Linear Stability of Thin Liquid Films flows down on an Inclined Plane using Integral Approximation

Hajar Farhan Ismael
hajar.math@yahoo.com
Faculty of Science, University of Zakho
Zakho – Kurdistan-Region, Iraq

Received on: 28/4/2013 **Accepted on: 11/10/2013**

ABSTRACT

In this paper, the stability and dynamics of a thin liquid films flowing down on an inclined plane are investigated by using integral approximation. The strong non-linear evolution equations are derived by the integral approximation with a specified velocity profile. The evolution equations are used to study the linear stability for liquid films. As a result, output of this research, we showed that the effect of inclination of films is an unstable factor.

Key words: Stability, Non-linear equations, Integral approximation.

الاستقرار الخطي للأغشية السائلة الرقيقة على المستوى المائل باستخدام التقريب التكاملي

هزار فرحان إسماعيل

كلية العلوم ، جامعة زاخو

تاريخ الاستلام: 2013/4/28 **تاريخ القبول: 2013/10/11**

الملخص

فى هذا البحث تمت دراسة الاستقرارية وديناميكية للأغشية الرقيقة السائلة والمائلة باستخدام طريقة التكامل . تم اشتقاق المعادلات الغير الخطية باستخدام التقريب التكاملي مع توزيع السرعة و هذه المعادلات استخدمت لدراسة الاستقرارية الخطية للاغشية السائلة. ان اهم نتائج هذه الدراسة يشير بان تأثير ميلان الغشاء يعتبرعامل غير مستقر.

الكلمات المفتاحية : الاستقرارية، المعادلات الغير خطية ، التقريب التكاملي

1. Introduction

Investigations involving the linear stability of thin liquid films down on an inclined plane are considered. The stability and dynamics of thin liquid films, in general, are of an immense scientific and technological importance. A liquid layer on a solid substrate becomes unstable when the layer is very thin. Hwang and Chen in [7] investigated the stability of thin liquid films on a horizontal plane by using long wave theory, integral approximation and numerical solution, then they compared among them, and they concluded that the evolution equations derived from integral theory could properly model a thin liquid film. Chen and Hwang [1] studied the inertia effect on rupture process of a thin liquid film. Hwang et al. in [9] derived strong non-linear partial differential equations of the thickness of a film on plate by using integral theory and concluded that van der Waals potential and the inertia of x-momentum equations are the unstable factors, while the surface tension and high-order viscous dissipation are the stable factors for the instability of the film. Erneux and Davis in [6] derived the non-linear partial differential equation on free thin liquid film by using long-wave theory and found that the non-linear terms contribute to the acceleration of the rupture phenomenon, but Hwang et al. in [10] used an integral method to derive the strong non-linear evolution equation of thin liquid films. Hwang et al. in [8] investigated the effects of insoluble surfactant on the dynamic rupture of a thin free film and compared there

results with results obtained by De Wit et al. in [4]. De Wit and Gallez in [3] studied the role of insoluble surfactants on the stability of free-liquid films, taking into account the influence of van der Waals attraction and surface tension, also investigated the linear stability of free thin and thin liquid films with substrate on horizontal by using long-wave theory and compared between them. A non-linear differential equation that describes the long-wave evolution of the interface shape is derived by Chen and Hwang [2] to investigate the dynamic rupture process of a thin liquid film on a cylinder.

2. Mathematical Formulation

This section introduces the physical model of a thin liquid film. Consider a thin liquid layer flowing down a plane inclined at angle θ to the horizontal (Figure 1) the film of initial thickness h_0 is bounded at the thin surface by a passive gas and is laterally unbounded. The liquid layer is assumed thin enough that Van der Waals forces are effective and thick enough that a continuum theory of the liquid is applicable and we assume that the liquid is a Newtonian viscous fluid.

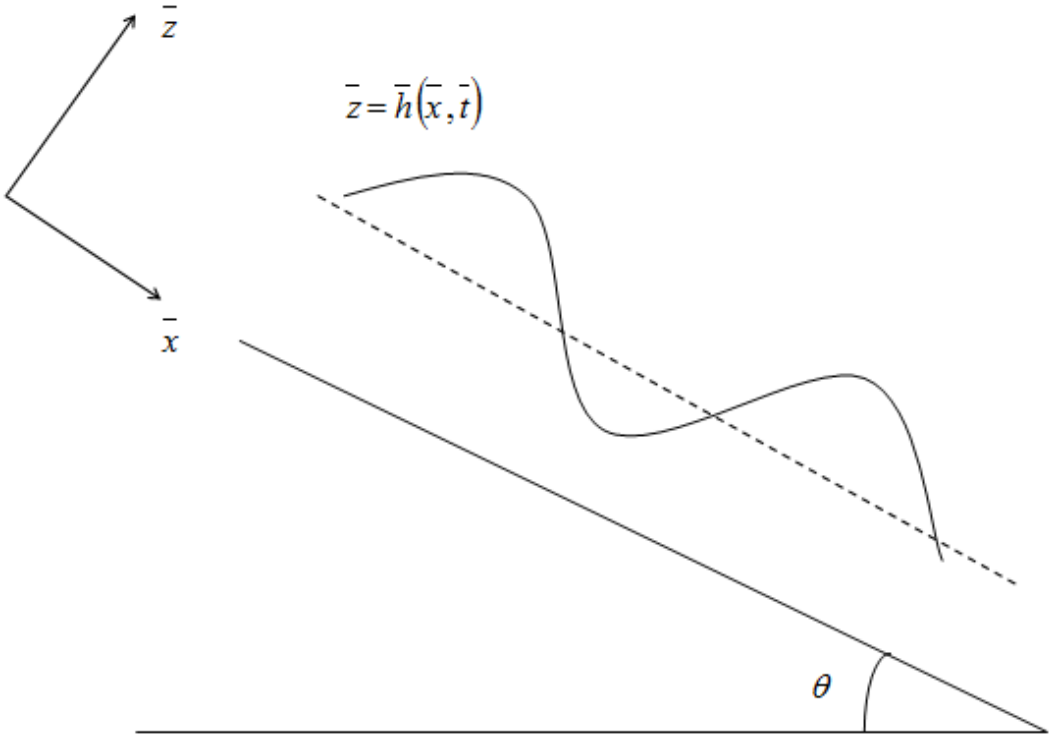

Figure (1) Thin Liquid Films with Substrate

For two-dimensional motions of the liquid film, we have the Navier-Stokes equations and the continuity equation given by [11, 13]:

$$\rho\left(\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right)=-\frac{\partial\bar{p}}{\partial\bar{x}}-\frac{\partial\bar{\phi}}{\partial\bar{x}}+\mu\left(\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right)+\rho g\sin(\theta) \quad ...(1)$$

$$\rho\left(\frac{\partial\bar{w}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{w}}{\partial\bar{x}}+\bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}\right)=-\frac{\partial\bar{p}}{\partial\bar{x}}-\frac{\partial\bar{\phi}}{\partial\bar{x}}+\mu\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right)-\rho g\cos(\theta) \quad ...(2)$$

$$\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{z}}=0 \quad ...(3)$$

where, $(\bar{u},\bar{w})$ are the velocity components in the $(\bar{x},\bar{z})$ direction, respectively. The quantities ρ, g, μ, $\bar{\Phi}$ and $\bar{p}$ are density, gravity, viscosity, Van der Waals potential and pressure of film respectively.

At the interface of the thin liquid film, we have the following boundary conditions [13]. The kinematics boundary conditions are given by:

$$\bar{w} = \frac{\partial \bar{h}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{h}}{\partial \bar{x}} \qquad \text{at} \quad \bar{z} = \bar{h} \qquad \ldots(4)$$

The shear-stress conditions on the interfaces have the form

$$2\frac{\partial \bar{h}}{\partial \bar{x}}\left(\frac{\partial \bar{u}}{\partial \bar{x}} - \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \left(\left(\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2 - 1\right)\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{x}}\right) = 0 \qquad \text{at} \quad \bar{z} = \bar{h} \qquad \ldots(5)$$

The normal-stress condition on the interfaces is obtained by

$$\bar{p} - 2\mu\left[\left(\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2 \frac{\partial \bar{u}}{\partial \bar{x}} - \frac{\partial \bar{h}}{\partial \bar{x}}\left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{x}}\right) + \frac{\partial \bar{w}}{\partial \bar{z}}\right]\left(1 + \frac{\partial \bar{h}}{\partial \bar{x}}\right)^{-1} = p_a - \frac{\sigma}{R} \qquad \text{at } \bar{z} = \bar{h} \qquad \ldots(6)$$

where, p_a denotes atmospheric pressure, σ is the dimensional coefficient of surface tension and $\frac{1}{R}$ is the radius of curvature defined as $\frac{1}{R} = \frac{\frac{\partial^2 \bar{h}}{\partial \bar{x}^2}}{\left(1 + \left(\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2\right)^{\frac{3}{2}}}$, The conditions at the substrate of thin liquid film are

$$\bar{u} = \bar{w} = 0 \qquad \text{at} \qquad \bar{z} = 0 \qquad \ldots(7)$$

3. Non-dimensional Analysis

To express the Navier-Stokes equations, the equation of continuity with the associated boundary conditions into non-dimensional form, we define the following dimensionless quantities [6, 8] as

$$\bar{h} = hh_0, \qquad \bar{z} = zh_0, \qquad \bar{x} = xh_0, \qquad \bar{u} = \frac{uv}{h_0},$$

$$\bar{w} = \frac{wv}{h_0}, \qquad \bar{p} = \frac{pv^2\rho}{h_0^2}, \qquad \bar{t} = \frac{th_0^2}{v}, \qquad \phi = \frac{\Phi h_0^2}{v^2\rho} \qquad \ldots(8)$$

where, the mean thickness of the liquid is h_0 and $v = \frac{\mu}{\rho}$ is the kinematics viscosity of the film fluid. The non-dimensional mean surface tension, S is defined as

$$S = \frac{3\rho v^2}{h_0 \sigma} \qquad \ldots(9)$$

Substituting the dimensionless variables and parameters given by equations (8), and (9) into equations (1-7) and simplifying the resulting equation, we obtain the dimensionless Navier-Stokes and the continuity equations as the form [12]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} - \frac{\partial \phi}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) + G_a \sin(\theta) \qquad \ldots(10)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}-\frac{\partial \phi}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-G_a\cos(\theta) \qquad \ldots(11)$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \qquad \ldots(12)$$

and the boundary conditions are derived to give

$$w=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \qquad \text{at} \quad z=h \qquad \ldots(13)$$

$$2\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right)+\left(\left(\frac{\partial h}{\partial x}\right)^2-1\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)=0 \qquad \text{at} \quad z=h \qquad \ldots(14)$$

$$p-2\left(\frac{\frac{\partial u}{\partial x}\left(\left(\frac{\partial h}{\partial x}\right)^2-1\right)-\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)}{N^2}\right)=-3S\frac{\frac{\partial^2 h}{\partial x^2}}{N^3} \qquad \text{at} \quad z=h \qquad \ldots(15)$$

$$u=w=0 \qquad \text{at} \quad z=0 \qquad \ldots(16)$$

where, $h(x,t)$ is the local thickness of the layer, and $N=\left(1+h_x^2\right)^{\frac{1}{2}}$.

4. Integral Approximation

Now, the method of multiple scales can be used to study the non-linear stability by using the notions given [1, 2] as

$$u=O(1), \qquad w=O(k), \qquad p=O\left(k^{-1}\right),$$
$$x=O\left(k^{-1}\right) \qquad z=O(1) \qquad t=O\left(k^{-1}\right) \qquad \ldots(17)$$

and the orders of the other dimensionless variables are

$$S=O(1) \qquad A=O\left(k^3\right) \qquad G_a=O(k)\ . \qquad \ldots(18)$$

Introducing equations (17) and (18) into equations (10-16), we get the following equations of motion and boundary conditions

$$\frac{\partial u}{\partial(k^{-1}t)}+u\frac{\partial u}{\partial(k^{-1}x)}+(kw)\frac{\partial u}{\partial z}=-\frac{\partial(k^{-1}p)}{\partial(k^{-1}x)}-\frac{\partial(k^3\phi)}{\partial(k^{-1}x)}+\frac{\partial}{\partial(k^{-1}x)}\frac{\partial u}{\partial(k^{-1}x)}+\frac{\partial}{\partial z}\frac{\partial u}{\partial z}+(k\,G_a)\sin(\theta)$$

$$\frac{\partial(kw)}{\partial(k^{-1}t)}+u\frac{\partial(kw)}{\partial(k^{-1}x)}+(kw)\frac{\partial(kw)}{\partial z}=-\frac{\partial(k^{-1}p)}{\partial z}-\frac{\partial(k^3\phi)}{\partial z}+\frac{\partial}{\partial(k^{-1}x)}\frac{\partial(kw)}{\partial(k^{-1}x)}+\frac{\partial}{\partial z}\frac{\partial(kw)}{\partial z}-(kG_a)\cos(\theta)$$

$$\frac{\partial u}{\partial(k^{-1}x)}+\frac{\partial(kw)}{\partial z}=0$$

$$u=kw=0 \qquad \text{at} \quad z=0$$

$$(kw)=\frac{\partial h}{\partial(k^{-1}t)}+u\frac{\partial h}{\partial(k^{-1}x)} \qquad \text{at} \quad z=h$$

$$4\frac{\partial h}{\partial(k^{-1}x)}\frac{\partial u}{\partial(k^{-1}x)}+\left(\left(\frac{\partial h}{\partial(k^{-1}x)}\right)^2-1\right)\left(\frac{\partial u}{\partial z}+\frac{\partial(kw)}{\partial(k^{-1}x)}\right)=0 \qquad \text{at} \quad z=h$$

$$(k^{-1}p)-2\left[\left(\left(\frac{\partial h}{\partial(k^{-1}x)}\right)^2-1\right)\frac{\partial u}{\partial(k^{-1}x)}-\frac{\partial h}{\partial(k^{-1}x)}\left(\frac{\partial u}{\partial z}+\frac{\partial(kw)}{\partial(k^{-1}x)}\right)\right]\left(1+\left(\frac{\partial h}{\partial(k^{-1}x)}\right)^2\right)^{-1}=$$

$$-3S\frac{\partial^2 h}{\partial(k^{-1}x)^2}\left(1+\left(\frac{\partial h}{\partial(k^{-1}x)}\right)^2\right)^{-\frac{3}{2}} \qquad \text{at} \quad z=h$$

Simplified the above equations, we can formulate the Navier-stokes equations and equation of continuity as

$$k\frac{\partial u}{\partial t}+ku\frac{\partial u}{\partial x}+kw\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}-k^4\frac{\partial \phi}{\partial x}+k^2\frac{\partial}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial}{\partial z}\frac{\partial u}{\partial z}+kG_a\sin(\theta)$$

$$k^2\frac{\partial w}{\partial t}+k^2u\frac{\partial w}{\partial x}+k^2w\frac{\partial w}{\partial z}=-k^{-1}\frac{\partial p}{\partial z}-k^3\frac{\partial \phi}{\partial z}+k^3\frac{\partial}{\partial x}\frac{\partial w}{\partial x}+k\frac{\partial}{\partial z}\frac{\partial w}{\partial z}-kG_a\cos(\theta)$$

$$k\frac{\partial u}{\partial x}+k\frac{\partial w}{\partial z}=0$$

Now, at $z=0$, we have $u=kw=0$ and at the interface $z=h(x,t)$

$$kw=k\frac{\partial h}{\partial t}+ku\frac{\partial h}{\partial x}$$

$$4k^4\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}+\left(k^2\left(\frac{\partial h}{\partial x}\right)^2-1\right)\left(\frac{\partial u}{\partial z}+k^2\frac{\partial w}{\partial x}\right)=0$$

$$k^{-1}p-2\left[\left(k^2\left(\frac{\partial h}{\partial x}\right)^2-1\right)k\frac{\partial u}{\partial x}-k\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial z}+k^2\frac{\partial w}{\partial x}\right)\right]\left(1+k^2\left(\frac{\partial h}{\partial x}\right)^2\right)^{-1}=$$

$$-3Sk^2\frac{\partial^2 h}{\partial x^2}\left(1+k^2\left(\frac{\partial h}{\partial x}\right)^2\right)^{-\frac{3}{2}}$$

Neglecting the terms higher than $O(\varepsilon^6)$ with $k=O\left(\varepsilon^{\frac{3}{2}}\right)$ and the term $\left(\frac{\partial h}{\partial x}\right)^2$ from all boundary conditions because they are very small in thin liquid film, then the reduced equations of the motion and the pertinent boundary conditions can be derived as follows

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}-\frac{\partial \phi}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}+G_a\sin(\theta) \qquad \ldots(19)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial^2 w}{\partial z^2}-G_a\cos(\theta) \qquad \ldots(20)$$

and the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \qquad \ldots(21)$$

also, the boundary condition at $z=0$ becomes

$$u=w=0 \qquad \text{at} \quad z=0 \qquad \ldots(22)$$

$$w=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \qquad \text{at} \quad z=h(x,t)$$

…(23)

$$\frac{\partial u}{\partial z} = 4\frac{\partial u}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial w}{\partial x} \qquad \text{at} \quad z = h(x,t) \qquad \ldots(24)$$

$$-p - 3S\frac{\partial^2 h}{\partial x^2} = 2\frac{\partial u}{\partial x} + 2\frac{\partial h}{\partial x}\frac{\partial u}{\partial z} \qquad \text{at} \quad z = h(x,t) \qquad \ldots(25)$$

Now, integrate equation (19) with respect to z over the film thickness, where $\frac{1}{2}\frac{\partial u^2}{\partial x} = u\frac{\partial u}{\partial x}$, we have

$$\int_0^h \frac{\partial u}{\partial t}dz + \frac{1}{2}\int_0^h \frac{\partial u}{\partial x}dz + \int_0^h w\frac{\partial u}{\partial z}dz = -\int_0^h \frac{\partial p}{\partial x}dz - \int_0^h \frac{\partial \phi}{\partial x}dz + \int_0^h \frac{\partial^2 u}{\partial x^2}dz + \int_0^h \frac{\partial^2 u}{\partial z^2}dz + \int_0^h G_a \sin(\theta)\,dz, \qquad \ldots(26)$$

integrating equations (20) with respect to z, we get

$$\int \frac{\partial w}{\partial t}dz + \int u\frac{\partial w}{\partial x}dz + \int w\frac{\partial w}{\partial z}dz = -\int \frac{\partial p}{\partial z}dz + \int \frac{\partial^2 w}{\partial z^2}dz - \int G_a \cos(\theta)\,dz \qquad \ldots(27)$$

and integrating continuity equation with respect to z over the film thickness, we have

$$\int_0^h \frac{\partial u}{\partial x}dz + \int_0^h \frac{\partial w}{\partial z}dz = 0 \qquad \ldots(28)$$

By using the general form of the Leibniz integral rule, the first term on the left hand side of equation (26), can be written as follows

$$\int_0^h \frac{\partial u}{\partial t}dz = \frac{\partial}{\partial t}\int_0^h u dz + u\frac{\partial h}{\partial t} \qquad \ldots(29)$$

also, we perform the similar transformations on other integrals of equation(26). Now, we reach at the integral condition

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad \ldots(30)$$

The velocity profile can be written as [7, 9]

$$u = \frac{3q}{h}\left[\left(\frac{z}{h}\right) - \frac{1}{2}\left(\frac{z}{h}\right)^2\right] - \frac{fh}{2}\left[\left(\frac{z}{h}\right) - \frac{3}{2}\left(\frac{z}{h}\right)^2\right] \qquad \ldots(31)$$

Let

$$G(x,t) = 3q\,h^{-2} - \frac{1}{2}f \qquad \ldots(32)$$

and

$$L(x,t) = -\frac{3}{2}qh^{-3} + \frac{3}{4}f\,h^{-1} \qquad \ldots(33)$$

then, equation (31) can be written as

$$u = Gz + Lz^2 \qquad \ldots(34)$$

from differentiation equation (31)

$$\frac{\partial u}{\partial z} = f(x,t) \qquad \ldots(35)$$

Now, by differentiation equation (34) with respect to x and substituting into equation (21), we get

$$\frac{\partial u}{\partial x} = \frac{\partial G}{\partial x}z + \frac{\partial L}{\partial x}z^2$$

$$\frac{\partial G}{\partial x}z+\frac{\partial L}{\partial x}z^2+\frac{\partial w}{\partial z}=0 \qquad \ldots(36)$$

now, integrating equation (36) with respect to z, we can write the value of w as

$$\int\frac{\partial w}{\partial z}dz=-\int\frac{\partial G}{\partial x}zdz-\int\frac{\partial L}{\partial x}z^2dz$$

$$w=-\frac{1}{2}\frac{\partial G}{\partial x}z^2-\frac{1}{3}\frac{\partial L}{\partial x}z^3+d(x,t)$$

Applying the boundary condition $w\big|_{z=0}=\frac{\partial u}{\partial x}\big|_{z=0}=0$,we get

$$w=-\frac{1}{2}\frac{\partial G}{\partial x}z^2-\frac{1}{3}\frac{\partial L}{\partial x}z^3 \qquad \ldots(37)$$

Applying equation (34) and equation (37), the shear stress balance boundary condition at free surface can be given as

$$f=4h\frac{\partial G}{\partial x}\frac{\partial h}{\partial x}+4h^2\frac{\partial L}{\partial x}\frac{\partial h}{\partial x}+\frac{1}{2}h^2\frac{\partial^2 G}{\partial x^2}+\frac{1}{3}h^3\frac{\partial^2 L}{\partial x^2} \qquad \ldots(38)$$

putting equation (29) into equation (19), we get

$$\frac{\partial}{\partial t}\int_0^h udz-u\frac{\partial h}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\int_0^h u^2dz-\frac{1}{2}u^2\frac{\partial h}{\partial x}+wu=-\int_0^h\frac{\partial p}{\partial x}dz-\int_0^h\frac{\partial \phi}{\partial x}dz+\int_0^h\frac{\partial^2 u}{\partial x^2}dz$$

$$+\frac{\partial}{\partial z}\int_0^h\frac{\partial u}{\partial z}dz+\int_0^h G_a\sin(\theta)dz$$

or

$$\frac{\partial}{\partial t}\int_0^h u\,dz-u\frac{\partial h}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\int_0^h u^2dz-\frac{1}{2}u^2\frac{\partial h}{\partial x}+wu=-\int_0^h\frac{\partial p}{\partial x}dz+3Ah^{-3}\frac{\partial h}{\partial x}-3qh^{-2}$$
$$+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}h^2+\frac{1}{3}\frac{\partial^2 L}{\partial x^2}h^3+\frac{\partial u}{\partial z}+G_a\sin(\theta)h \qquad \ldots(39)$$

by substituting the boundary condition (23) at $z=h$ into equation (39), we obtain

$$\frac{\partial}{\partial t}\int_0^h u\,dz+\frac{\partial}{\partial x}\int_0^h u^2dz=-\int_0^h\frac{\partial p}{\partial x}dz+3Ah^{-3}\frac{\partial h}{\partial x}-3qh^{-2}$$
$$+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}h^2+\frac{1}{3}\frac{\partial^2 L}{\partial x^2}h^3+\frac{\partial u}{\partial z}+f+G_a\sin(\theta)h \qquad \ldots(40)$$

Putting equations (34) and (37) into equation (27), we find the value of p as

$$p=-\frac{\partial G}{\partial x}z-\frac{\partial L}{\partial x}z^2+\frac{1}{6}\frac{\partial^2 G}{\partial(xt)}z^3+\frac{1}{12}\frac{\partial^2 L}{\partial(xt)}z^4+\frac{1}{8}G\frac{\partial^2 G}{\partial x^2}z^4+\frac{1}{15}G\frac{\partial^2 L}{\partial x^2}z^5$$
$$+\frac{1}{10}L\frac{\partial^2 G}{\partial x^2}z^5+\frac{1}{18}L\frac{\partial^2 L}{\partial x^2}z^6-\frac{1}{8}\left(\frac{\partial G}{\partial x}\right)^2z^4-\frac{1}{6}\frac{\partial G}{\partial x}\frac{\partial L}{\partial x}z^5-\frac{1}{18}\left(\frac{\partial L}{\partial x}\right)^2z^6 \qquad \ldots(41)$$
$$-G_a\cos(\theta)+c(x,t)$$

from the boundary condition (25), we have

$$p=-3S\frac{\partial^2 h}{\partial x^2}-2\frac{\partial h}{\partial x}\frac{\partial u}{\partial z}-2\frac{\partial u}{\partial x} \qquad \ldots(42)$$

Now, we put equation (32) into equation (41), one can get

$$c(x,t) = -3S\frac{\partial^2 h}{\partial x^2} - 2G\frac{\partial h}{\partial x} - 4L\frac{\partial h}{\partial x}h - 2\frac{\partial G}{\partial x}h - 2\frac{\partial L}{\partial x}h^2 + \frac{\partial G}{\partial x}h + \frac{\partial L}{\partial x}h^2 - \frac{1}{6}\frac{\partial^2 G}{\partial(xt)}h^3$$
$$-\frac{1}{12}\frac{\partial^2 L}{\partial(xt)}h^4 - \frac{1}{8}G\frac{\partial^2 G}{\partial x^2}h^4 - \frac{1}{15}G\frac{\partial^2 L}{\partial x^2}h^5 - \frac{1}{10}L\frac{\partial^2 G}{\partial x^2}h^5 - \frac{1}{18}L\frac{\partial^2 L}{\partial x^2}h^6$$
$$+\frac{1}{8}\left(\frac{\partial G}{\partial x}\right)^2 h^4 + \frac{1}{6}\frac{\partial G}{\partial x}\frac{\partial L}{\partial x}h^5 + \frac{1}{18}\left(\frac{\partial L}{\partial x}\right)^2 h^6 + G_a\cos(\theta)h$$

...(43)

Substituting the value of $c(x,t)$ into equation (41), we have

$$p = -3S\frac{\partial^2 h}{\partial x^2} - 2G\frac{\partial h}{\partial x} - 4L\frac{\partial h}{\partial x}h - 2\frac{\partial G}{\partial x}h - 2\frac{\partial L}{\partial x}h^2 - \frac{\partial G}{\partial x}z + \frac{\partial G}{\partial x}h - \frac{\partial L}{\partial x}z^2 + \frac{\partial L}{\partial x}h^2$$
$$+\frac{1}{6}\frac{\partial^2 G}{\partial(xt)}z^3 - \frac{1}{6}\frac{\partial^2 G}{\partial(xt)}h^3 + \frac{1}{12}\frac{\partial^2 L}{\partial(xt)}z^4 - \frac{1}{12}\frac{\partial^2 L}{\partial(xt)}h^4 + \frac{1}{8}G\frac{\partial^2 G}{\partial x^2}z^4 - \frac{1}{8}G\frac{\partial^2 G}{\partial x^2}h^4$$
$$+\frac{1}{15}G\frac{\partial^2 L}{\partial x^2}z^5 - \frac{1}{15}G\frac{\partial^2 L}{\partial x^2}h^5 + \frac{1}{10}L\frac{\partial^2 G}{\partial x^2}z^5 - \frac{1}{10}L\frac{\partial^2 G}{\partial x^2}h^5 + \frac{1}{18}L\frac{\partial^2 L}{\partial x^2}z^6 - \frac{1}{18}L\frac{\partial^2 L}{\partial x^2}h^6$$
$$-\frac{1}{8}\left(\frac{\partial G}{\partial x}\right)^2 z^4 + \frac{1}{8}\left(\frac{\partial G}{\partial x}\right)^2 h^4 - \frac{1}{6}\frac{\partial G}{\partial x}\frac{\partial L}{\partial x}z^5 + \frac{1}{6}\frac{\partial G}{\partial x}\frac{\partial L}{\partial x}h^5 - \frac{1}{18}\left(\frac{\partial L}{\partial x}\right)^2 z^6 + \frac{1}{18}\left(\frac{\partial L}{\partial x}\right)^2 h^6$$
$$-G_a\cos(\theta)z + G_a\cos(\theta)h \qquad \text{...(44)}$$

now, by differentiating equation (44) and, then integrating equation (44) with respect to z at $z=0$ to h with the use of equation (34), we get

$$\int_0^h p_x dz = -3Shh_{xxx} - 2(GH)_{xx} - 2(Lh^2)_{xx} + G_a h\cos(\theta)h_x$$
$$+\begin{pmatrix} -\frac{1}{2}G_x h^2 - \frac{1}{8}G_{xt}h^4 - \frac{1}{15}L_{xt}h^5 - \frac{1}{10}GG_{xx}h^5 \\ -\frac{1}{18}GL_{xx}h^6 - \frac{1}{12}LG_{xx}h^6 - \frac{1}{21}LL_{xx}h^7 \\ +\frac{1}{10}(G_x)^2 h^5 + \frac{5}{36}G_x L_x h^6 + \frac{1}{21}(L_x)^2 h^7 \end{pmatrix}_x \qquad \ldots(45)$$

now, putting equations (45) and (34) into equation (40), we get

$$q_t + \left(\frac{1}{3}G^2h^3 + \frac{1}{2}GLh^4 - \frac{1}{5}L^2h^5\right) = 3Shh_{xxx} + \frac{1}{2}G_{xx}h^2 + \frac{1}{3}L_{xx}h^3 + 3Ah^{-3}h_x$$
$$-3qh^{-2} + \frac{3}{2}f + G_a\sin(\theta)h - G_a\cos(\theta)hh_x + 2h(Gh)_{xx} + 3h(Lh^2)_{xx} \qquad \ldots(46)$$
$$+\begin{bmatrix} -\frac{1}{2}G_x h^2 - \frac{2}{3}L_x h^3 + \frac{1}{8}G_{xt}h^4 + \frac{1}{15}L_{xt}h^5 + \frac{1}{10}\left((GG_x)_x - (G_x)^2\right)h^5 \\ +\frac{1}{36}\left(3(LG_x)_x + 2(GL_x)_x - 10G_x L_x\right)h^6 + \frac{1}{21}\left((LL_x)_x - (L_x)^2\right)h^7 \end{bmatrix}_x$$

where, $q = \int_0^h u\, dz$.

5. Linear Stability Analysis

The normal mode method [8, 12] can be applied to equations (30), (38) and (46)

$$h = h_0 + h', \qquad f = f_0 + f', \qquad q = q_0 + q' \qquad \ldots(47)$$

The equilibrium states of equations (30), (38) and (46) are [9]

$$(h_0, q_0, f_0) = (1, 0, 0) \qquad \ldots(48)$$

Putting the equilibrium states into equations (30), (32), (33), (38) and (46), we get the following equations

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad , \qquad \ldots(49)$$

$$G(x,t) = 3q - \frac{1}{2} f \qquad \ldots(50)$$

and

$$L(x,t) = -\frac{3}{2} q + \frac{3}{4} f \qquad \ldots(51)$$

The shear-stress boundary condition at free surface can be rewritten as

$$f = 4\frac{\partial G}{\partial x}\frac{\partial h}{\partial x} + 4\frac{\partial L}{\partial x}\frac{\partial h}{\partial x} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} + \frac{1}{3}\frac{\partial^2 L}{\partial x^2} \qquad \ldots(52)$$

and the averaged x-momentum equation (46) gives the form:

$$q_t + \left(\frac{1}{3}G^2 + \frac{1}{2}GL - \frac{1}{5}L^2\right)_x = 3Sh_{xxx} + \frac{1}{2}G_{xx} + \frac{1}{3}L_{xx} + 3Ah_x - 3q + \frac{3}{2} f$$

$$+ G_a \sin(\theta) - G_a \cos(\theta) h_x + 2G_{xx} + 2L_{xx} \qquad \ldots(53)$$

$$+ \left[\begin{array}{l} -\frac{1}{2}G_x - \frac{2}{3}L_x + \frac{1}{8}G_{xt} + \frac{1}{15}L_{xt} + \frac{1}{10}\left((GG_x)_x - (G_x)^2\right) \\ + \frac{1}{36}\left(3(LG_x)_x + 2(GL_x)_x - 10G_xL_x\right) + \frac{1}{21}\left((LL_x)_x - (L_x)^2\right) \end{array}\right]_x$$

Now, substituting equations (50) and (51) into equations (52) and (53), we obtain the following system

$$f = 4\frac{\partial}{\partial x}\left(3q - \frac{1}{2} f\right)\frac{\partial h}{\partial x} + 4\frac{\partial}{\partial x}\left(-\frac{3}{2}q + \frac{3}{4} f\right)\frac{\partial h}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(3q - \frac{1}{2} f\right) + \frac{1}{3}\frac{\partial^2}{\partial x^2}\left(-\frac{3}{2}q + \frac{3}{4} f\right)$$

$$f = 12\frac{\partial q}{\partial x}\frac{\partial h}{\partial x} - 2\frac{\partial f}{\partial x}\frac{\partial h}{\partial x} - 6\frac{\partial q}{\partial x}\frac{\partial h}{\partial x} + 3\frac{\partial f}{\partial x}\frac{\partial h}{\partial x} + \frac{3}{2}\frac{\partial^2 q}{\partial x^2} - \frac{1}{4}\frac{\partial^2 f}{\partial x^2} - \frac{1}{2}\frac{\partial^2 q}{\partial x^2} + \frac{1}{4}\frac{\partial^2 f}{\partial x^2}$$

$$f = 6\frac{\partial q}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial^2 q}{\partial x^2} \qquad \ldots(54)$$

and

$$q_t+\left(\frac{6}{5}q^2+\frac{1}{120}f^2+\frac{1}{20}qf\right)_x=3Sh_{xxx}+3Ah_x-3q+\frac{3}{2}f+\frac{5}{2}q_{xx}-\frac{1}{4}f_{xx}+\frac{1}{8}G_{xxt}$$

$$+\frac{1}{15}L_{xxt}+2G_{xx}+2L_{xx}+G_a\sin(\theta)-G_a\cos(\theta)h_x \qquad \ldots(55)$$

$$+\left[\begin{array}{l}-\frac{3}{7}q_{xx}-\frac{2}{7}f_{xx}+\frac{529}{560}qq_{xx}-\frac{1}{3360}ff_{xx}-\frac{9}{560}qf_{xx}\\+\frac{49}{1680}fq_{xx}-\frac{43}{60}q_xf_x-\frac{47}{80}(q_x)^2+\frac{37}{480}(f_x)^2\end{array}\right]_x$$

from equations (47) and (48) , we have

$$h'=h-1,\qquad q'=q,\qquad f'=f\ . \qquad \ldots(56)$$

Putting equations (51), (52) and (56) into equations (49), (54) and (55) with the neglect of the non-linear terms of equations (49) and (54), the linearization of equation is obtained and has the form

$$\frac{\partial h'}{\partial t}+\frac{\partial q'}{\partial x}=0 \qquad \ldots(57)$$

$$f'=q'_{xx} \qquad \ldots(58)$$

and

$$q'_t=3Sh'_{xxx}+3Ah'_x-3q'+\frac{3}{2}f'+\frac{7}{2}q'_{xx}+\frac{1}{4}f'_{xx}+\frac{11}{40}q'_{xxt}-\frac{1}{80}f'_{xxt} \qquad \ldots(59)$$

$$+G_a\sin(\theta)-G_a\cos(\theta)h'_x$$

The solutions of those disturbances are assumed to be [1, 3, 4]

$$(h',q',f')=(H_0,Q_0,f_0)\exp(wt+ikx) \qquad \ldots(60)$$

$$h'=H_0e^{wt+ikx} \qquad \ldots(61)$$

$$q'=Q_0e^{wt+ikx} \qquad \ldots(62)$$

$$f'=f_0e^{wt+ikx} \qquad \ldots(63)$$

putting equations (61), (62) and (63) into equations (57), (58) and (59), we get

$$\frac{\partial}{\partial t}\left(H_0e^{wt+ikx}\right)+\frac{\partial}{\partial x}\left(Q_0e^{wt+ikx}\right)=0$$

$$H_0w+Q_0ik=0 \qquad \ldots(64)$$

$$f_0=-k^2Q_0 \qquad \ldots(65)$$

$$Q_0w=-3Sik^3H_0+3AikH_0-3Q_0+\frac{3}{2}f_0-\frac{7}{2}k^2Q_0-\frac{1}{4}k^2f_0$$

$$-\frac{11}{40}k^2wQ_0+\frac{1}{80}k^2wf_0+G_a\sin(\theta)-G_a\cos(\theta)H_0ik \qquad \ldots(66)$$

$$\text{where, }\ H_0=-ik\frac{Q_0}{w}\ \text{ and }\ f_0=-k^2Q_0 \qquad \ldots(67)$$

Substituting equation (67) into equation (66), we get

$$\left(1+\frac{11}{40}k^2+\frac{1}{80}k^4\right)w^2+\left(3+5k^2+\frac{1}{4}k^4\right)w+\left(3Sk^2-3A+G_a\cos(\theta)\right)k^2=0 \qquad \ldots(68)$$

Hence, we found that the film is unstable if $w > 0$ only when $k < k_c$, where k_c, is a critical 'cut-off' wave number, and this is clearly shown in Figure 2. For neutrally stable wave $w = 0$, then k_c is given by

$$k_c = \left(\left(\frac{A}{S} - \frac{1}{3S} G_a \cos(\theta) \right) \right)^{\frac{1}{2}} \qquad \ldots(69)$$

which is the best result that agrees with the linear results of Hwang and Chen in [7] and Hwang et al. in [9], when $G_a \to 0$.

The maximum growth rate, w of the linear waves occurs for the dominant wave number, k which is obtained by setting $\frac{dw}{dk} = 0$ from equation (68). Thus

$$k = \left(\frac{1}{2S} \left(A - \frac{1}{3} G_a \cos(\theta) \right) \right)^{\frac{1}{2}}$$

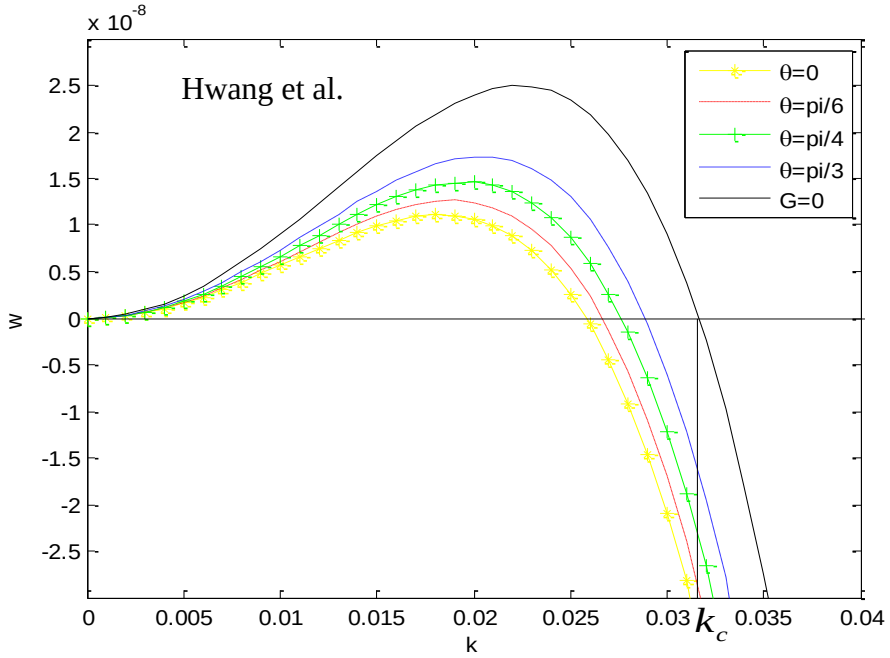

Figure 2. The Growth Rate w vs. Wave Number k plotted after Equation (68) for S=0.1, A=0.0001 and $G_a = 0.0001$ under various θ .

6. Conclusion

This study analyzed the stability of thin liquid film. Linear stability analysis reveals the qualitative results. As shown in Figure (2), the film becomes stable to short-wave-perturbation if $k > k_c$ and unstable to long-wave –perturbation when $k < k_c$, then we conclude that the effect of inclination of thin liquid films is an unstable factor.

REFERENCES

[1] Chen, J. L. and Hwang, C. C.; (1994), Effects of inertia on the rupture process of a thin liquid film, Journal of Colloid and Interface Science 167, pp. 214-216.

[2] Chen, J. L. and Hwang, C. C.; (1996), nonlinear rupture theory of thin liquid film on a cylinder, Journal of Colloid and Interface Science 182, pp. 564-569.

[3] De Wit, A. and Gallez, D.; (1996), Interfacial instabilities in thin liquid films with insoluble surfactants, Advances in multi-fluid flows, pp. 112-130.

[4] De Wit, A., Gallez, D. and Christov, C. I.; (1994), Nonlinear evolution equations for thin liquid films with insoluble surfactants, Phys. Fluids 6 (10), pp. 3256-3266.

[5] Durst, F.; (2008), Fluid Mechanics, Springer-Verlag Berlin Heidelberg.

[6] Erneux, T. and Davis, S. H.; (1993), Nonlinear rupture of free films, Phys. Fluids A 5 (5), pp. 1117-1122, American Institute of Physics.

[7] Hwang, C. C. and Chen, J. L.; (2005), Stability of Thin Liquid Films, Encyclopedia of Surface and Colloid Science DOI: 10.108/E-Escs-120028146.

[8] Hwang, C. C. and Chen, J. L. and Shen, L. F.; (1996), Nonlinear Rupture of a thin free liquid film with Insoluble Surfactant, Journal of the Physical Society of Japan, Vol. 65, No. 8, pp. 2494-2501.

[9] Hwang, C. C. and Chen, J. L. and Shen, L. F.; (1996), Strong nonlinear dynamic rupture theory of thin liquid films, Physical review E, Vol. 54, No. 3, pp. 3013-3016, American Physical Society.

[10] Hwang, C. C., Chen, J. L., Shen, L. F. and Weng, C. I.; (1996), Strong nonlinear rupture theory of thin free liquid films, Physica A 231, pp. 448-460.

[11] Jameel, A. T.; (1994), nonlinear stability of thin films on solid surfaces, PhD dissertation, Indian institute of technology, Kanpur, India.

[12] Kao, J. C.T., Golvin, A. A. and Davis, S.H.; (2006), Rupture of thin films with resonant substrate patterning, Journal of Colloid and Interface Science, 303, pp. 532-545.

[13] White, F. M.; (2008), Fluid Mechanics, McGraw-Hill Companies.