \dots ,11; t = 1,2, \dots ,55$ .
For example, $D_1 = C_1 \cup C_2$ ={1,2,15,29,49,51,61,62,69,79,84,86,104,105,111,117,124, 128,138,139,148,154,162 and 181} it is a (24,n)-arc, but this set of points needs some operations like{-148-154-139-111-162-124-138+162+164+113+119}, to get the complete (k,3)-arc (here the minus sign means eliminating the following point while the plus sign means adding the following point). This means the operations which had been done on $D_1$ , are first step: eliminates the point 148 from $D_1$, second step: eliminates the point 154 from result of first step, and so on …,eighth step: adds the point 162 to the result of seventh step, and so on … .The (k,3)-arc will be (21,3)-arc which its points are {1,2,15,29,49,51,61,62,69,79,84,86,104,105,117,128,181,162,164,113 and 119},which has no points of index zero, so it is complete (21,3)-arc. The spectrum for this complete (21,3)-arc is $\tau_{0,1,2,3}$ =[42,45,39 and 57] respectively.

Another example : $D_{55} = C_{10} \cup C_{11}$={1,2,15,29,45,64,72,78,87,88,98,102,109, 115,121,122,140,142,147,157,164,165, 175 and 177} it is a (24,n)-arc, but this set of points needs some operations like {-165-98-177-164-175-142-157+48+104}, to get the complete (k,3)-arc. The (k,3)-arc will be (19,3)-arc which its points are{1,2,15,29,45,64, 72,78,87,88,102,109,115,121,122,140,147,48 and 104},checking the plane with this (19,3)-arc we can see it has no points of index zero, so it is complete (19,3)-arc. The spectrum for this complete (19,3)-arc is $\tau_{0,1,2,3}$ =[47,47,48 and 41] respectively.

### 6.2 Construction of Complete (k,4)-arcs

Let $E_t = D_i \cup D_j$ where $i < j, \forall\, i = 1,3,5, \dots ,53, j = i + 1;\ t = 1,2,\ \dots ,28$ and let $E_{28} = D_{55} \cup D_{54}$ .This selection of *i's* and *j's* makes using of all different $D_i$ in minimum number of *t* cases.

<u>Example</u> : Take t=12$\Longrightarrow E_{12} = D_{23} \cup D_{24}$ ={1,2,15,29,26,32,48,50,53,56,59,60,70,73, 77,89,92,95,99,100,110,114,118,124,129,135,136,165 and 182} it is (29,n)-arc, but this set of points needs some operations like {-182-165+24+5+6+130} to get the complete (k,4)-arc. The (29,n)-arc will be (31,4)-arc which its points are : {1,2,15,29,26,32,48,50, 53,56,59,60,70,73,77,89,92,95,99,100,110,114,118,124,129,135,136,24,5,6 and 130},

which has zero number of points of index zero, so it is complete (31,4)-arc. The spectrum for this complete (31,4)-arc is $\tau_{o,\dots,4}$ =[24,27,36,49 and 47] respectively.

### 6.3 Construction of Complete (k,5)-arcs

Let $F_t = E_i \cup E_j$ where $i < j, i, j = 1,2, \dots ,11;\ t = 1,2, \dots ,55.$

Example : Take t=17$\Longrightarrow F_{17} = E_2 \cup E_9$ ={1,2,15,29,7,8,9,18,19,28,30,44,45,47,49,51, 52,58,61,62,63,66,69,72,74,76,79,83,84,86,88,90,91,96,97,101,102,104,105,109,111, 112,117,126,127,135,137,140,142,150,154,160,177 and 180} it is (54,n)-arc, but this set of points needs some operations like{-160-180-137-154-177-150-97-135-109-140-127 -142-105-79-66-91-90-52-51-88+118+21+146+87+134+154+120} to get the complete (k,5)-arc. The (54,n)-arc will be (41,5)-arc which its points are : {1,2,15,29,7,8,9,18,19, 28,30,44,45,47,49,58,61,62,63,69,72,74,76,83,84,86,96,101,102,104,111,112,117,126, 118,21,146,87,134,154 and 120},which has zero number of points of index zero, so it is complete (41,5)-arc. The spectrum for this complete (41,5)-arc is $\tau_{o,\dots,5}$ =[11,18,33,37, 41 and 43] respectively.

### 6.4 Construction of Complete (k,6)-arcs

Let $G_t = F_i \cup F_j$ where $i < j, \forall\ i = 1,3,5, \dots ,53, j = i + 1;\ t = 1,2,\ \dots ,28$ and let $G_{28} = F_{55} \cup F_{54}$ .

Example : Take t=23$\Longrightarrow G_{23} = F_{45} \cup F_{46} =$ {1,2,15,29,4,6,8,14,16,18,20,22,24,26,28, 31,33,36,38,40,46,47,48,49,50,52,56,58,59,60,61,65,69,70,73,74,75,81,82,83,84,89,93, 95,97,100,101,105,112,114,115,118,120,124,129,134,145,147,149,157,162,168,176 and 180} it is (64,n)-arc, but this set of points needs some operations like {-26-24-124-120-162-176-168-147-134-105-157-112-65-129-52-180-145-100-40-38+159+146+105+110 +152+163+147+119+158+128} to get the complete (k,6)-arc. The (64,n)-arc will be (54,6)-arc which its points are:{ 1,2,15,29,4,6,8,14,16,18,20,22,28,31,33,36,46,47,48, 49,50,56,58,59,60,61,69,70,73,74,75,81,82,83,84,89,93,95,97,101,114,115,118,149,159 ,146,105,110,152,163,147,119,158 and 128},which has zero number of points of index zero, so it is complete (54,6)-arc. The spectrum for this complete (54,6)-arc is $\tau_{o,\dots,6}$ =[4,10,19,27,41,29 and 53] respectively.

### 6.5 Construction of Complete (k,7)-arcs

Let $H_t = G_i \cup G_j$ where $i < j, i, j = 1,2, \dots ,11;\ t = 1,2, \dots ,55.$

Example : Take t=6$\Longrightarrow H_6 = G_1 \cup G_7$ ={1,2,15,29,3,4,6,7,8,16,18,19,24,26,28,30,31, 34,35,39,41,45,46,47,49,50,51,56,58,59,61,62,64,69,72,73,74,75,76,82,83,84,86,87,88, 89,96,97,99,100,101,102,104,107,108,111,112,113,115,119,127,128,129,133,135,137, 144,146,158,160,162,163,164,167,168,181 and 182} it is (77,n)-arc,but this set of points needs some operations like{-163-182-167-162-164-128-104-168-181-89-51-133-158-129-115-119-39+110+161} to get the complete (k,7)-arc. The (77,n)-arc will be (62,7)-arc which its points are : {1,2,15,29,3,4,6,7,8,16,18,19,24,26,28,30,31,34,35,41, 45,46,47,49,50,56,58,59,61,62,64,69,72,73,74,75,76,82,83,84,86,87,88,96,97,99,100, 101,102,107,108,111,112,113,127,135,137,144,146,160,110 and 161},which has zero number of points of index zero, so it is complete (62,7)-arc. The spectrum for this complete (62,7)-arc is $\tau_{o,\dots,7}$ =[ 0,6,16,20,40,36,25 and 40] respectively.

### 6.6 Construction of Complete (k,8)-arcs

Let $I_t = H_i \cup H_j$ where $i < j, \forall\ i = 1,3,5, \dots ,53, j = i + 1;\ t = 1,2,\ \dots ,28$ and let $I_{28} = H_{55} \cup H_{54}$ .

Example : Take t=14⟹ $I_{14} = H_{27} \cup H_{28} =$ {1,2,15,29,3,4,5,6,8,9,10,16,18,19,20,21, 22,24,28,30,31,33,34,35,38,41,43,44,45,46,47,48,49,51,56,58,59,60,61,62,65,69,70,72, 73,74,75,76,81,82,83,84,86,87,88,95,96,97,98,100,101,106,107,108,110,111,113,114, 115,116,117,118,119,122,123,127,140,145,146,154,159,164,174,180 and 181} it is (85,n)-arc, but this set of points needs some operations like {-24-174-10-140-113-164-114-181-51-49-159-145-118-117-38-146+155+140+145+151+133+132} to get the complete (k,8)-arc. The (85,n)-arc will be (75,8)-arc which its points are:{1,2,15,29,3,4, 5,6,8,9,16,18,19,20,21,22,28,30,31,33,34,35,41,43,44,45,46,47,48,56,58,59,60,61,62,65 ,69,70,72,73,74,75,76,81,82,83,84,86,87,88,95,96,97,98,100,101,106,107,108,110,111, 115,116,119,122,123,127,154,180,155,140,145,151,133 and 132}, which has zero number of points of index zero, so it is complete (75,8)-arc. The spectrum for this complete (75,8)-arc is $\tau_{o,\ldots,8}$ =[0,2,6,11,31,29,34,30 and 40] respectively.

### 6.7 Construction of Complete (k,9)-arcs

Let $J_t = I_i \cup I_j$ where $i < j, i, j = 1,2,\ldots,11;\ t = 1,2,\ldots,55$.

Example : Take t=9⟹ $J_9 = I_1 \cup I_{10}$ ={1,2,15,29,3,4,5,6,7,8,9,10,16,18,19,21,22,24,26, 28,30,31,32,33,34,35,41,44,45,46,47,48,49,56,58,59,60,61,62,64,65,69,70,72,73,74,75, 76,81,82,83,84,86,87,88,89,95,96,97,98,100,101,102,106,107,108,109,111,112,113,115 ,117,118,119,120,121,122,123,124,127,131,132,133,137,140,141,147,150,151,157,168, 178 and 179} it is (93,n)-arc, but this set of points needs some operations like{-168-10-150-178-147-179-132-118-117-157+183+160+170+149+159+67+172+53} to get the complete (k,9)-arc. The (93,n)-arc will be (91,9)-arc which its points are : {1,2,15, 29,3,4,5,6,7,8,9,16,18,19,21,22,24,26,28,30,31,32,33,34,35,41,44,45,46,47,48,49,56,58, 59,60,61,62,64,65,69,70,72,73,74,75,76,81,82,83,84,86,87,88,89,95,96,97,98,100,101, 102,106,107,108,109,111,112,113,115,119,120,121,122,123,124,127,131,133,137,140, 141,151,183,160,170,149,159,67,172 and 53},which has no points of index zero, so it is complete (91,9)-arc. The spectrum for this complete (91,9)-arc is $\tau_{o,\ldots,9}$ =[0,0,2,6,11,23, 27,30,35 and 49] respectively.

### 6.8 Construction of Complete (k,10)-arcs

Let $K_t = J_i \cup J_j$ where $i < j, \forall\ i = 1,3,5,\ldots,53, j = i + 1;\ t = 1,2,\ \ldots,28$ and let $K_{28} = J_{55} \cup J_{54}$ .

Example : Take t=20⟹ $K_{20} = J_{39} \cup J_{40} =$ {1,2,15,29,3,4,5,6,7,8,9,16,17,18,19,20,21, 22,24,28,30,31,32,33,34,35,41,43,44,45,46,47,48,49,51,54,56,58,59,60,61,62,64,65,69, 70,72,73,74,75,76,77,81,82,83,84,86,87,88,89,95,96,97,98,100,101,102,103,104,106, 107,108,109,110,111,112,113,119,120,121,122,123,124,127,128,129,132,133,136,138, 140,141,144,149,153,159,163,165,167,168,169,171 and 176} it is (103,n)-arc, but this set of points needs some operations like {-149-176-171-168-169-163+132+167+ 38+146+157+92+167+68+114+154} to get the complete (k,10)-arc. The (103,n)-arc will be (107,10)-arc which its points are:{1,2,15,29,3,4,5,6,7,8,9,16,17,18, 19,20,21, 22,24,28,30,31,32,33,34,35,41,43,44,45,46,47,48,49,51,54,56,58,59,60,61,62,64,65,69, 70,72,73,74,75,76,77,81,82,83,84,86,87,88,89,95,96,97,98,100,101,102,103,104,106, 107,108,109,110,111,112,113,119,120,121,122,123,124,127,128,129,132,133,136,138, 140,141,144,153,159,165,167,132,167,38,146,157,92,167,68,114 and 154},which has no points of index zero, so it is complete (107,10)-arc. The spectrum for this complete (107,10)-arc is $\tau_{o,\ldots,10}$ =[0,1,0,1,3,12,26,24,37,30 and 49] respectively.

### 6.9 Construction of Complete (k,11)-arcs

Let $L_t = K_i \cup K_j$ where $i < j, i, j = 1,2,\ldots,11;\ t = 1,2,\ldots,55$.

Example : Take t=13⟹ $L_{13} = K_2 \cup K_5$ ={1,2,15,29,3,4,5,6,7,8,9,13,14, 16,17,18,19, 20,21,22,24,26,28,30,31,32,33,34,35,36,41,43,44,45,46,47,48,49,53,54,55,56,58,59,60, 61,62,63,64,65,67,69,70,72,73,74,75,76,77,79,80,81,82,83,84,86,87,88,89,93,95,96,97, 98,100,101,102,103,104,106,107,108,109,110,111,112,113,115,116,119,120,121,122, 123,124,125,127,128,129,131,133,135,136,137,140,141,142,144,147,148,150,151,153, 154,156,158,164,165,166,167,169,170 and 178} it is (123,n)-arc,but this set of points needs some operations like{-170-165-65-147-158-131-166-156+145+149+155+52} to get the complete (k,11)-arc. The (123,n)-arc will be (119,11)-arc which its points are {1,2,15,29,3,4,5,6,7,8,9,13,14,16,17,18,19,20,21,22,24,26,28,30,31,32,33,34,35,36,41, 43,44,45,46,47,48,49,53,54,55,56,58,59,60,61,62,63,64,67,69,70,72,73,74,75,76,77,79, 80,81,82,83,84,86,87,88,89,93,95,96,97,98,100,101,102,103,104,106,107,108,109,110, 111,112,113,115,116,119,120,121,122,123,124,125,127,128,129,133,135,136,137,140, 141,142,144,148,150,151,153,154,164,167,169,178,145,149,155 and 52} ,which has no points of index zero, so it is complete (119,11)-arc. The spectrum for this complete (119,11)-arc is $\tau_{o,\dots,11}$ =[0,0,1,0,2,2,9,18,29,33,42 and 47] respectively.

### 6.10 Construction of Complete (k,12)-arcs

Let $M_t = L_i \cup L_j$ where $i < j, \forall\ i = 1,3,5, \dots ,53, j = i + 1;\ t = 1,2,\ \dots ,28$ and let $M_{28} = L_{55} \cup L_{54}$ .

Example : Take t=15⟹ $M_{15} = L_{29} \cup L_{30} =$ {1,2,15,29,3,4,5,6,7,8,9,10,11,16,17,18, 19,20,21,22,24,25,28,30,31,32,33,34,35,36,37,40,41,42,43,44,45,46,47,48,49,52,53,54, 56,58,59,60,61,62,63,64,65,67,68,69,70,71,72,73,74,75,76,78,81,82,83,84,86,87,88,89, 90,91,93,94,95,96,97,98,99,100,101,102,106,107,108,109,110,111,112,113,114,115, 119,120,121,122,123,124,127,128,129,130,132,133,135,136,137,139,140,141,142,143, 144,145,148,150,151,152,153,154,155,159,160,161,169 and 178} it is (128,n)-arc, but this set of points needs some operations like {+171+118+23+56+131+164+85+66+168 +149} to get the complete (k,12)-arc. The (128,n)-arc will be (138,12)-arc which its points are:{ 1,2,15,29,3,4,5,6,7,8,9,10,11,16,17,18,19,20,21,22,24,25,28,30,31,32,33, 34,35,36,37,40,41,42,43,44,45,46,47,48,49,52,53,54,56,58,59,60,61,62,63,64,65,67,68, 69,70,71,72,73,74,75,76,78,81,82,83,84,86,87,88,89,90,91,93,94,95,96,97,98,99,100, 101,102,106,107,108,109,110,111,112,113,114,115,119,120,121,122,123,124,127,128, 129,130,132,133,135,136,137,139,140,141,142,143,144,145,148,150,151,152,153,154, 155,159,160,161,169,178,171,118,23,56,131,164,85,66,168 and 149}, which has no points of index zero, so it is complete (138,12)-arc. The spectrum for this complete (138,12)-arc is $\tau_{o,\dots,12}$ =[0,0,0,1,0,0,2,7,8,28,28,50 and 59] respectively.

### 6.11 Construction of Complete (k,13)-arcs

It is clearly from [7] that the simple way to get the complete (k,13)-arcs is to eliminate one line from the plane PG(2,13) and get the complete(169,13)-arc. So, using this idea on any line that does not contain any point from the fundamental arc will get the result.

### 6.12 Construction of Complete (k,14)-arcs

Clearly from [7], the whole plane is the only one choice to get the complete (183,14)-arc in PG(2,13).

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# Appendix

## Points and Lines for PG(2,13)

| $P_i$ | $X_0$ | $X_1$ | $X_2$ | $L_i$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 15 | 28 | 41 | 54 | 67 | 80 | 93 | 106 | 119 | 132 | 145 | 158 | 171 |
| 2 | 0 | 1 | 0 | 1 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 3 | 1 | 1 | 0 | 14 | 15 | 40 | 52 | 64 | 76 | 88 | 100 | 112 | 124 | 136 | 148 | 160 | 172 |
| 4 | 2 | 1 | 0 | 8 | 15 | 34 | 53 | 59 | 78 | 84 | 103 | 109 | 128 | 134 | 153 | 159 | 178 |
| 5 | 3 | 1 | 0 | 6 | 15 | 32 | 49 | 66 | 70 | 87 | 104 | 108 | 125 | 142 | 146 | 163 | 180 |
| 6 | 4 | 1 | 0 | 5 | 15 | 31 | 47 | 63 | 79 | 82 | 98 | 114 | 130 | 133 | 149 | 165 | 181 |
| 7 | 5 | 1 | 0 | 7 | 15 | 33 | 51 | 56 | 74 | 92 | 97 | 115 | 120 | 138 | 156 | 161 | 179 |
| 8 | 6 | 1 | 0 | 4 | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 107 | 122 | 137 | 152 | 167 | 182 |
| 9 | 7 | 1 | 0 | 13 | 15 | 39 | 50 | 61 | 72 | 83 | 94 | 118 | 129 | 140 | 151 | 162 | 173 |
| 10 | 8 | 1 | 0 | 10 | 15 | 36 | 44 | 65 | 73 | 81 | 102 | 110 | 131 | 139 | 147 | 168 | 176 |
| 11 | 9 | 1 | 0 | 12 | 15 | 38 | 48 | 58 | 68 | 91 | 101 | 111 | 121 | 144 | 154 | 164 | 174 |
| 12 | 10 | 1 | 0 | 11 | 15 | 37 | 46 | 55 | 77 | 86 | 95 | 117 | 126 | 135 | 157 | 166 | 175 |
| 13 | 11 | 1 | 0 | 9 | 15 | 35 | 42 | 62 | 69 | 89 | 96 | 116 | 123 | 143 | 150 | 170 | 177 |
| 14 | 12 | 1 | 0 | 3 | 15 | 29 | 43 | 57 | 71 | 85 | 99 | 113 | 127 | 141 | 155 | 169 | 183 |
| 15 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 16 | 1 | 0 | 1 | 2 | 27 | 40 | 53 | 66 | 79 | 92 | 105 | 118 | 131 | 144 | 157 | 170 | 183 |
| 17 | 2 | 0 | 1 | 2 | 21 | 34 | 47 | 60 | 73 | 86 | 99 | 112 | 125 | 138 | 151 | 164 | 177 |
| 18 | 3 | 0 | 1 | 2 | 19 | 32 | 45 | 58 | 71 | 84 | 97 | 110 | 123 | 136 | 149 | 162 | 175 |
| 19 | 4 | 0 | 1 | 2 | 18 | 31 | 44 | 57 | 70 | 83 | 96 | 109 | 122 | 135 | 148 | 161 | 174 |
| 20 | 5 | 0 | 1 | 2 | 20 | 33 | 46 | 59 | 72 | 85 | 98 | 111 | 124 | 137 | 150 | 163 | 176 |
| 21 | 6 | 0 | 1 | 2 | 17 | 30 | 43 | 56 | 69 | 82 | 95 | 108 | 121 | 134 | 147 | 160 | 173 |
| 22 | 7 | 0 | 1 | 2 | 26 | 39 | 52 | 65 | 78 | 91 | 104 | 117 | 130 | 143 | 156 | 169 | 182 |
| 23 | 8 | 0 | 1 | 2 | 23 | 36 | 49 | 62 | 75 | 88 | 101 | 114 | 127 | 140 | 153 | 166 | 179 |
| 24 | 9 | 0 | 1 | 2 | 25 | 38 | 51 | 64 | 77 | 90 | 103 | 116 | 129 | 142 | 155 | 168 | 181 |
| 25 | 10 | 0 | 1 | 2 | 24 | 37 | 50 | 63 | 76 | 89 | 102 | 115 | 128 | 141 | 154 | 167 | 180 |
| 26 | 11 | 0 | 1 | 2 | 22 | 35 | 48 | 61 | 74 | 87 | 100 | 113 | 126 | 139 | 152 | 165 | 178 |
| 27 | 12 | 0 | 1 | 2 | 16 | 29 | 42 | 55 | 68 | 81 | 94 | 107 | 120 | 133 | 146 | 159 | 172 |
| 28 | 0 | 1 | 1 | 1 | 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 |
| 29 | 1 | 1 | 1 | 14 | 27 | 39 | 51 | 63 | 75 | 87 | 99 | 111 | 123 | 135 | 147 | 159 | 171 |
| 30 | 2 | 1 | 1 | 8 | 21 | 40 | 46 | 65 | 71 | 90 | 96 | 115 | 121 | 140 | 146 | 165 | 171 |
| 31 | 3 | 1 | 1 | 6 | 19 | 36 | 53 | 57 | 74 | 91 | 95 | 112 | 129 | 133 | 150 | 167 | 171 |
| 32 | 4 | 1 | 1 | 5 | 18 | 34 | 50 | 66 | 69 | 85 | 101 | 117 | 120 | 136 | 152 | 168 | 171 |
| 33 | 5 | 1 | 1 | 7 | 20 | 38 | 43 | 61 | 79 | 84 | 102 | 107 | 125 | 143 | 148 | 166 | 171 |
| 34 | 6 | 1 | 1 | 4 | 17 | 32 | 47 | 62 | 77 | 92 | 94 | 109 | 124 | 139 | 154 | 169 | 171 |
| 35 | 7 | 1 | 1 | 13 | 26 | 37 | 48 | 59 | 70 | 81 | 105 | 116 | 127 | 138 | 149 | 160 | 171 |

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **36** | **8** | **1** | **1** | **10** | **23** | **31** | **52** | **60** | **68** | **89** | **97** | **118** | **126** | **134** | **155** | **163** | **171** |
| **37** | **9** | **1** | **1** | **12** | **25** | **35** | **45** | **55** | **78** | **88** | **98** | **108** | **131** | **141** | **151** | **161** | **171** |
| **38** | **10** | **1** | **1** | **11** | **24** | **33** | **42** | **64** | **73** | **82** | **104** | **113** | **122** | **144** | **153** | **162** | **171** |
| **39** | **11** | **1** | **1** | **9** | **22** | **29** | **49** | **56** | **76** | **83** | **103** | **110** | **130** | **137** | **157** | **164** | **171** |
| **40** | **12** | **1** | **1** | **3** | **16** | **30** | **44** | **58** | **72** | **86** | **100** | **114** | **128** | **142** | **156** | **170** | **171** |
| **41** | **0** | **2** | **1** | **1** | **93** | **94** | **95** | **96** | **97** | **98** | **99** | **100** | **101** | **102** | **103** | **104** | **105** |
| **42** | **1** | **2** | **1** | **13** | **27** | **38** | **49** | **60** | **71** | **82** | **93** | **117** | **128** | **139** | **150** | **161** | **172** |
| **43** | **2** | **2** | **1** | **14** | **21** | **33** | **45** | **57** | **69** | **81** | **93** | **118** | **130** | **142** | **154** | **166** | **178** |
| **44** | **3** | **2** | **1** | **10** | **19** | **40** | **48** | **56** | **77** | **85** | **93** | **114** | **122** | **143** | **151** | **159** | **180** |
| **45** | **4** | **2** | **1** | **8** | **18** | **37** | **43** | **62** | **68** | **87** | **93** | **112** | **131** | **137** | **156** | **162** | **181** |
| **46** | **5** | **2** | **1** | **12** | **20** | **30** | **53** | **63** | **73** | **83** | **93** | **116** | **126** | **136** | **146** | **169** | **179** |
| **47** | **6** | **2** | **1** | **6** | **17** | **34** | **51** | **55** | **72** | **89** | **93** | **110** | **127** | **144** | **148** | **165** | **182** |
| **48** | **7** | **2** | **1** | **11** | **26** | **35** | **44** | **66** | **75** | **84** | **93** | **115** | **124** | **133** | **155** | **164** | **173** |
| **49** | **8** | **2** | **1** | **5** | **23** | **39** | **42** | **58** | **74** | **90** | **93** | **109** | **125** | **141** | **157** | **160** | **176** |
| **50** | **9** | **2** | **1** | **9** | **25** | **32** | **52** | **59** | **79** | **86** | **93** | **113** | **120** | **140** | **147** | **167** | **174** |
| **51** | **10** | **2** | **1** | **7** | **24** | **29** | **47** | **65** | **70** | **88** | **93** | **111** | **129** | **134** | **152** | **170** | **175** |
| **52** | **11** | **2** | **1** | **3** | **22** | **36** | **50** | **64** | **78** | **92** | **93** | **107** | **121** | **135** | **149** | **163** | **177** |
| **53** | **12** | **2** | **1** | **4** | **16** | **31** | **46** | **61** | **76** | **91** | **93** | **108** | **123** | **138** | **153** | **168** | **183** |
| **54** | **0** | **3** | **1** | **1** | **67** | **68** | **69** | **70** | **71** | **72** | **73** | **74** | **75** | **76** | **77** | **78** | **79** |
| **55** | **1** | **3** | **1** | **12** | **27** | **37** | **47** | **57** | **67** | **90** | **100** | **110** | **120** | **143** | **153** | **163** | **173** |
| **56** | **2** | **3** | **1** | **7** | **21** | **39** | **44** | **62** | **67** | **85** | **103** | **108** | **126** | **144** | **149** | **167** | **172** |
| **57** | **3** | **3** | **1** | **14** | **19** | **31** | **43** | **55** | **67** | **92** | **104** | **116** | **128** | **140** | **152** | **164** | **176** |
| **58** | **4** | **3** | **1** | **11** | **18** | **40** | **49** | **58** | **67** | **89** | **98** | **107** | **129** | **138** | **147** | **169** | **178** |
| **59** | **5** | **3** | **1** | **4** | **20** | **35** | **50** | **65** | **67** | **82** | **97** | **112** | **127** | **142** | **157** | **159** | **174** |
| **60** | **6** | **3** | **1** | **8** | **17** | **36** | **42** | **61** | **67** | **86** | **105** | **111** | **130** | **136** | **155** | **161** | **180** |
| **61** | **7** | **3** | **1** | **9** | **26** | **33** | **53** | **60** | **67** | **87** | **94** | **114** | **121** | **141** | **148** | **168** | **175** |
| **62** | **8** | **3** | **1** | **13** | **23** | **34** | **45** | **56** | **67** | **91** | **102** | **113** | **124** | **135** | **146** | **170** | **181** |
| **63** | **9** | **3** | **1** | **6** | **25** | **29** | **46** | **63** | **67** | **84** | **101** | **118** | **122** | **139** | **156** | **160** | **177** |
| **64** | **10** | **3** | **1** | **3** | **24** | **38** | **52** | **66** | **67** | **81** | **95** | **109** | **123** | **137** | **151** | **165** | **179** |
| **65** | **11** | **3** | **1** | **10** | **22** | **30** | **51** | **59** | **67** | **88** | **96** | **117** | **125** | **133** | **154** | **162** | **183** |
| **66** | **12** | **3** | **1** | **5** | **16** | **32** | **48** | **64** | **67** | **83** | **99** | **115** | **131** | **134** | **150** | **166** | **182** |
| **67** | **0** | **4** | **1** | **1** | **54** | **55** | **56** | **57** | **58** | **59** | **60** | **61** | **62** | **63** | **64** | **65** | **66** |
| **68** | **1** | **4** | **1** | **11** | **27** | **36** | **45** | **54** | **76** | **85** | **94** | **116** | **125** | **134** | **156** | **165** | **174** |
| **69** | **2** | **4** | **1** | **13** | **21** | **32** | **43** | **54** | **78** | **89** | **100** | **111** | **122** | **133** | **157** | **168** | **179** |
| **70** | **3** | **4** | **1** | **5** | **19** | **35** | **51** | **54** | **70** | **86** | **102** | **118** | **121** | **137** | **153** | **169** | **172** |
| **71** | **4** | **4** | **1** | **14** | **18** | **30** | **42** | **54** | **79** | **91** | **103** | **115** | **127** | **139** | **151** | **163** | **175** |
| **72** | **5** | **4** | **1** | **9** | **20** | **40** | **47** | **54** | **74** | **81** | **101** | **108** | **128** | **135** | **155** | **162** | **182** |
| **73** | **6** | **4** | **1** | **10** | **17** | **38** | **46** | **54** | **75** | **83** | **104** | **112** | **120** | **141** | **149** | **170** | **178** |
| **74** | **7** | **4** | **1** | **7** | **26** | **31** | **49** | **54** | **72** | **90** | **95** | **113** | **131** | **136** | **154** | **159** | **177** |

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| 75 | 8 | 4 | 1 | 8 | 23 | 29 | 48 | 54 | 73 | 92 | 98 | 117 | 123 | 142 | 148 | 167 | 173 |
| 76 | 9 | 4 | 1 | 3 | 25 | 39 | 53 | 54 | 68 | 82 | 96 | 110 | 124 | 138 | 152 | 166 | 180 |
| 77 | 10 | 4 | 1 | 12 | 24 | 34 | 44 | 54 | 77 | 87 | 97 | 107 | 130 | 140 | 150 | 160 | 183 |
| 78 | 11 | 4 | 1 | 4 | 22 | 37 | 52 | 54 | 69 | 84 | 99 | 114 | 129 | 144 | 146 | 161 | 176 |
| 79 | 12 | 4 | 1 | 6 | 16 | 33 | 50 | 54 | 71 | 88 | 105 | 109 | 126 | 143 | 147 | 164 | 181 |
| 80 | 0 | 5 | 1 | 1 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 |
| 81 | 1 | 5 | 1 | 10 | 27 | 35 | 43 | 64 | 72 | 80 | 101 | 109 | 130 | 138 | 146 | 167 | 175 |
| 82 | 2 | 5 | 1 | 6 | 21 | 38 | 42 | 59 | 76 | 80 | 97 | 114 | 131 | 135 | 152 | 169 | 173 |
| 83 | 3 | 5 | 1 | 9 | 19 | 39 | 46 | 66 | 73 | 80 | 100 | 107 | 127 | 134 | 154 | 161 | 181 |
| 84 | 4 | 5 | 1 | 4 | 18 | 33 | 48 | 63 | 78 | 80 | 95 | 110 | 125 | 140 | 155 | 170 | 172 |
| 85 | 5 | 5 | 1 | 14 | 20 | 32 | 44 | 56 | 68 | 80 | 105 | 117 | 129 | 141 | 153 | 165 | 177 |
| 86 | 6 | 5 | 1 | 12 | 17 | 40 | 50 | 60 | 70 | 80 | 103 | 113 | 123 | 133 | 156 | 166 | 176 |
| 87 | 7 | 5 | 1 | 5 | 26 | 29 | 45 | 61 | 77 | 80 | 96 | 112 | 128 | 144 | 147 | 163 | 179 |
| 88 | 8 | 5 | 1 | 3 | 23 | 37 | 51 | 65 | 79 | 80 | 94 | 108 | 122 | 136 | 150 | 164 | 178 |
| 89 | 9 | 5 | 1 | 13 | 25 | 36 | 47 | 58 | 69 | 80 | 104 | 115 | 126 | 137 | 148 | 159 | 183 |
| 90 | 10 | 5 | 1 | 8 | 24 | 30 | 49 | 55 | 74 | 80 | 99 | 118 | 124 | 143 | 149 | 168 | 174 |
| 91 | 11 | 5 | 1 | 11 | 22 | 31 | 53 | 62 | 71 | 80 | 102 | 111 | 120 | 142 | 151 | 160 | 182 |
| 92 | 12 | 5 | 1 | 7 | 16 | 34 | 52 | 57 | 75 | 80 | 98 | 116 | 121 | 139 | 157 | 162 | 180 |
| 93 | 0 | 6 | 1 | 1 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 |
| 94 | 1 | 6 | 1 | 9 | 27 | 34 | 41 | 61 | 68 | 88 | 95 | 115 | 122 | 142 | 149 | 169 | 176 |
| 95 | 2 | 6 | 1 | 12 | 21 | 31 | 41 | 64 | 74 | 84 | 94 | 117 | 127 | 137 | 147 | 170 | 180 |
| 96 | 3 | 6 | 1 | 13 | 19 | 30 | 41 | 65 | 76 | 87 | 98 | 109 | 120 | 144 | 155 | 166 | 177 |
| 97 | 4 | 6 | 1 | 7 | 18 | 36 | 41 | 59 | 77 | 82 | 100 | 118 | 123 | 141 | 146 | 164 | 182 |
| 98 | 5 | 6 | 1 | 6 | 20 | 37 | 41 | 58 | 75 | 92 | 96 | 113 | 130 | 134 | 151 | 168 | 172 |
| 99 | 6 | 6 | 1 | 14 | 17 | 29 | 41 | 66 | 78 | 90 | 102 | 114 | 126 | 138 | 150 | 162 | 174 |
| 100 | 7 | 6 | 1 | 3 | 26 | 40 | 41 | 55 | 69 | 83 | 97 | 111 | 125 | 139 | 153 | 167 | 181 |
| 101 | 8 | 6 | 1 | 11 | 23 | 32 | 41 | 63 | 72 | 81 | 103 | 112 | 121 | 143 | 152 | 161 | 183 |
| 102 | 9 | 6 | 1 | 10 | 25 | 33 | 41 | 62 | 70 | 91 | 99 | 107 | 128 | 136 | 157 | 165 | 173 |
| 103 | 10 | 6 | 1 | 4 | 24 | 39 | 41 | 56 | 71 | 86 | 101 | 116 | 131 | 133 | 148 | 163 | 178 |
| 104 | 11 | 6 | 1 | 5 | 22 | 38 | 41 | 57 | 73 | 89 | 105 | 108 | 124 | 140 | 156 | 159 | 175 |
| 105 | 12 | 6 | 1 | 8 | 16 | 35 | 41 | 60 | 79 | 85 | 104 | 110 | 129 | 135 | 154 | 160 | 179 |
| 106 | 0 | 7 | 1 | 1 | 158 | 159 | 160 | 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170 |
| 107 | 1 | 7 | 1 | 8 | 27 | 33 | 52 | 58 | 77 | 83 | 102 | 108 | 127 | 133 | 152 | 158 | 177 |
| 108 | 2 | 7 | 1 | 5 | 21 | 37 | 53 | 56 | 72 | 88 | 104 | 107 | 123 | 139 | 155 | 158 | 174 |
| 109 | 3 | 7 | 1 | 4 | 19 | 34 | 49 | 64 | 79 | 81 | 96 | 111 | 126 | 141 | 156 | 158 | 173 |
| 110 | 4 | 7 | 1 | 10 | 18 | 39 | 47 | 55 | 76 | 84 | 105 | 113 | 121 | 142 | 150 | 158 | 179 |
| 111 | 5 | 7 | 1 | 11 | 20 | 29 | 51 | 60 | 69 | 91 | 100 | 109 | 131 | 140 | 149 | 158 | 180 |
| 112 | 6 | 7 | 1 | 3 | 17 | 31 | 45 | 59 | 73 | 87 | 101 | 115 | 129 | 143 | 157 | 158 | 172 |
| 113 | 7 | 7 | 1 | 14 | 26 | 38 | 50 | 62 | 74 | 86 | 98 | 110 | 122 | 134 | 146 | 158 | 183 |

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| **114** | **8** | **7** | **1** | **6** | **23** | **40** | **44** | **61** | **78** | **82** | **99** | **116** | **120** | **137** | **154** | **158** | **175** |
| **115** | **9** | **7** | **1** | **7** | **25** | **30** | **48** | **66** | **71** | **89** | **94** | **112** | **130** | **135** | **153** | **158** | **176** |
| **116** | **10** | **7** | **1** | **13** | **24** | **35** | **46** | **57** | **68** | **92** | **103** | **114** | **125** | **136** | **147** | **158** | **182** |
| **117** | **11** | **7** | **1** | **12** | **22** | **32** | **42** | **65** | **75** | **85** | **95** | **118** | **128** | **138** | **148** | **158** | **181** |
| **118** | **12** | **7** | **1** | **9** | **16** | **36** | **43** | **63** | **70** | **90** | **97** | **117** | **124** | **144** | **151** | **158** | **178** |
| **119** | **0** | **8** | **1** | **1** | **119** | **120** | **121** | **122** | **123** | **124** | **125** | **126** | **127** | **128** | **129** | **130** | **131** |
| **120** | **1** | **8** | **1** | **7** | **27** | **32** | **50** | **55** | **73** | **91** | **96** | **114** | **119** | **137** | **155** | **160** | **178** |
| **121** | **2** | **8** | **1** | **11** | **21** | **30** | **52** | **61** | **70** | **92** | **101** | **110** | **119** | **141** | **150** | **159** | **181** |
| **122** | **3** | **8** | **1** | **8** | **19** | **38** | **44** | **63** | **69** | **88** | **94** | **113** | **119** | **138** | **157** | **163** | **182** |
| **123** | **4** | **8** | **1** | **13** | **18** | **29** | **53** | **64** | **75** | **86** | **97** | **108** | **119** | **143** | **154** | **165** | **176** |
| **124** | **5** | **8** | **1** | **3** | **20** | **34** | **48** | **62** | **76** | **90** | **104** | **118** | **119** | **133** | **147** | **161** | **175** |
| **125** | **6** | **8** | **1** | **5** | **17** | **33** | **49** | **65** | **68** | **84** | **100** | **116** | **119** | **135** | **151** | **167** | **183** |
| **126** | **7** | **8** | **1** | **12** | **26** | **36** | **46** | **56** | **79** | **89** | **99** | **109** | **119** | **142** | **152** | **162** | **172** |
| **127** | **8** | **8** | **1** | **14** | **23** | **35** | **47** | **59** | **71** | **83** | **95** | **107** | **119** | **144** | **156** | **168** | **180** |
| **128** | **9** | **8** | **1** | **4** | **25** | **40** | **42** | **57** | **72** | **87** | **102** | **117** | **119** | **134** | **149** | **164** | **179** |
| **129** | **10** | **8** | **1** | **9** | **24** | **31** | **51** | **58** | **78** | **85** | **105** | **112** | **119** | **139** | **146** | **166** | **173** |
| **130** | **11** | **8** | **1** | **6** | **22** | **39** | **43** | **60** | **77** | **81** | **98** | **115** | **119** | **136** | **153** | **170** | **174** |
| **131** | **12** | **8** | **1** | **10** | **16** | **37** | **45** | **66** | **74** | **82** | **103** | **111** | **119** | **140** | **148** | **169** | **177** |
| **132** | **0** | **9** | **1** | **1** | **145** | **146** | **147** | **148** | **149** | **150** | **151** | **152** | **153** | **154** | **155** | **156** | **157** |
| **133** | **1** | **9** | **1** | **6** | **27** | **31** | **48** | **65** | **69** | **86** | **103** | **107** | **124** | **141** | **145** | **162** | **179** |
| **134** | **2** | **9** | **1** | **4** | **21** | **36** | **51** | **66** | **68** | **83** | **98** | **113** | **128** | **143** | **145** | **160** | **175** |
| **135** | **3** | **9** | **1** | **12** | **19** | **29** | **52** | **62** | **72** | **82** | **105** | **115** | **125** | **135** | **145** | **168** | **178** |
| **136** | **4** | **9** | **1** | **3** | **18** | **32** | **46** | **60** | **74** | **88** | **102** | **116** | **130** | **144** | **145** | **159** | **173** |
| **137** | **5** | **9** | **1** | **8** | **20** | **39** | **45** | **64** | **70** | **89** | **95** | **114** | **120** | **139** | **145** | **164** | **183** |
| **138** | **6** | **9** | **1** | **7** | **17** | **35** | **53** | **58** | **76** | **81** | **99** | **117** | **122** | **140** | **145** | **163** | **181** |
| **139** | **7** | **9** | **1** | **10** | **26** | **34** | **42** | **63** | **71** | **92** | **100** | **108** | **129** | **137** | **145** | **166** | **174** |
| **140** | **8** | **9** | **1** | **9** | **23** | **30** | **50** | **57** | **77** | **84** | **104** | **111** | **131** | **138** | **145** | **165** | **172** |
| **141** | **9** | **9** | **1** | **14** | **25** | **37** | **49** | **61** | **73** | **85** | **97** | **109** | **121** | **133** | **145** | **170** | **182** |
| **142** | **10** | **9** | **1** | **5** | **24** | **40** | **43** | **59** | **75** | **91** | **94** | **110** | **126** | **142** | **145** | **161** | **177** |
| **143** | **11** | **9** | **1** | **13** | **22** | **33** | **44** | **55** | **79** | **90** | **101** | **112** | **123** | **134** | **145** | **169** | **180** |
| **144** | **12** | **9** | **1** | **11** | **16** | **38** | **47** | **56** | **78** | **87** | **96** | **118** | **127** | **136** | **145** | **167** | **176** |
| **145** | **0** | **10** | **1** | **1** | **132** | **133** | **134** | **135** | **136** | **137** | **138** | **139** | **140** | **141** | **142** | **143** | **144** |
| **146** | **1** | **10** | **1** | **5** | **27** | **30** | **46** | **62** | **78** | **81** | **97** | **113** | **129** | **132** | **148** | **164** | **180** |
| **147** | **2** | **10** | **1** | **10** | **21** | **29** | **50** | **58** | **79** | **87** | **95** | **116** | **124** | **132** | **153** | **161** | **182** |
| **148** | **3** | **10** | **1** | **3** | **19** | **33** | **47** | **61** | **75** | **89** | **103** | **117** | **131** | **132** | **146** | **160** | **174** |
| **149** | **4** | **10** | **1** | **6** | **18** | **35** | **52** | **56** | **73** | **90** | **94** | **111** | **128** | **132** | **149** | **166** | **183** |
| **150** | **5** | **10** | **1** | **13** | **20** | **31** | **42** | **66** | **77** | **88** | **99** | **110** | **121** | **132** | **156** | **167** | **178** |
| **151** | **6** | **10** | **1** | **9** | **17** | **37** | **44** | **64** | **71** | **91** | **98** | **118** | **125** | **132** | **152** | **159** | **179** |
| **152** | **7** | **10** | **1** | **8** | **26** | **32** | **51** | **57** | **76** | **82** | **101** | **107** | **126** | **132** | **151** | **170** | **176** |

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| 153 | 8 | 10 | 1 | 4 | 23 | 38 | 53 | 55 | 70 | 85 | 100 | 115 | 130 | 132 | 147 | 162 | 177 |
| 154 | 9 | 10 | 1 | 11 | 25 | 34 | 43 | 65 | 74 | 83 | 105 | 114 | 123 | 132 | 154 | 163 | 172 |
| 155 | 10 | 10 | 1 | 14 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | 157 | 169 | 181 |
| 156 | 11 | 10 | 1 | 7 | 22 | 40 | 45 | 63 | 68 | 86 | 104 | 109 | 127 | 132 | 150 | 168 | 173 |
| 157 | 12 | 10 | 1 | 12 | 16 | 39 | 49 | 59 | 69 | 92 | 102 | 112 | 122 | 132 | 155 | 165 | 175 |
| 158 | 0 | 11 | 1 | 1 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 |
| 159 | 1 | 11 | 1 | 4 | 27 | 29 | 44 | 59 | 74 | 89 | 104 | 106 | 121 | 136 | 151 | 166 | 181 |
| 160 | 2 | 11 | 1 | 3 | 21 | 35 | 49 | 63 | 77 | 91 | 105 | 106 | 120 | 134 | 148 | 162 | 176 |
| 161 | 3 | 11 | 1 | 7 | 19 | 37 | 42 | 60 | 78 | 83 | 101 | 106 | 124 | 142 | 147 | 165 | 183 |
| 162 | 4 | 11 | 1 | 9 | 18 | 38 | 45 | 65 | 72 | 92 | 99 | 106 | 126 | 133 | 153 | 160 | 180 |
| 163 | 5 | 11 | 1 | 5 | 20 | 36 | 52 | 55 | 71 | 87 | 103 | 106 | 122 | 138 | 154 | 170 | 173 |
| 164 | 6 | 11 | 1 | 11 | 17 | 39 | 48 | 57 | 79 | 88 | 97 | 106 | 128 | 137 | 146 | 168 | 177 |
| 165 | 7 | 11 | 1 | 6 | 26 | 30 | 47 | 64 | 68 | 85 | 102 | 106 | 123 | 140 | 157 | 161 | 178 |
| 166 | 8 | 11 | 1 | 12 | 23 | 33 | 43 | 66 | 76 | 86 | 96 | 106 | 129 | 139 | 149 | 159 | 182 |
| 167 | 9 | 11 | 1 | 8 | 25 | 31 | 50 | 56 | 75 | 81 | 100 | 106 | 125 | 144 | 150 | 169 | 175 |
| 168 | 10 | 11 | 1 | 10 | 24 | 32 | 53 | 61 | 69 | 90 | 98 | 106 | 127 | 135 | 156 | 164 | 172 |
| 169 | 11 | 11 | 1 | 14 | 22 | 34 | 46 | 58 | 70 | 82 | 94 | 106 | 131 | 143 | 155 | 167 | 179 |
| 170 | 12 | 11 | 1 | 13 | 16 | 40 | 51 | 62 | 73 | 84 | 95 | 106 | 130 | 141 | 152 | 163 | 174 |
| 171 | 0 | 12 | 1 | 1 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 172 | 1 | 12 | 1 | 3 | 27 | 28 | 42 | 56 | 70 | 84 | 98 | 112 | 126 | 140 | 154 | 168 | 182 |
| 173 | 2 | 12 | 1 | 9 | 21 | 28 | 48 | 55 | 75 | 82 | 102 | 109 | 129 | 136 | 156 | 163 | 183 |
| 174 | 3 | 12 | 1 | 11 | 19 | 28 | 50 | 59 | 68 | 90 | 99 | 108 | 130 | 139 | 148 | 170 | 179 |
| 175 | 4 | 12 | 1 | 12 | 18 | 28 | 51 | 61 | 71 | 81 | 104 | 114 | 124 | 134 | 157 | 167 | 177 |
| 176 | 5 | 12 | 1 | 10 | 20 | 28 | 49 | 57 | 78 | 86 | 94 | 115 | 123 | 144 | 152 | 160 | 181 |
| 177 | 6 | 12 | 1 | 13 | 17 | 28 | 52 | 63 | 74 | 85 | 96 | 107 | 131 | 142 | 153 | 164 | 175 |
| 178 | 7 | 12 | 1 | 4 | 26 | 28 | 43 | 58 | 73 | 88 | 103 | 118 | 120 | 135 | 150 | 165 | 180 |
| 179 | 8 | 12 | 1 | 7 | 23 | 28 | 46 | 64 | 69 | 87 | 105 | 110 | 128 | 133 | 151 | 169 | 174 |
| 180 | 9 | 12 | 1 | 5 | 25 | 28 | 44 | 60 | 76 | 92 | 95 | 111 | 127 | 143 | 146 | 162 | 178 |
| 181 | 10 | 12 | 1 | 6 | 24 | 28 | 45 | 62 | 79 | 83 | 100 | 117 | 121 | 138 | 155 | 159 | 176 |
| 182 | 11 | 12 | 1 | 8 | 22 | 28 | 47 | 66 | 72 | 91 | 97 | 116 | 122 | 141 | 147 | 166 | 172 |
| 183 | 12 | 12 | 1 | 14 | 16 | 28 | 53 | 65 | 77 | 89 | 101 | 113 | 125 | 137 | 149 | 161 | 173 |