On Π – Pure Ideals

Shaimaa Hatem Ahmad
shaima.hatem1977@gmail.com
Mathematics Department
College of Computer Science and Mathematics
University of Mosul, Mosul, Iraq

Received on: 20/10/2013 **Accepted on: 12/2/2014**

ABSTRACT

As a generalization of right pure ideals, we introduce the notion of right Π – pure ideals. A right ideal I of R is said to be Π – pure, if for every $a \in I$ there exists $b \in I$ and a positive integer n such that $a^n \neq 0$ and $a^n b = a^n$. In this paper, we give some characterizations and properties of Π – pure ideals and it is proved that:

If every principal right ideal of a ring R is Π – pure then,

a). $L(a^n) = L(a^{n+1})$ for every $a \in R$ and for some positive integer n .

b). R is directly finite ring.

c). R is strongly Π – regular ring.

Keywords: Pure, strongly regular, Π – ring.

حول المثاليات النقية من النمط – Π

شيماء حاتم أحمد

كلية علوم الحاسوب والرياضيات، جامعة الموصل

تاريخ استلام البحث: 2013/10/20 **تاريخ قبول البحث: 2014/02/12**

الملخص

كتعميم للمثاليات النقية اليمنى أعطينا مفهوم المثالي النقي الأيمن من النمط – Π وهو أن كل مثالي أيمن I في R يكون نقياً من النمط – Π , إذا كان لكل عنصر a في I يوجد عنصر b في I ولكل عدد موجب n بحيث أن $a^n \neq 0$ فان $a^n b = a^n$. في هذا البحث قدمت تمييزا لبعض الخواص للمثاليات من النمط –Π .

وبرهنا ما يلي :اذا كان كل مثالي ايمن خصوصي في الحلقةR من النمط –Π فان

a) $L(a^n) = L(a^{n+1})$ لكل $a \in R$ ولبعض عدد صحيح n .

b) R حلقة منتهية الاتجاه .

c) R حلقة منتظمة بقوة من النمط –Π .

الكلمات المفتاحية: نقي، منتظمة بقوة، حلقات من النمط – Π.

1. Introduction

Throughout this paper , a ring R denotes as associative ring with identity and all modules are unitary . We write J (R) for Jacobson radical of R . L (x) (Y (x)) denotes the left (right) annihilator of x in R .

Recall the following definitions and facts :

1- A ring R is called Π – regular [3] , if for any $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^n b a^n$. A ring R is called strongly Π – regular if for any $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^{2n} b$.

2- A ring R is called a quasi ZI – ring [8], if for any non-zero elements a , b $\in$ R, ab = 0 implies that there exists a positive integer n such that $a^n \neq 0$ and $a^n R b^n = 0$.
3- A ring R is said to be reduced [9] , if it contains no non-zero nilpotent element .

A ring R is called right SXM if for each $0 \neq a \in R$, r (a) = r (a^n) for a positive integer n satisfying $a^n \neq 0$. For example , reduced rings are right SXM rings [7] .

Pure ideals have been extensively studied for several years. Many authors studied some properties and connections between pure ideals and regular rings [2] , [4],and [5] .

2. Π – Pure Ideals

In this section ,we introduce the notion of a right Π – pure ideals , with some of their basic properties . Also, we give a connection between Π – pure ideals and pure ideals .

Following [1],an ideal I of a ring R is said to be right(left) pure ideal, if for any a $\in$ I , there exists b $\in$ I such that a = ab . (a = ba) .

Following [6] , an ideal of a ring R is said to be G P-ideal, if for every a $\in$ I , there exists b $\in$ I and a positive integer n such that $a^n = a^n b$.

Definition (2.1) :

An ideal I of a ring R is said to be right Π – pure ideal if for every a $\in$ I , there exists a positive integer n and b $\in$ I , such that $a^n \neq 0$ and $a^n = a^n b$.

Clearly , every right pure ideal is a right Π – pure ideal but the converse is not true .

Example (1) :

Let Z_{12} be the ring of integers modulo 12 and I = (3) , J = (4) . Then, both I and J are Π – pure ideals of Z_{12} .Obviously, Π – pure ideal implies GP- ideal .

It is clear that in the case of reduced rings , GP – ideals coincide. with Π – pure .

Example (2) :

Let Z_9 be the ring of integers modulo 9 and the (3) is not Π–pure , but GP – ideal .

We now consider a necessary and sufficient condition for Π–pure to be pure ideal .

Proposition (2.2) :

Let R be right SXM ring . Then, every Π – pure ideal is pure ideal.

Proof : Let I be a right Π – pure ideal , and let a $\in$ I . Then, there exists b $\in$ I and a positive integer n such that $a^n \neq 0$, and $a^n = a^n$ b, this implies that (1 – b) $\in$ r (a^n) = r (a) . (R is right SXM) . Therefore, (1 – b) $\in$ r (a) and a = ab . So, I is pure ideal.

Proposition (2.3) :

Let R be a ring with every principal ideal is Π – pure ideal. Then ,

1- Every non - zero divisor element of R is invertible .
2- J (R) is a nil ideal .

Proof: It proved the same method as [5 , Proposition. 3 . 2 . 6] .

3. The Connection Between Π–Pure Ideals and Other Rings

In this section, we study the connection between rings whose every principal ideal is Π – pure and strongly Π – regular rings and other rings .

Proposition (3.1) :

Let R be a ring such that every principal left ideal is right Π – pure . Then, L (a^n) = L (a^{n+1}) for every a ∈ R and for some a positive integer n .

Proof :

Let a ∈ I . Then, there exists b ∈ I , and a positive integer n , such that $a^n \neq 0$ and $a^n = a^n b$ where, b = ax for some x ∈ R .

Therefore $a^n = a^{n+1} x$. Let y ∈ L (a^{n+1}) , y a^{n+1} = 0 , then y $(a^{n+1} x)$ = 0 So ya^n = 0 and y ∈ L (a^n) . Therefore, L $(a^{n+1}) \subseteq$ L (a^n). Clearly

L $(a^n) \subseteq$ L (a^{n+1}) . So, L (a^n) = L (a^{n+1}) .

Following [3] , a ring R is called directly finite if ab = 1 implies ba = 1 for all a , b ∈ R .

As a parallel result to [3 , Proposition 2.1.13],the following result was obtained.

Proposition (3.2) :

Let R be a ring with every principal right ideal is Π – pure . Then, R is directly finite .

Proof :

Let x , y ∈ R such that xy = 1 . It is clear that $x^n y^n = 1$ and $x^{n+1} y^{n+1} = 1$ multiple by y^{n+1} . So $y^{n+1} x^{n+1} y^{n+1} = y^{n+1}$,and $(1 - y^{n+1} x^{n+1}) \in L(y^{n+1}) = L(y^n)$ (Proposition 3.1).Hence, $y^n = y^{n+1} x^{n+1} y^n = (y^{n+1} x)(x^n y^n) = y^{n+1} x$

Now, $yx = (x^n y^n) yx = x^n (y^{n+1} x) = x^n y^n = 1$.

Theorem (3.3) :

Let R be a ring with every principal right ideal is right Π – pure . Then, R is strongly Π – regular .

Proof :

For any a ∈ R , aR is Π – pure . Since a ∈ aR. There exists b∈R and a positive integer n such that $a^n \neq 0$,

and $a^n = a^{n+1} x$ for some x in R

$= a^{n+1} x a^{n+2} x^2 = ... a^{2n} x^n = a^{2n} y$

Therefore, R is Strongly Π – regular .

Aright R – modulo M is said to be YJ – injective [9] , if for any $0 \neq a \in R$, there exists appositive integer n such that $a^n \neq 0$ and any right R – homomorphism . From a^n R into M , extends to one from R into M .

A ring R is called a right YJ – injective ring , if R is YJ – injective ring .

Proposition (3.4) : [8]

Let R be a quasi ZI ring . If every simple singular right R – modulo is YJ – injective . Then ,

1- R is reduced .
2- I + r (a) = R for any non - zero ideal I of R and every a ∈ I .

Theorem (3.5) :

Let R be a quasi ZI – ring . If every simple singular right R – modulo is YJ – injective . Then, every ideal of R is right Π – pure .

Proof :

From Proposition (3.4) , I + r (a) = R for every non - zero ideal I of R , and a ∈ R So, b + d = 1 , b ∈ I , d ∈ r (a) , ab + ad = a . Therefore, ab = a , and $a^n b = a^n$ for some positive integer n and $a^n \neq 0$. So, I is Π – pure .

REFERENCES

[1] Al- Ezeh , H ., (1989) , "Pure ideals in commutative reduced Gelfand rings with Unitry" , Arch . Math ., Vol. 53 , 266-269.

[2] Al- Ezeh , H ., (1998) , "The pure spectrum PF - rings" comm. Math Univ . s . p . vol . 37 , No . 2 , 18-29 .

[3] Al- Kouri , M . R ., (2000) , "On strongly Π– regular Rings", ph . D . thesis Mosul of University .

[4] Field house D , J ., (1969) , "Pure theories" , Math . Ann . 184 , 1-18 .

[5] Mahmood , R . D ., (2000) , "On pure Ideals and pure Submodules" , ph . D . thesis Mosul of University .

[6] Shuker , N . H . and Mahmood , R . D . (2000) , "On generalization of pure ideals" J . Edu and Sci . Vol . 48 , 86-90 .

[7] Wei , J . c . (2007) , "On Simple Singular YJ - injective modules" Southeast Asian Bu . of Math . Vol . 31 , 1009-1018 .

[8] Yin , X . B ., Wang , R . and Long , X ., (2011) , "The non Singularity and regularity of GP - V - rings" , J . of Math . B . and E . Vol . 31 , No . 6 , 1004-1008 .

[9] Yue chi Ming , R . (1997) "Anote of YJ - injectivity" , Dem . Math . xxx (3) 31-37 .