A Comparative Study of Wavelets Methods for Solving Non-Linear Two-Dimensional Boussinesq System of Type BBM-BBM

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ABSTRACT

In this paper, numerical techniques based on the wavelets methods are proposed for the numerical solution of non-linear two-dimensional BBM-BBM system and we compared between them. Two methods used in numerical solutions, are the Haar wavelets and Legendre wavelets methods.In addition, we derived formulas of integrals for Legendre wavelets analytically. Its efficiency is tested by solving an example for which the exact solution is known. The accuracy of the numerical solutions is quite high even if the number of calculation points is small, by increasing the number of collocation points, the error of the solution rapidly decreases. We have found that the Legendre wavelets method is better and closer to the exact solution than the Haar wavelets method.

Keywords:*Boussinesq systems, BBM-BBM system, Haar wavelets, Legendre wavelets***.**

دراسة مقارنة طرائق المويجات لحل نظام Boussinesq اللاخطي ببعدين من نوع BBM-BBM

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جامعة الموصل

الملخص

في هذا البحث, تم تطبيق طرائق المويجات في الحل العددي لنظام BBM-BBM غير الخطي ببعدين ومقارنة النتائج حيث تم استخدام طريقة مويجاتHaar وطريقة مويجاتLegendre في الحل العددي. بالإضافة إلى ذلك قمنا باشتقاق صيغة التكاملات لمويجاتLegendre تحليليا. تأثير الحل للطريقتين اختبر بواسطة حل مثال ومقارنته مع الحل المضبوط. إن دقة الحلول العددية عالية وإن كانت عدد نقاط الشبكة المحسوبة صغيرة وكلما زادت عدد نقاط الشبكة المحسوبة فان الخطأ يتناقص. لقد تبين لنا أيضا أن طريقة مويجاتLegendre أفضل واقرب إلى الحل المضبوط من طريقة مويجاتHaar.

الكلمات المفتاحية : نظام *Boussinesq* ، نظام BBM-BBM ، مويجاتHaar، مويجاتLegendre

1.Introduction

Boussinesq developed the original formulation of the governing equations for a free surface flow, which included the effects of surface waves, but in which the vertical dimension was eliminated. The formulation was in terms of the bottom velocity and was restricted to simulating waves moving over bathymetry with a flat bottom. The governing equations consist of one continuity equation and two momentum equations (in x and y directions). The governing equations were then called as Boussinesq equations [13].

Hmidi and Keraani [10] are proved the global well-posedness of the two-dimensional Boussinesq system with zero diffusivity for rough initial data. Ataie.andNajafi [3] are studied a higher-order two-dimensional Boussinesq wave model and they used the finite difference method in higher-order scheme for time and space in derived equations. Chen and Goubet [6] obtained the long time asymptotes of the solutions for a long class of the two-dimensional dissipative Boussinesq system which is model surface waves in three space dimensions. Also Chen [7] studied a highly efficient and accurate numerical scheme for initial and boundary value problems of a two-dimensional Boussinesqsystem. Mitsotakis [14] is derived and solved numerically by the standard Galerkin-finite element method the Boussinesq system in two space dimensions. Mera [13] focuses on the development of a set of two-dimensional boundary conditions for specific governing equations which is existing Boussinesq type equations. Sadaka [15] is using the FreeFem++ code to solve a three-parameter family of Boussinesq type systems in two space dimensions which approximate the three-dimensional Euler equations over an horizontal bottom.

Many authors have studied the solution for partial differential equations by using the Haar wavelets method.

AL-Rawi and Qasem [2] found the numerical solution for non-linear Murray equation by the operational matrices of Haar wavelet method and compared the results of this method with the exact solution, they transformed the non-linear Murray equation into a linear algebraic equations that can be solved by Gauss-Jordan method. Hariharanand Kannan [9] develop an accurate and efficient Haar transformorHaarwaveletmethodforsomeofthewell-knownnon-linear parabolic partialdifferentialequations.The equationsincludetheNowell-whiteheadequation, Cahn-Allenequation,FitzHugh-Nagumoequation, and other equations.

Lepik, U., [11] applied the two-dimensional Haar wavelets for solution of the partial differential equations. To demonstrate the efficiency of the method, two test problems are discussed. Celik, I. [5] studied an efficient numerical method for solution of non-linear generalized Burgers-Huxley equation based on the Haar wavelets approach, approximate solutions are compared with exact solutions.

Liu, N. and Lin, E-B. [12] introduced an orthogonal basis on the square [-1,-1]x[-1,1] generated by Legendre polynomials on [-1,1], and defined an associated expression for the expansion of a Riemann integrable function. They described some properties and derived a uniform convergence theorem. Abbas, Z. et al. [1] used the continuous Legendre multi-wavelets on the interval [0,1] to solve Fredholm integral equations of the second kind. To do so, they reduced the solution of Fredholm integral equation to the algebraic equations.

In this paper, we study a comparision between Haar wavelets and Legendre wavelets for non-linear two-dimensional BBM-BBM system.

We organized our paper as follows. In section 2, the Haarwavelet is introduced and an operational matrix and function approximation is presented. Section 3 Legendre wavelet approximation is presented. Section 4 we explain the 2D BBM-BBM system and we use Haar wavelets to solve this system. Section 5 numerical solution by

Legendre wavelets is presented. Section 6 numerical results are presented. Concluding remarks are given in section 7.

2. Haarwavelets

As a powerful mathematical tool, Wavelet analysis has been widely used in image digital processing, quantum field theory, numerical analysis and many other fields in recent years.

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1) by [9]:

$$h_i(x)=\begin{cases}1 & \frac{k}{m}\le x<\frac{k+1/2}{m}\\ -1 & \frac{k+1/2}{m}\le x<\frac{k+1}{m}\\ 0 & otherwise \quad in \quad [0,1)\end{cases} \qquad \ldots(1)$$

The integer $m=2^j$, $j=0,1,2,....,J$ indicates the level of the wavelet; k=0,1,2,…,m-1 is the translation parameter. Maximal level of resolution is J. The indix i is calculated according to the formula i=m+k+1; in the case of minimal values. m=1,k=0 we have i=2, the maximal value of i is $i=2M=2^{j+1}$. It is assumed that the value i=1 corresponds to the scaling function for which $h_1(x)=1 \quad in\,[0,1]$. Let us define the collocation points $x_l=(l-0.5)/2M,\quad (l=1,2,...,2M)$ and discredits the Haarfunction $h_i(x)$; in this way, we get the coefficient matrix $H(i,l)=(h_i(x_l))$, which has the dimension 2M*2M.

The operational matrix of integration P, which is a 2M square matrix, is defined by the equation: [11]

$$P_{i,1}(x)=\int_0^{x_l} h_i(x)\,dx \qquad \ldots(2)$$

$$P_{i,v+1}(x)=\int_0^{x} P_{i,v}(x)\,dx \quad , \qquad v=1,2,... \qquad \ldots(3)$$

Lipek, U. found the general form of v-times of integrals [11]:

$$P_{i,v}(x)=\begin{cases}0 & for\ x<\xi_1(i)\\ \frac{1}{v!}[x-\xi_1(i)]^v & for\ x\in[\xi_1(i),\xi_2(i)]\\ \frac{1}{v!}\{[x-\xi_1(i)]^v-2[x-\xi_2(i)]^v\} & for\ x\in[\xi_2(i),\xi_3(i)]\\ \frac{1}{v!}\{[x-\xi_1(i)]^v-2[x-\xi_2(i)]^v+[x-\xi_3(i)]^v\} & for\ x>\xi_3(i)\end{cases} \qquad \ldots(4)$$

such that $\xi_1(i)=\frac{k}{m}$, $\xi_2(i)=\frac{k+1/2}{m}$, $\xi_3(i)=\frac{k+1}{m}$.

Integrate equation (3) from (0) to (1), we get the following notation:

$$R_{i,v}=\int_0^1 P_{i,v}(x)\,dx=P_{i,v+1}(1) \qquad \ldots(5)$$

$$R_{i,1}(x) = P_{i,2}(1) = \begin{cases} \frac{1}{2}(1-\alpha)^2 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - 1)^2 & \text{for } x \in [\beta, \gamma) \\ \frac{1}{4m^2} & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(6)$$

Any square integrable function u(x) in the interval [0,1) can be expanded by a Haar series of infinite terms [9]:

$$u(x) = \sum_{i=0}^{\infty} c_i \, h_i(x) \qquad i \in \{0\} \cup N \qquad \ldots(7)$$

Where, the Haar coefficients c_i are determined as follows:

$$c_0 = \int_0^1 u(x) h_0(x) dx \quad , \quad c_n = 2^j \int_0^1 u(x) h_i(x) dx \qquad \ldots(8)$$

$$i = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad x \in [0,1)$$

Usually, the series expansion of (7) contains infinite terms for a general smooth function u(x). However, If u(x) is approximated as piecewise constant during each subinterval, then u(x) will be terminated at finite m terms, that is:

$$u(x) = \sum_{i=0}^{m-1} c_i \, h_i(x) = c_{(m)}^T \, h_{(m)}(x)$$

Where, the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as:

$c_{(m)}^T = [c_0, \quad c_1, \quad \ldots., \quad c_{m-1}]$ And $h_{(m)}(x) = [h_0(x), h_1(x), \ldots\ldots, h_{m-1}(x)]^T$

Where, T is referring to the transpose.

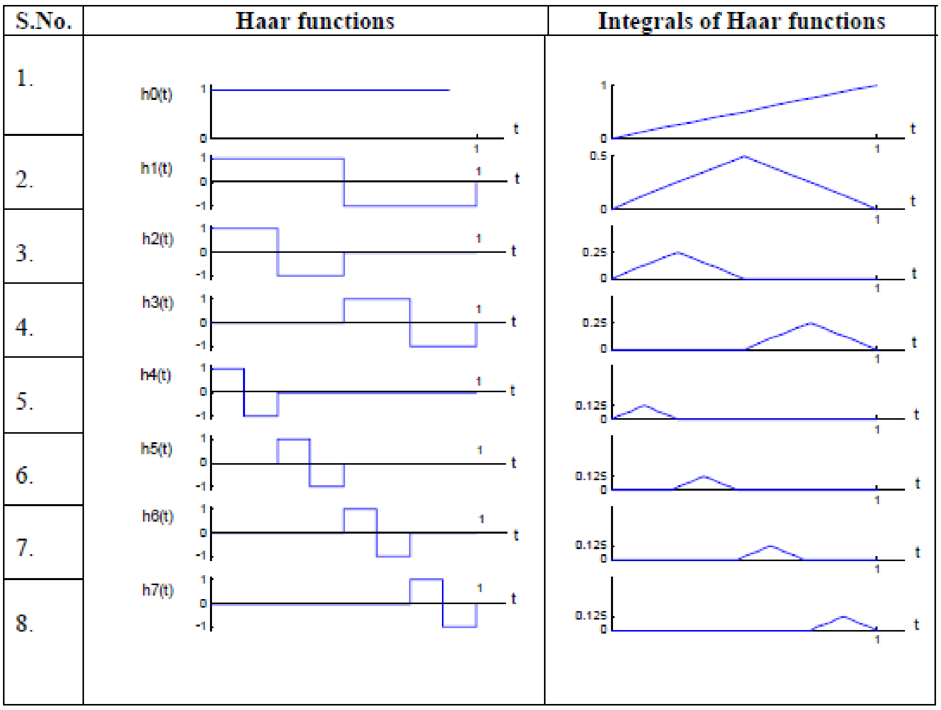

Fig.1. First eight Haar functions

Similarly, a two dimensional function $u(x, y)$ which is square integrable in the interval $0 \leq x < 1 \quad and \quad 0 \leq y < 1$ can be expanded into Haar series by [16]:

$$u(x,y)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{i,j}\,h_i(x)\,h_j(y) \qquad \ldots(9)$$

Where the coefficient matrix $C_{i,j}$ and the Haar function $h_i(x)$ *and* $h_j(y)$ are defined as:

$$C=\begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & \cdots & c_{1,m-1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & \cdots & c_{m-1,m-1} \end{bmatrix}, H=\begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & \cdots & h_{1,m-1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & \cdots & h_{m-1,m-1} \end{bmatrix}$$

Equation (9) can be written into the discrete form by:

$$u(x,y)=H^T(x)\cdot C\cdot H(y) \qquad \ldots(10)$$

Now, integrate with respect to variable (x) of $u(x,y)$ by using equations (2) and (3), we get [16]:

$$\int_0^x u(x,y)\,dx=\int_0^x H^T(x)\cdot C\cdot H(y)\,dx$$
$$=P_{i,1}^T(x)\cdot C\cdot H(y) \qquad \ldots(11)$$

also

$$\int_0^y u(x,y)\,dy=\int_0^y H^T(x)\cdot C\cdot H(y)\,dy$$
$$=H^T(x)\cdot C\cdot P_{i,1}(y) \qquad \ldots(12)$$

performing the double integration, we obtain:

$$\int_0^y\int_0^x u(x,y)\,dx\,dy=P_{i,1}^T(x)\cdot C\cdot P_{i,1}(y) \qquad \ldots(13)$$

3. Legendre Wavelets

Legendre wavelets $\psi_{n,m}(t)=\psi(k,\hat{n},m,t)$ have four arguments ; $\hat{n}=2n-1, n=1,2,3,4,\ldots,2^{k-1}$, k can assume any positive integer, m is the order for legendre polynomials and t is the normalized time. They are defined on the interval [0,1) by: [1,12]

$$\psi_{n,m}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\;2^{\frac{k}{2}}\;L_m\left(2^k t-\hat{n}\right) & \text{for} \quad \frac{\hat{n}-1}{2^k}\le t<\frac{\hat{n}+1}{2^k} \\ 0 & \text{otherwise}\end{cases} \qquad \ldots(14)$$

where m=0,1,….,M-1. in equation (1), the coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality. [1]

Here Lm(t) are the well-known legendre polynomials of the order m, which are orthogonal to the weight function w(t)=1 and satisfy the following recursive formula:

$$L_0(t)=1$$
$$L_1(t)=t$$
$$\vdots \qquad \ldots(15)$$
$$L_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)t\,L_m(t)-\left(\frac{m}{m+1}\right)L_{m-1}(t) \qquad m=1,2,3,\ldots.$$

such that the set of legendre wavelets are an orthonormal set.

Any function $f(t) \in L^2[0,1)$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\,\psi_{n,m}(t) \qquad \ldots(16)$$

where $C_{n,m} = \left(f(t), \psi_{n,m}(t)\right)$, in which (.,.) denotes the inner product.

If the infinite series in (16) are truncated, then (16) can be written as:

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\,\psi_{n,m}(t) = C^T\,\psi(t) \qquad \ldots(17)$$

where C and $\psi(t)$ are $2^{k-1}M*1$ matrices given by:

$$C = \left[c_{1,0}, c_{1,1}, \ldots\ldots, c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots\ldots, c_{2^{k-1},0}, \ldots\ldots, c_{2^{k-1},M-1}\right]^T$$

$$\psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \ldots\ldots, \psi_{1,M-1}, \psi_{2,0}, \psi_{2,1}, \ldots\ldots, \psi_{2^{k-1},0}, \ldots\ldots, \psi_{2^{k-1},M-1}\right]^T$$

similarly, any function $u(x,y) \in L^2[0,1) \times [0,1)$ may be expanded as:

$$u(x,y) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} C_{n,m,n',m'}\,\psi_{n,m}(x)\psi_{n',m'}(y) \qquad \ldots(18)$$

where

$$C_{n,m,n',m'} = \int_0^1\int_0^1 u(x,y)\psi_{n,m,n',m'}(x,y)\,dx\,dy$$

$n = 1,2,3,\ldots.,2^{k-1}$ $\quad ,n' = 1,2,3,4,\ldots,2^{k'-1}$ $m = 0,1,2,\ldots., M-1$ $\quad ,m' = 0,1,2,\ldots, M'-1$

k *and* k' are positive integers.

For convenience equation (18) can be re-written as follows: [1]

$$u(x,y) = \sum_{j=1}^{2^{k-1}M}\ \sum_{l=1}^{2^{k'-1}M'} a_{j,l}\,\psi_{2^{k-1}M,j}(x)\psi_{2^{k'-1}M',l}(y) \qquad \ldots(19)$$

Now, we derive the operational matrix of integration P, which is a ($2^{k-1}M*1$) square matrix is defined by the equation:

$$P_{i,1}(t) = \int_0^t \psi_i(t')\,dt' \qquad \ldots(20)$$

$$P_{i,v+1}(t) = \int_0^t P_{i,v}(t')\,dt' \qquad \ldots(21)$$

These integrals can be evaluated by using equation (14), we get:

$$P_{i,1}(t) = \begin{cases} 0 & 0 \le t < \frac{2n-2}{2^k} \\ \int_{\frac{2n-2}{2^k}}^{t} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}\,L_m\left(2^k t' - \hat{n}\right)dt' & \frac{2n-2}{2^k} \le t < \frac{2n}{2^k} \\ \int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}\,L_m\left(2^k t' - \hat{n}\right)dt' & \frac{2n}{2^k} \le t < 1 \end{cases} \qquad \ldots(22)$$

$$P_{i,2}(t)=\begin{cases} 0 & 0\le t<\frac{2n-2}{2^k} \\ \int\limits_{\frac{2n-2}{2^k}}^{t}\int\limits_{\frac{2n-2}{2^k}}^{t}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}\,L_m\left(2^k t'-\hat{n}\right)\left(dt'\right)^2 & \frac{2n-2}{2^k}\le t<\frac{2n}{2^k} \\ \begin{cases} \int\limits_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}}\int\limits_{\frac{2n-2}{2^k}}^{t}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}\,L_m\left(2^k t'-\hat{n}\right)\left(dt'\right)^2+ \\ +\left(t-\frac{2n}{2^k}\right)\int\limits_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}\,L_m\left(2^k t'-\hat{n}\right)dt' \end{cases} & \frac{2n}{2^k}\le t<1 \end{cases} \quad \ldots(23)$$

We also introduce the following notation:

$$D_{i,v}(t)=\int_0^1 P_{i,v}(t')\,dt' \quad \ldots(24)$$

4. Mathematical Model

We consider the non-linear two-dimensional coupled Benjamin-Bona-Mahony (BBM-BBM) system which has the form [8,15]:

$$\begin{aligned} &\eta_{t_*}+\nabla.V+\nabla.\eta V-b\,\Delta\eta_{t_*}=0 \\ &V_{t_*}+\nabla\,\eta+\frac{1}{2}\nabla\,|V|^2-d\,\Delta V_{t_*}=0 \end{aligned} \quad \ldots(25)$$

for $X_*=[x_*,y_*]\in\Omega$, $t_*>0$, *where* Ω is a bounded open set in IR^2, and b, d > 0 and the initial data:

$$\eta(X_*,0)=\eta_0(X_*)\quad,\quad V(X_*,0)=V_0(X_*)\qquad t_*\ge 0\quad,\quad X_*\in\partial\Omega \quad \ldots(26)$$

and zero Dirichlet homogenous boundary conditions [8,15]:

$$\eta(X_*,t_*)=0\quad,\quad V(X_*,t_*)=0\qquad t_*\ge 0\quad,\quad X_*\in\partial\Omega \quad \ldots(27)$$

This system is of type of Boussinesq systems derived as approximations to the three-dimensional Euler equations describing irrotational free surface flow of an ideal fluid over a horizontal bottom. The independent variables $X_*=[x_*,y_*]$ and t represent the position and elapsed time, respectively, $\eta(X_*,t_*)$ is proportional to the deviation of the free surface from its rest, while $V(X_*,t_*)=(u(X_*,t_*),v(X_*,t_*))$ is proportional to the horizontal velocity of the fluid at some height. Specifically, we have $b=d=\frac{1}{2}(\theta^2-\frac{1}{3})$, $\frac{1}{3}<\theta^2\le 1$ the so-called BBM-BBM system corresponds to $\theta^2=\frac{2}{3}$, $b=d=\frac{1}{6}$ [15].

Since, the Haar and Legendre wavelets are defined for $x \in [0,1]$, we must first normalize of the system (25) with regard to $X_* = [x_*, y_*]$ and the domain $\Omega = [a_x, b_x] \times [a_y, b_y]$, we change the variables:

$$x = \frac{1}{L_x}(x_* - a_x), \qquad t = t_* - 0, \qquad L = b_x - a_x$$

$$y = \frac{1}{L_y}(y_* - a_y), \qquad t = t_* - 0, \qquad L = b_y - a_y$$

Then, the system (25) becomes:

$$\eta_t + \frac{1}{L_x}u_x + \frac{1}{L_y}v_y + \frac{1}{L_x}\eta u_x + \frac{1}{L_x}\eta_x u + \frac{1}{L_y}\eta v_y + \frac{1}{L_y}\eta_y v - \frac{b}{L_x^2}\eta_{xxt} - \frac{b}{L_y^2}\eta_{yyt} = 0$$

$$u_t + \frac{1}{L_x}\eta_x + \frac{1}{L_x}u u_x + \frac{1}{L_x}v v_x - \frac{d}{L_x^2}u_{xxt} - \frac{d}{L_y^2}u_{yyt} = 0 \qquad ...(28)$$

$$v_t + \frac{1}{L_y}\eta_y + \frac{1}{L_y}u u_y + \frac{1}{L_y}v v_y - \frac{d}{L_x^2}v_{xxt} - \frac{d}{L_y^2}v_{yyt} = 0$$

With the initial and boundary conditions:

$$\eta(X,0) = \eta_0(X) \quad , \quad V(X,0) = V_0(X) \qquad t \geq 0 \quad , \quad X \in \partial\Omega \quad \ldots(29)$$

$$\eta(X,t) = 0 \quad , \quad V(X,t) = 0 \qquad t \geq 0 \quad , \quad X \in \partial\Omega \quad \ldots(30)$$

The solution by the Haar wavelets method is started by dividing the interval (0,T] into N equal parts of length $\Delta t = T/N$ and denoting to $t_s = (s-1)\Delta t$ s=1,2,…,N.

We assume that $\eta_{xxyyt}(x,y,t)$, $u_{xxyyt}(x,y,t)$ and $v_{xxyyt}(x,y,t)$ can be expanded in terms of Haar wavelets as follows:

$$\eta_{xxyyt}(x,y,t) = \sum_{m_2=1}^{2M_2}\sum_{m_1=1}^{2M_1} E_{m_1,m_2}\, h_{m_1}(x)\, h_{m_2}(y)$$

$$u_{xxyyt}(x,y,t) = \sum_{m_2=1}^{2M_2}\sum_{m_1=1}^{2M_1} C_{m_1,m_2}\, h_{m_1}(x)\, h_{m_2}(y)$$

$$v_{xxyyt}(x,y,t) = \sum_{m_2=1}^{2M_2}\sum_{m_1=1}^{2M_1} D_{m_1,m_2}\, h_{m_1}(x)\, h_{m_2}(y)$$

where the elements E_{m_1,m_2}, C_{m_1,m_2} *and* D_{m_1,m_2} are constants in the subinterval $t \in (t_s, t_{s+1}]$.

Assume that $m_1 = m_2 = m$, above equations can be written into the matrices form by:

$$\eta_{xxyyt}(x,y,t) = H_m^T(x)\, E_m\, H_m(y) \qquad t \in (t_s, t_{s+1}] \qquad \ldots(31a)$$

$$u_{xxyyt}(x,y,t) = H_m^T(x)\, C_m\, H_m(y) \qquad t \in (t_s, t_{s+1}] \qquad \ldots(31b)$$

$$v_{xxyyt}(x,y,t) = H_m^T(x)\, D_m\, H_m(y) \qquad t \in (t_s, t_{s+1}] \qquad \ldots(31c)$$

wewell focus on the function $u(x,y,t)$ and the functions $\eta(x,y,t)$ *and* $v(x,y,t)$ are computed by the same way.

Integrating (31b) with respect to (t) from (t_s) to (t) and double integrating with respect to (x) from (0) to (x) , and double integrating with respect to (y) from (0) to (y) ,we obtain:

$$u_{xxyy}(x,y,t)=(t-t_s)\,H_m^T(x)\,C_m\,H_m(y)+u_{xxyy}(x,y,t_s) \quad \dots(32)$$

$$u_{xxy}(x,y,t)=(t-t_s)\,H_m^T(x)\,C_m\,P_{i,1}(y)+u_{xxy}(x,y,t_s) + \left[u_{xxy}(x,0,t)-u_{xxy}(x,0,t_s)\right] \quad \dots(33)$$

$$u_{xx}(x,y,t)=(t-t_s)\,H_m^T(x)\,C_m\,P_{i,2}(y)+u_{xx}(x,y,t_s) + y\left[u_{xxy}(x,0,t)-u_{xxy}(x,0,t_s)\right]+\left[u_{xx}(x,0,t)-u_{xx}(x,0,t_s)\right] \quad \dots(34)$$

$$\begin{aligned} u_x(x,y,t)=&(t-t_s)\,P_{i,1}^T(x)\,C_m\,P_{i,2}(y)+u_x(x,y,t_s)\\ &+y\left[u_{xy}(x,0,t)-u_{xy}(x,0,t_s)\right]-y\left[u_{xy}(0,0,t)-u_{xy}(0,0,t_s)\right]\\ &+\left[u_x(x,0,t)-u_x(x,0,t_s)\right]+\left[u_x(0,y,t)-u_x(0,y,t_s)\right]\\ &-\left[u_x(0,0,t)-u_x(0,0,t_s)\right] \end{aligned} \quad \dots(35)$$

$$\begin{aligned} u(x,y,t)=&(t-t_s)\,P_{i,2}^T(x)\,C_m\,P_{i,2}(y)+u(x,y,t_s)\\ &+y\left[u_y(x,0,t)-u_y(x,0,t_s)\right]-y\left[u_y(0,0,t)-u_y(0,0,t_s)\right]\\ &-x\left[u_x(0,0,t)-u_x(0,0,t_s)\right]+x\left[u_x(0,y,t)-u_x(0,y,t_s)\right]\\ &-xy\left[u_{xy}(0,0,t)-u_{xy}(0,0,t_s)\right]+\left[u(x,0,t)-u(x,0,t_s)\right]\\ &+\left[u(0,y,t)-u(0,y,t_s)\right]-\left[u(0,0,t)-u(0,0,t_s)\right] \end{aligned} \quad \dots(36)$$

We can reduce the order of boundary conditions used in equations (34)-(36) by using the boundary condition at x=1 and notation (6) instead of the derivatives $u_x(0,y,t), u_x(0,y,t_s), u_x(0,0,t)$ and $u_x(0,0,t_s)$.

The values of unknown term $u_x(0,y,t), u_x(0,y,t_s), u_x(0,0,t)$ and $u_x(0,0,t_s)$ can be calculated by integrating equation (36) from 0 to 1 which is given by:

$$\begin{aligned} &\left[u_x(0,y,t)-u_x(0,y,t_s)\right]-\left[u_x(0,0,t)-u_x(0,0,t_s)\right]=\\ &-(t-t_s)\,P_{i,2}^T(1)\,C_m\,P_{i,2}(y)+\left[u(1,y,t)-u(1,y,t_s)\right]\\ &-y\left[u_y(1,0,t)-u_y(1,0,t_s)\right]+y\left[u_y(0,0,t)-u_y(0,0,t_s)\right]\\ &+y\left[u_{xy}(0,0,t)-u_{xy}(0,0,t_s)\right]-\left[u(1,0,t)-u(1,0,t_s)\right]\\ &-\left[u(0,y,t)-u(0,y,t_s)\right]+\left[u(0,0,t)-u(0,0,t_s)\right] \end{aligned} \quad \dots(37)$$

Such that $P_{i,2}(1)$ is defined in equation (6).By substituting equation (37) in equation (36), we get:

$$\begin{aligned} u(x,y,t)=&(t-t_s)\,P_{i,2}^T(x)\,C_m\,P_{i,2}(y)+u(x,y,t_s)+y\left[u_y(x,0,t)-u_y(x,0,t_s)\right]\\ &-y\left[u_y(0,0,t)-u_y(0,0,t_s)\right]+\left[u(x,0,t)-u(x,0,t_s)\right]\\ &+\left[u(0,y,t)-u(0,y,t_s)\right]-\left[u(0,0,t)-u(0,0,t_s)\right]\\ &+x\,\{\ -(t-t_s)\,R_{i,1}^T(x)\,C_m\,P_{i,2}(y)+\left[u(1,y,t)-u(1,y,t_s)\right]\\ &-y\left[u_y(1,0,t)-u_y(1,0,t_s)\right]+y\left[u_y(0,0,t)-u_y(0,0,t_s)\right]\\ &-\left[u(1,0,t)-u(1,0,t_s)\right]-\left[u(0,y,t)-u(0,y,t_s)\right]+\left[u(0,0,t)-u(0,0,t_s)\right]\ \} \end{aligned} \quad \dots(39)$$

Similarly, and by using the boundary condition at y=1 and notation (6), we get:

$$\begin{aligned}&\left[u_y(x,0,t)-u_y(x,0,t_s)\right]-\left[u_y(0,0,t)-u_y(0,0,t_s)\right]\\&-x\left[u_y(1,0,t)-u_y(1,0,t_s)\right]+x\left[u_y(0,0,t)-u_y(0,0,t_s)\right]=\\&-(t-t_s)\,P_{i,2}^T(x)\,C_m\,R_{i,1}(y)+\left[u(x,1,t)-u(x,1,t_s)\right]\\&-\left[u(x,0,t)-u(x,0,t_s)\right]-\left[u(0,1,t)-u(0,1,t_s)\right]+\left[u(0,0,t)-u(0,0,t_s)\right]\\&+x\,(t-t_s)\,R_{i,1}^T(x)\,C_m\,R_{i,1}(y)-x\left[u(1,1,t)-u(1,1,t_s)\right]\\&+x\left[u(1,0,t)-u(1,0,t_s)\right]+x\left[u(0,1,t)-u(0,1,t_s)\right]-x\left[u(0,0,t)-u(0,0,t_s)\right]\end{aligned}\qquad \ldots(40)$$

By substituting equation (40) in equation (39), we get:

$$\begin{aligned}u(x,y,t)=&(t-t_s)\,P_{i,2}^T(x)\,C_m\,P_{i,2}(y)+u(x,y,t_s)+\left[u(x,0,t)-u(x,0,t_s)\right]\\&+\left[u(0,y,t)-u(0,y,t_s)\right]-\left[u(0,0,t)-u(0,0,t_s)\right]\\&-x(t-t_s)\,R_{i,1}^T(x)\,C_m\,P_{i,2}(y)+x\left[u(1,y,t)-u(1,y,t_s)\right]\\&-x\left[u(1,0,t)-u(1,0,t_s)\right]-x\left[u(0,y,t)-u(0,y,t_s)\right]+x\left[u(0,0,t)-u(0,0,t_s)\right]\\&-y(t-t_s)\,P_{i,2}^T(x)\,C_m\,R_{i,1}(y)+y\left[u(x,1,t)-u(x,1,t_s)\right]\\&-y\left[u(x,0,t)-u(x,0,t_s)\right]-y\left[u(0,1,t)-u(0,1,t_s)\right]+y\left[u(0,0,t)-u(0,0,t_s)\right]\\&+x\,y\,(t-t_s)\,R_{i,1}^T(x)\,C_m\,R_{i,1}(y)-x\,y\left[u(1,1,t)-u(1,1,t_s)\right]\\&+x\,y\left[u(1,0,t)-u(1,0,t_s)\right]+x\,y\left[u(0,1,t)-u(0,1,t_s)\right]\\&-x\,y\left[u(0,0,t)-u(0,0,t_s)\right]\end{aligned}\qquad \ldots(41)$$

Now, the derivatives of equation (41) with respect (t),(x) and (y), we get:

$$\begin{aligned}u_t(x,y,t)=&P_{i,2}^T(x)\,C_m\,P_{i,2}(y)+u_t(0,y,t)+u_t(x,0,t)-u_t(0,0,t)\\&-x\;R_{i,1}^T(x)\,C_m\,P_{i,2}(y)+xu_t(1,y,t)\\&-xu_t(1,0,t)-xu_t(0,y,t)+xu_t(0,0,t)\\&-y\;P_{i,2}^T(x)\,C_m\,R_{i,1}(y)+yu_t(x,1,t)\\&-yu_t(0,1,t)-yu_t(x,0,t)+yu_t(0,0,t)\\&+x\,y\;R_{i,1}^T(x)\,C_m\,R_{i,1}(y)-x\,yu_t(1,1,t)\\&+x\,yu_t(1,0,t)+x\,yu_t(0,1,t)-x\,yu_t(0,0,t)\end{aligned}\qquad \ldots(42)$$

$$\begin{aligned}u_x(x,y,t)=&(t-t_s)\,P_{i,1}^T(x)\,C_m\,P_{i,2}(y)+u_x(x,y,t_s)+\left[u_x(x,0,t)-u_x(x,0,t_s)\right]\\&-(t-t_s)\,R_{i,1}^T(x)\,C_m\,P_{i,2}(y)+\left[u(1,y,t)-u(1,y,t_s)\right]\\&-\left[u(1,0,t)-u(1,0,t_s)\right]-\left[u(0,y,t)-u(0,y,t_s)\right]+\left[u(0,0,t)-u(0,0,t_s)\right]\\&-y(t-t_s)\,P_{i,1}^T(x)\,C_m\,R_{i,1}(y)+y\left[u_x(x,1,t)-u_x(x,1,t_s)\right]\\&-y\left[u_x(x,0,t)-u_x(x,0,t_s)\right]\\&+y\,(t-t_s)\,R_{i,1}^T(x)\,C_m\,R_{i,1}(y)-y\left[u(1,1,t)-u(1,1,t_s)\right]\\&+y\left[u(1,0,t)-u(1,0,t_s)\right]+y\left[u(0,1,t)-u(0,1,t_s)\right]\\&-y\left[u(0,0,t)-u(0,0,t_s)\right]\end{aligned}\qquad \ldots(43)$$

$$\begin{aligned}u_{xx}(x,y,t)=&(t-t_s)\,H_m^T(x)\,C_m\,P_{i,2}(y)+u_{xx}(x,y,t_s)\\&+\left[u_{xx}(x,0,t)-u_{xx}(x,0,t_s)\right]-y(t-t_s)\,H_m^T(x)\,C_m\,R_{i,1}(y)\\&+y\left[u_{xx}(x,1,t)-u_{xx}(x,1,t_s)\right]-y\left[u_{xx}(x,0,t)-u_{xx}(x,0,t_s)\right]\end{aligned}\qquad \ldots(44)$$

$$u_{xxt}(x,y,t) = H_m^T(x)\, C_m\, P_{i,2}(y) - y\, H_m^T(x)\, C_m\, R_{i,1}(y) + \left[u_{xxt}(x,0,t)\right] + y\left[u_{xxt}(x,1,t) - u_{xxt}(x,0,t)\right] \quad \ldots(45)$$

Similarly, we find $u_y, u_{yy}, and\ u_{yyt}$. By substituting equations (42)-(45) in system (28), we get:

$$P_{i,2}^T(x)\, E_m\, P_{i,2}(y) - x\, R_{i,1}^T(x)\, E_m\, P_{i,2}(y) - y\, P_{i,2}^T(x)\, E_m\, R_{i,1}(y) + x\, y\, R_{i,1}^T(x)\, E_m\, R_{i,1}(y) - \left(\frac{b}{L_x^2}\right) H_m^T(x)\, E_m\, P_{i,2}(y) + \left(\frac{b}{L_x^2}\right) y\, H_m^T(x)\, E_m\, R_{i,1}(y) - \left(\frac{b}{L_y^2}\right) P_{i,2}^T(x)\, E_m\, H_m(y) + \left(\frac{b}{L_y^2}\right) x\, R_{i,1}^T(x)\, E_m\, H_m(y) = G_1 \quad \ldots(46a)$$

$$P_{i,2}^T(x)\, C_m\, P_{i,2}(y) - x\, R_{i,1}^T(x)\, C_m\, P_{i,2}(y) - y\, P_{i,2}^T(x)\, C_m\, R_{i,1}(y) + x\, y\, R_{i,1}^T(x)\, C_m\, R_{i,1}(y) - \left(\frac{d}{L_x^2}\right) H_m^T(x)\, C_m\, P_{i,2}(y) + \left(\frac{d}{L_x^2}\right) y\, H_m^T(x)\, C_m\, R_{i,1}(y) - \left(\frac{d}{L_y^2}\right) P_{i,2}^T(x)\, C_m\, H_m(y) + \left(\frac{d}{L_y^2}\right) x\, R_{i,1}^T(x)\, C_m\, H_m(y) = G_2 \quad \ldots(46b)$$

$$P_{i,2}^T(x)\, D_m\, P_{i,2}(y) - x\, R_{i,1}^T(x)\, D_m\, P_{i,2}(y) - y\, P_{i,2}^T(x)\, D_m\, R_{i,1}(y) + x\, y\, R_{i,1}^T(x)\, D_m\, R_{i,1}(y) - \left(\frac{d}{L_x^2}\right) H_m^T(x)\, D_m\, P_{i,2}(y) + \left(\frac{d}{L_x^2}\right) y\, H_m^T(x)\, D_m\, R_{i,1}(y) - \left(\frac{d}{L_y^2}\right) P_{i,2}^T(x)\, D_m\, H_m(y) + \left(\frac{d}{L_y^2}\right) x\, R_{i,1}^T(x)\, D_m\, H_m(y) = G_3 \quad \ldots(46c)$$

such that:

$$\begin{aligned} G_1 = & -\left(\frac{1}{L_x}\right) u_x(x,y,t_s) - \left(\frac{1}{L_y}\right) v_y(x,y,t_s) - \left(\frac{1}{L_x}\right) \eta(x,y,t_s)\, u_x(x,y,t_s) \\ & -\left(\frac{1}{L_x}\right) \eta_x(x,y,t_s)\, u(x,y,t_s) - \left(\frac{1}{L_y}\right) \eta(x,y,t_s)\, v_y(x,y,t_s) - \left(\frac{1}{L_y}\right) \eta_y(x,y,t_s)\, v(x,y,t_s) \\ & + (x-1)\,\eta_t(0,y,t) + (y-1)\,\eta_t(x,0,t) + (1-x-y+x\,y)\,\eta_t(0,0,t)) - x\,\eta_t(1,y,t) \\ & + (x-x\,y)\,\eta_t(1,0,t) - y\,\eta_t(x,1,t) + (y-x\,y)\,\eta_t(0,1,t) + x\,y\,\eta_t(1,1,t) \\ & + \left(\frac{b}{L_x^2}\right)(1-y)\eta_{xxt}(x,0,t) + \left(\frac{b}{L_x^2}\right) y\,\eta_{xxt}(x,1,t) \\ & + \left(\frac{b}{L_y^2}\right)(1-x)\eta_{yyt}(0,y,t) + \left(\frac{b}{L_y^2}\right) x\,\eta_{yyt}(1,y,t) \end{aligned}$$

$$\begin{aligned} G_2 = & -\left(\frac{1}{L_x}\right) \eta_x(x,y,t_s) - \left(\frac{1}{L_x}\right) u(x,y,t_s)\, u_x(x,y,t_s) - \left(\frac{1}{L_x}\right) v(x,y,t_s)\, v_x(x,y,t_s) \\ & + (x-1)\, u_t(0,y,t) + (y-1)\, u_t(x,0,t) + (1-x-y+x\,y)\, u_t(0,0,t)) - x\, u_t(1,y,t) \\ & + (x-x\,y)\, u_t(1,0,t) - y\, u_t(x,1,t) + (y-x\,y)\, u_t(0,1,t) + x\,y\, u_t(1,1,t) \\ & + \left(\frac{d}{L_x^2}\right)(1-y)u_{xxt}(x,0,t) + \left(\frac{d}{L_x^2}\right) y\, u_{xxt}(x,1,t) \\ & + \left(\frac{d}{L_y^2}\right)(1-x)u_{yyt}(0,y,t) + \left(\frac{d}{L_y^2}\right) x\, u_{yyt}(1,y,t) \end{aligned}$$

$$G_3 = -\left(\frac{1}{L_y}\right)\eta_y(x,y,t_s) - \left(\frac{1}{L_y}\right)u(x,y,t_s)\,u_y(x,y,t_s) - \left(\frac{1}{L_y}\right)v(x,y,t_s)\,v_y(x,y,t_s)$$
$$+ (x-1)\,v_t(0,y,t) + (y-1)\,v_t(x,0,t) + (1-x-y+x\,y)\,v_t(0,0,t)) - x\,v_t(1,y,t)$$
$$+ (x - x\,y)\;v_t(1,0,t) - y\,v_t(x,1,t) + (y - x\,y)\;v_t(0,1,t) + x\,y\,v_t(1,1,t)$$
$$+\left(\frac{d}{L_x^2}\right)(1-y)v_{xxt}(x,0,t) + \left(\frac{d}{L_x^2}\right)y\,v_{xxt}(x,1,t)$$
$$+\left(\frac{d}{L_y^2}\right)(1-x)v_{yyt}(0,y,t) + \left(\frac{d}{L_y^2}\right)x\,v_{yyt}(1,y,t)$$

We can write system (46) by the form:

$$\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{b}{L_x^2}\right)H_m^T(x)\right]\cdot E_m\cdot\left[P_{i,2}(y) - y\,R_{i,1}(y) - \left(\frac{b}{L_y^2}\right)H_m(y)\right]$$
$$-\left(\frac{b^2}{L_x^2\,L_y^2}\right)H_m^T(x)\,E_m\,H_m(y) = G_1 \qquad \ldots(47a)$$

$$\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{d}{L_x^2}\right)H_m^T(x)\right]\cdot C_m\cdot\left[P_{i,2}(y) - y\,R_{i,1}(y) - \left(\frac{d}{L_y^2}\right)H_m(y)\right]$$
$$-\left(\frac{d^2}{L_x^2\,L_y^2}\right)H_m^T(x)\,C_m\,H_m(y) = G_2 \qquad \ldots(47b)$$

$$\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{d}{L_x^2}\right)H_m^T(x)\right]\cdot D_m\cdot\left[P_{i,2}(y) - y\,R_{i,1}(y) - \left(\frac{d}{L_y^2}\right)H_m(y)\right]$$
$$-\left(\frac{d^2}{L_x^2\,L_y^2}\right)H_m^T(x)\,D_m\,H_m(y) = G_3 \qquad \ldots(47c)$$

By multiplying $\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{b}{L_x^2}\right)H_m^T(x)\right]^{-1}$ to the right hand side and $\left[H_m(y)\right]^{-1}$ to the left hand side of each term in equation (47a), we obtain:

$$-\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{b}{L_x^2}\right)H_m^T(x)\right]^{-1}\cdot\left(\frac{b^2}{L_x^2\,L_y^2}\right)H_m^T(x)\cdot E_m$$
$$+ E_m\cdot\left[P_{i,2}(y) - y\,R_{i,1}(y) - \left(\frac{b}{L_y^2}\right)H_m(y)\right]\cdot\left[H_m(y)\right]^{-1} \qquad \ldots(48a)$$
$$-\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{b}{L_x^2}\right)H_m^T(x)\right]^{-1}\cdot G_1\cdot\left[H_m(y)\right]^{-1} = 0$$

Also, we get:

$$-\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{d}{L_x^2}\right)H_m^T(x)\right]^{-1}\cdot\left(\frac{d^2}{L_x^2\,L_y^2}\right)H_m^T(x)\cdot C_m$$
$$+ C_m\cdot\left[P_{i,2}(y) - y\,R_{i,1}(y) - \left(\frac{d}{L_y^2}\right)H_m(y)\right]\cdot\left[H_m(y)\right]^{-1} \qquad \ldots(48b)$$
$$-\left[P_{i,2}^T(x) - x\,R_{i,1}^T(x) - \left(\frac{d}{L_x^2}\right)H_m^T(x)\right]^{-1}\cdot G_2\cdot\left[H_m(y)\right]^{-1} = 0$$

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{d}{L_x^2}\right)H_m^T(x)\right]^{-1}\cdot\left(\frac{d^2}{L_x^2\;L_y^2}\right)H_m^T(x)\cdot D_m$$

$$+D_m\cdot\left[P_{i,2}(y)-y\,R_{i,1}(y)-\left(\frac{d}{L_y^2}\right)H_m(y)\right]\cdot\left[H_m(y)\right]^{-1} \qquad \ldots(48c)$$

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{d}{L_x^2}\right)H_m^T(x)\right]^{-1}\cdot G_2\cdot\left[H_m(y)\right]^{-1}=0$$

The system (48) is the Lyapunov matrix equations which can be solved by one of the packages [4] or by using MATLAB Language:
X=Lyap(A,B,C)

To solve the equation AX+XB+C=0 ,such that the matrices A,B and C must have compatible dimensions but need not be square.Finally, The solution of the problem is found according to (41).

Now, we use the Legendre wavelets to solve the system, that is, we can replace the Legendre wavelets instead of Haar wavelets in equations (41)-(45) and by substituting in the system (28), we obtain:

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{b}{L_x^2}\right)\psi_{n,m}^T(x)\right]^{-1}\cdot\left(\frac{b^2}{L_x^2\;L_y^2}\right)\psi_{n,m}^T(x)\cdot E_m$$

$$+E_m\cdot\left[P_{i,2}(y)-y\,R_{i,1}(y)-\left(\frac{b}{L_y^2}\right)\psi_{n,m}(y)\right]\cdot\left[\psi_{n,m}(y)\right]^{-1} \qquad \ldots(49a)$$

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{b}{L_x^2}\right)\psi_{n,m}^T(x)\right]^{-1}\cdot G_1\cdot\left[\psi_{n,m}(y)\right]^{-1}=0$$

Also

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{d}{L_x^2}\right)\psi_{n,m}^T(x)\right]^{-1}\cdot\left(\frac{d^2}{L_x^2\;L_y^2}\right)\psi_{n,m}^T(x)\cdot C_m$$

$$+C_m\cdot\left[P_{i,2}(y)-y\,R_{i,1}(y)-\left(\frac{d}{L_y^2}\right)\psi_{n,m}(y)\right]\cdot\left[\psi_{n,m}(y)\right]^{-1} \qquad \ldots(49b)$$

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{d}{L_x^2}\right)\psi_{n,m}^T(x)\right]^{-1}\cdot G_2\cdot\left[\psi_{n,m}(y)\right]^{-1}=0$$

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{d}{L_x^2}\right)\psi_{n,m}^T(x)\right]^{-1}\cdot\left(\frac{d^2}{L_x^2\;L_y^2}\right)\psi_{n,m}^T(x)\cdot D_m$$

$$+D_m\cdot\left[P_{i,2}(y)-y\,R_{i,1}(y)-\left(\frac{d}{L_y^2}\right)\psi_{n,m}(y)\right]\cdot\left[\psi_{n,m}(y)\right]^{-1} \qquad \ldots(49c)$$

$$-\left[P_{i,2}^{T}(x)-x\,R_{i,1}^{T}(x)-\left(\frac{d}{L_x^2}\right)\psi_{n,m}^T(x)\right]^{-1}\cdot G_2\cdot\left[\psi_{n,m}(y)\right]^{-1}=0$$

such that $\psi_{n,m}(y)$, $P_{i,2}^T(x)$ and $R_{i,1}^T(x)$ are matrices defined in equations (14), (23) and (24), respectively.

5. Numerical Experiments

In this section, we present the results of two-dimensional BBM-BBM system (28) which solved numerically by using the wavelets technique.

In the first example, we took zero Dirichlet homogenous boundary conditions for $\eta, u \text{ and } v$ on the whole boundary in the square $[0,1]\mathrm{X}[0,1]$ with exacts solutions: [15]

$\eta(x,y,t) = e^t \,.\, \sin(\pi x).(y-1).\,y$

$u(x,y,t) = e^t \,.\, x.\cos(3\pi x/2).\sin(\pi y)$

$v(x,y,t) = e^t \,.\, \sin(\pi x).\cos(3\pi y/2).\,y$

Then, we compute in table (1) and figure (2), the corresponding right hand side in order to obtain the L^2 norm of the error between the exact solution and the numerical solution by using the Haar wavelets and Legendre wavelets, respectively.

Table (1) Compared between the Haar wavelets method and Legendre wavelets method when $b = 1/6$, $d = 1/6$, $L = 1$, $\Delta t = 0.0001$

The method	k, M	$\|\eta_{ex} - \eta\|_{L^2}$	$\|u_{ex} - u\|_{L^2}$	$\|v_{ex} - v\|_{L^2}$
Haar	2M=8	1.4264e-007	1.4810e-006	1.4828e-006
Legendre	k=2, M=4	5.0914e-009	1.0131e-007	1.0032e-007
Haar	2M=16	2.6805e-008	3.8652e-007	3.8842e-007
Legendre	k=2, M=8	1.2779e-008	8.4647e-009	6.6351e-009
Legendre	k=3, M=4	1.2180e-008	1.0966e-008	1.2176e-008
Haar	2M=32	3.6457e-009	9.1942e-008	9.3859e-008
Legendre	k=3, M=8	1.2779e-008	8.4559e-009	6.6241e-009

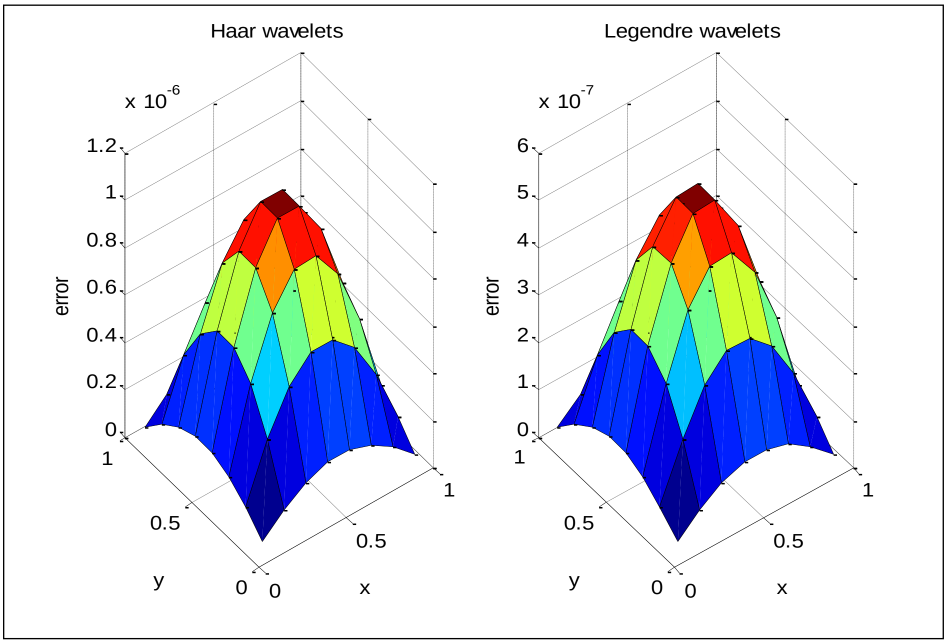

Figure (2)The error δ_8 **at** $t = 0.001$**by using 2D wavelets methods.**

In the second example, we consider the numerical solution of 2D BBM-BBM system (28) with initial and homogeneous Dirichlet boundary conditions [15]:

$\eta_0(x,y) = 0.2\, e^{-(x^2+y^2)/5} \quad , \quad V_0(x,y) = 0$

$\eta(X,t) = 0 \quad , \quad V(X,t) = 0 \quad , \quad X \in \partial\Omega \quad t \ge 0$

on the square $[-40,40]\mathrm{X}[-40,40]$. Figure (3) shows the generation and propagation of Tsunami wave $\eta - wave$ by the Haar wavelets method when the step of space $2M = 2^{J+1} = 16$ and time step $\Delta t = 0.1$.

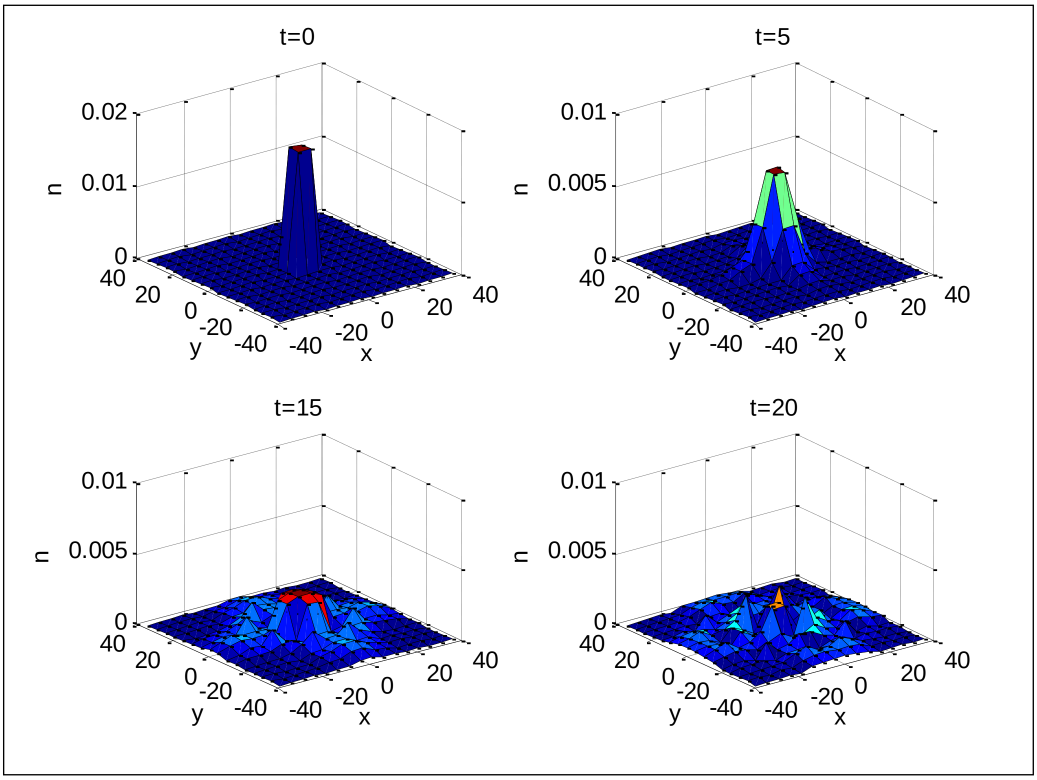

Figure (3a) the generation of tsunami wave $\eta - wave$ by 2D Haar wavelets method when 2M=16, $b = 1/6$, $d = 1/6$, $L_x = L_y = 80$, $\Delta t = 0.1$.

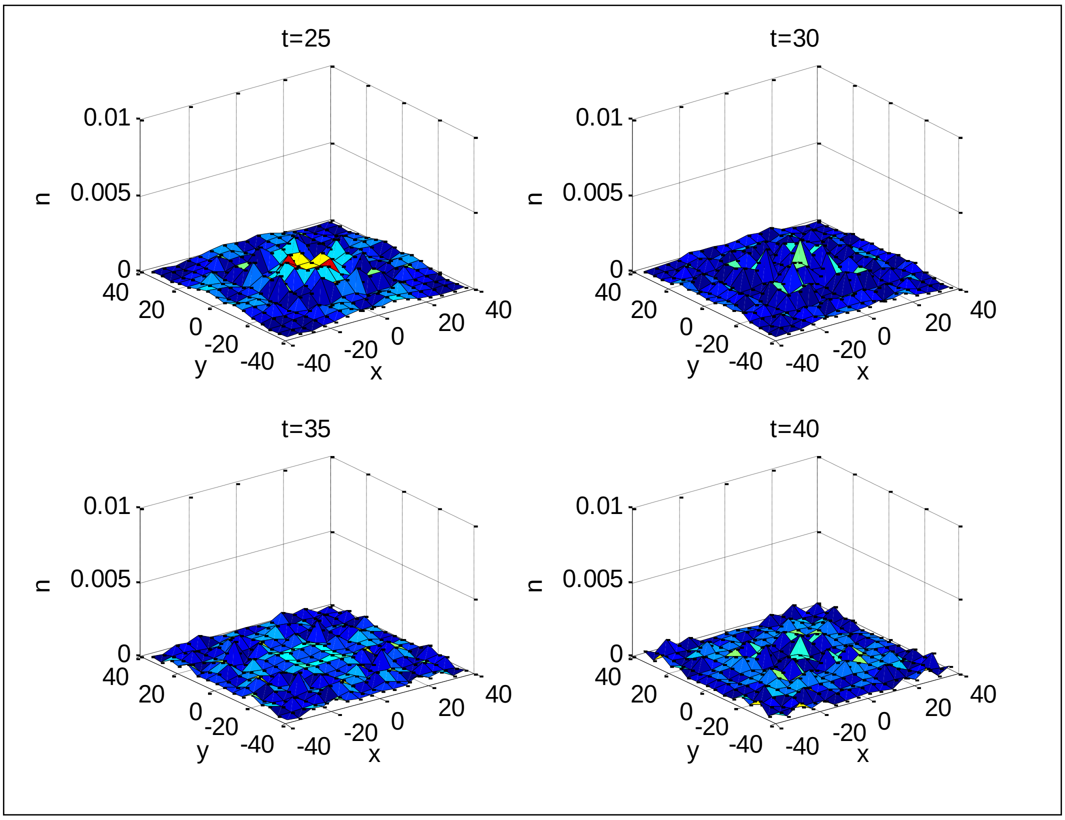

Figure (3b) the generation of tsunami wave $\eta - wave$ by 2D Haar wavelets method when 2M=16, $b = 1/6$, $d = 1/6$, $L_x = L_y = 80$, $\Delta t = 0.1$.

6. Conclusions

In this paper, we develop an accurate and efficient the wavelets methods for solving non-linear two-dimensional BBM-BBM system by convert the partial differential equation into a simple Lyapunov matrix equation.

The benefits of the wavelets approach are sparse matrices of representation, fast transformation and possibility of implementation of fast algorithms. It's worth mentioning that the wavelets solution provides excellent results even for small values of (2M) as noted in table (1). Also, when 2M=64 , 2M=128 , …, we can obtain the results closer to the exact values. We have also been reducing the boundary conditions used in the solution by using the notation (6) when x=L respect to space and the results were a high resolution. Matlab language is used in finding the results and figure draw, its characteristic at high accuracy and large speed.

Also, we compared between the wavelet methods in the numerical solution for non-linear BBM-BBM system and we have found that the Legendre wavelets method is better and closer to the exact solution of the Haar wavelets method as shown in table (1).

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