Alternating Direction Implicit Method for Solving Parabolic Partial Differential Equations in Three Dimensions

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ABSTRACT

In this paper, the parabolic partial differential equations in three-dimensions are solved by two types of finite differences, such as, Alternating Direction Explicit (ADE) method and Alternating Direction Implicit (ADI) method. By the comparison of the numerical results for the previous two methods with the Exact solution, we observe that the results of Alternating Direction Implicit (ADI) method is better and nearest to the exact solution compared with the results of Alternating Direction Explicit (ADE) method. we also studied the numerical stability of both methods by Von-Neumann Method.

Keywords: Parabolic Partial Differential Equations in Three Dimensions**,** finite difference methods, Alternating direction explicit method, Alternating direction implicit method, Von-Neumann Method.

طريقة المتجهات الضمنية المتعاقبة لحل المعادلات التفاضلية الجزية من نوع القطع المكافيء في ثلاث ابعاد

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الملخص

في هذا البحث تم حل المعادلات التفاضلية الجزئية في ثلاث أبعاد من نوع القطع المكافئ باستخدام نوعين من طرائق الفروقات المنتهية، طريقة المتجهات المتعاقبة الصريحة (ADE) وطريقة المتجهات المتعاقبة الضمنية (ADI). بمقارنة النتائج العددية لكل من الطريقتين السابقتين مع نتائج الحل المضبوط لوحظ أن النتائج بطريقة المتجهات المتعاقبة الضمنية (ADI) هو أفضل واقرب إلى الحل المضبوط من نتائج طريقة المتجهات المتعاقبة الصريحة (ADE). كما تم دراسة استقرارية كل من الطريقتين السابقتين باستخدام طريقة -Von Neumann.

الكلمات المفتاحية: المعادلات التفاضلية الجزئية في ثلاث أبعاد من نوع القطع المكافئ, الفروقات المنتهية, طريقة المتجهات المتعاقبة الصريحة, طريقة المتجهات المتعاقبة الضمنية, طريقة ڤون-نيومان.

1. Introduction:

Partial differential equations (PDEs) form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently they spread into economics, financial forecasting, image processing and other fields. To investigate the predictions of PDE models of such phenomena, it is often necessary to approximate their solution numerically, commonly in combination with the analysis of simple special cases; while in some of the recent instances the numerical models play an almost independent role [10].

Parabolic partial differential equations in two or three space dimensions with over-specified boundary data feature in the mathematical modeling of many important phenomena. While a significant body of knowledge about the theory and numerical methods for parabolic partial differential equations with classical boundary conditions

has been accumulated, not much has been extended to parabolic partial differential equations with over-specified boundary data [4]. We often meet the problem of solving equation of parabolic type in many fields such as seepage, diffusion, heat conduction and so on [9].

B.J. Noye and K.J. Hayman in [11] used ADI to solve the two dimensional time-dependent heat equations subject to a constant coefficient, J.M. McDonough in [12] used ADI methods for solving elliptic problems and Norma Alias and Md. Rajibul Islam in [1] used alternating group explicit (AGE) method and Iterative alternating decomposition explicit (IADE) method to solve a two-dimensional and three-dimensional in PDE problems. Mohamed A. Antar and Esmail M. Mokheimer in [2] used spreadsheet programs to solve a three dimensional equation for numerical solutions by using finite difference solutions which are the most appropriate.

In this paper, we study and apply the finite difference methods to approximate the solution and study the stability of the numerical solution of a model of parabolic partial differential equation in three dimensions.
These methods are combinations of finite difference method with
   - Alternating direction explicit method (ADE)
   - Alternating direction implicit method (ADI)

First, we derive the finite differential form of ADE and ADI methods for the given model and then present an algorithm for each method. Also we compare between them. The stability for the above methods has been examined .

2. Model of Equation

In the case of three dimensions, the mathematical model is such an initial and boundary value problem is given by [9] as follows :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad (0 \le x, y, z \le 1, t \ge 0) \qquad \text{...(1)}$$

$u(x, y, z, 0) = g(x, y, z), \quad (0 \le x, y, z \le 1)$ ...(2)

$u(0, y, z, t) = f_1(y, z, t), u(1, y, z, t) = f_2(y, z, t), \quad (0 \le y, z \le 1, t \ge 0)$ ..(3)

$u(x, 0, z, t) = f_3(x, z, t), u(x, 1, z, t) = f_4(x, z, t), \quad (0 \le x, z \le 1, t \ge 0)$ ...(4)

$u(x, y, 0, t) = f_5(x, y, t), u(x, y, 1, t) = f_6(x, y, t), \quad (0 \le x, y \le 1, t \ge 0)$ ...(5)

where $u(x, y, z, t)$ denoting temperature or concentration of chemical [15], while g, f_1, f_2, f_3, f_4, f_5 and f_6 are known functions. and where heat transferred in three dimension system of length L, width W and depth D as shows in fig. (1) [2]. Fig. (2) shows grid points in cubic.

(a)   (b)

80
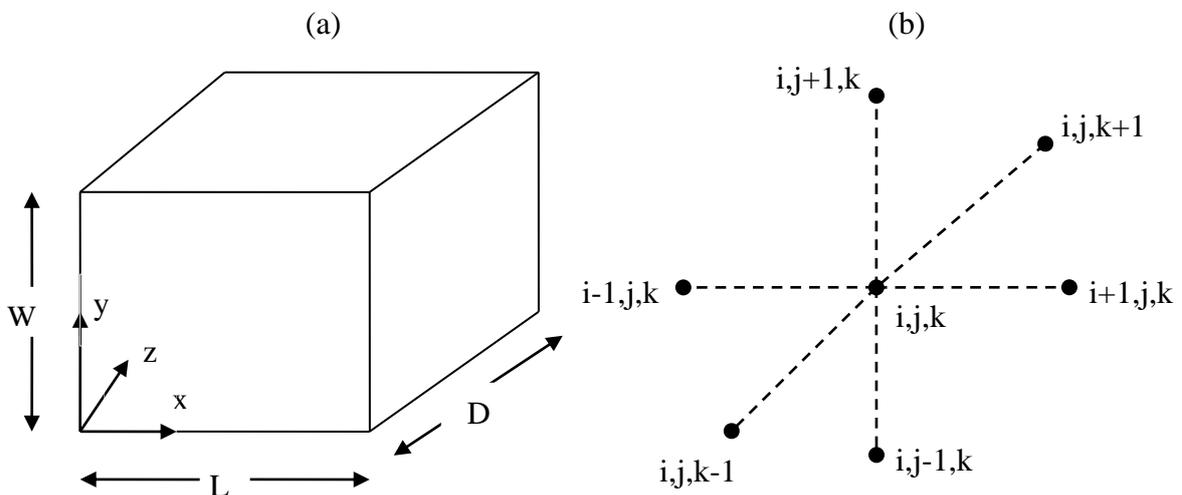
Fig. (1) (a) Problem Domain (b) Nodes layout

Outer points for cubic is
boundary conditions

Inner points for cubic
is initial conditions
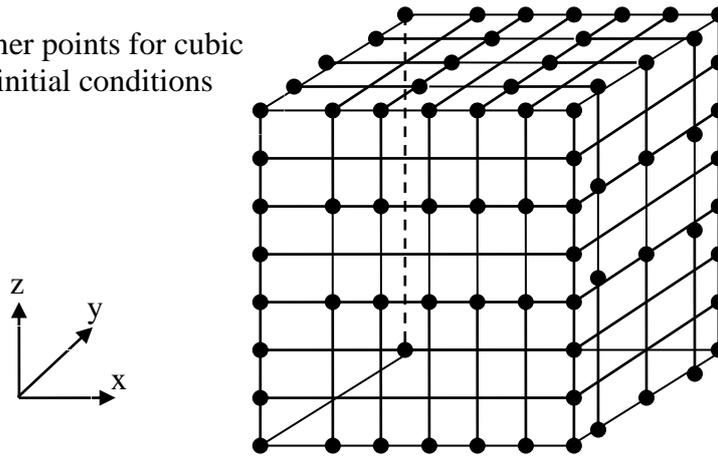
Fig. (2) Grid points for Cubic

3. Numerical Methods

We solve the mathematical model in (1) with the combination of the finite difference methods with ADE and ADI methods.

3.1 ADE Method

The alternating direction explicit (ADE) method for generating numerical solutions to the diffusion equation is stable for some time because it is an explicit method; it holds a speed advantage over implicit methods for computations over a single time level [7] the explicit methods in which the solution at the new time step is formed by a combination of pervious time step solutions [13, 14].

When we consider a square region $(0 \le x \le 1)$, $(0 \le y \le 1)$, $(0 \le z \le 1)$ and that u is known at all points within and on the boundary of the square region; we draw lines parallel to x, y, z, t – axis as

$x = i\Delta x$ $i = 0, 1, 2, ...$
$y = j\Delta y$ $j = 0, 1, 2, ...$
$z = k\Delta z$ $k = 0, 1, 2, ...$
$t = n\Delta t$ $n = 0, 1, 2, ...$

Then, the explicit finite difference approximation to parabolic partial differential equation in three dimensional equation is given by

$$\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n}}{\Delta t} = \frac{u_{i-1,j,k}^{n} - 2u_{i,j,k}^{n} + u_{i+1,j,k}^{n}}{(\Delta x)^2} + \frac{u_{i,j-1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j+1,k}^{n}}{(\Delta y)^2}$$

$$+ \frac{u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}}{(\Delta z)^2} \qquad \qquad \ldots(6)$$

Multiply eq. (6) by Δt and set $\Delta x = \Delta y = \Delta z$, then we have a square region and $r = \dfrac{\Delta t}{(\Delta x)^2}$ then we get

$$u_{i,j,k}^{n+1} = (1 - 6r)u_{i,j,k}^{n} + r\left(u_{i-1,j,k}^{n} + u_{i+1,j,k}^{n} + u_{i,j-1,k}^{n} + u_{i,j+1,k}^{n} + u_{i,j,k-1}^{n} + u_{i,j,k+1}^{n}\right) \qquad \ldots(7)$$

3.2 ADI Method

The ADI was first suggested by Douglas , Peaceman and Rachford [3, 4, 11] for solving the heat equation in two spatial variables and alternating direction implicit (ADI) methods have proved valuable in the approximation of the solutions of parabolic and elliptic differential equations in two and three variables [6, 7]

In the ADI approach, the finite difference equations are written in terms of quantities at three x levels. However, three different finite difference approximations are used alternately, one to advance the calculations from the plane n to a plane (n+1), the second to advance the calculations from (n+1) plane to the (n+2) plane and the third to advance the calculations from (n+2) plane to the (n+3) plane [10].

Then, we advance the solution of the parabolic partial differential equation in three dimensions from nth plane to (n+1)th plane by replacing $\dfrac{\partial^2 u}{\partial x^2}$ by implicit finite difference approximation at the (n+1)th plane. Similarly, $\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial^2 u}{\partial z^2}$ are replaced by an explicit finite difference approximation at the nth plane. With these approximation eq.(1) in parabolic model can be written as.

$$\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n}}{\Delta t} = \frac{u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}}{(\Delta x)^2} + \frac{u_{i,j-1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j+1,k}^{n}}{(\Delta y)^2}$$

$$+ \frac{u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}}{(\Delta z)^2} \qquad \ldots(8)$$

We set $\Delta x = \Delta y = \Delta z$ then we have a square region and multiply eq.(8) by Δt then we get

$$u_{i,j,k}^{n+1} - u_{i,j,k}^{n} = \frac{\Delta t}{(\Delta x)^2}\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1} + u_{i,j-1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j+1,k}^{n}\right.$$

$$\left. + u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}\right)$$

Let $r = \dfrac{\Delta t}{(\Delta x)^2}$ we get

$$u_{i,j,k}^{n+1} - u_{i,j,k}^{n} = r\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}\right) + r\left(u_{i,j-1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j+1,k}^{n}\right)$$

$$+ r\left(u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}\right)$$

And

$$u_{i,j,k}^{n+1} - r\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}\right) = u_{i,j,k}^{n} + r\left(u_{i,j-1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j+1,k}^{n}\right)$$
$$+ r\left(u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}\right)$$

We simplify and rearrange the above equation and we get

$$-ru_{i-1,j,k}^{n+1} + (1+2r)u_{i,j,k}^{n+1} - ru_{i+1,j,k}^{n+1} = (1-4r)u_{i,j,k}^{n} + r\left(u_{i,j-1,k}^{n} + u_{i,j+1,k}^{n}\right)$$
$$+ r\left(u_{i,j,k-1}^{n} + u_{i,j,k+1}^{n}\right) \qquad \ldots(9)$$

Also, we advance the solution from the (n+1)th plane to (n+2)th plane by replacing $\dfrac{\partial^2 u}{\partial y^2}$ by implicit finite difference approximation at (n+2)th plane. Similarly, $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial z^2}$ are replaced by an explicit finite difference approximation at the (n+1)th plane. With these approximation eqs.(1) in parabolic model can be written as follows:

$$\frac{u_{i,j,k}^{n+2} - u_{i,j,k}^{n+1}}{\Delta t} = \frac{u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}}{(\Delta x)^2} + \frac{u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}}{(\Delta y)^2}$$
$$+ \frac{u_{i,j,k-1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k+1}^{n+1}}{(\Delta z)^2} \qquad \ldots(10)$$

We set $\Delta x = \Delta y = \Delta z$ then, we have a square region and multiply eq.(10) by Δt then we get

$$u_{i,j,k}^{n+2} - u_{i,j,k}^{n+1} = \frac{\Delta t}{(\Delta x)^2}\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1} + u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}\right.$$
$$\left. + u_{i,j,k-1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k+1}^{n+1}\right)$$

Let $r = \dfrac{\Delta t}{(\Delta x)^2}$ we get

$$u_{i,j,k}^{n+2} - u_{i,j,k}^{n+1} = r\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}\right) + r\left(u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}\right)$$
$$+ r\left(u_{i,j,k-1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k+1}^{n+1}\right)$$

And

$$u_{i,j,k}^{n+2} - r\left(u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}\right) = u_{i,j,k}^{n+1} + r\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j+1,k}^{n+1}\right)$$
$$+ r\left(u_{i,j,k-1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k+1}^{n+1}\right)$$

We simplify and rearrange the above equation and we get

$$-ru_{i,j-1,k}^{n+2} + (1+2r)u_{i,j,k}^{n+2} - ru_{i,j+1,k}^{n+2} = (1-4r)u_{i,j,k}^{n+1} + r\left(u_{i-1,j,k}^{n+1} + u_{i+1,j,k}^{n+1}\right)$$
$$+ r\left(u_{i,j,k-1}^{n+1} + u_{i,j,k+1}^{n+1}\right) \qquad \ldots(11)$$

Now, we advance the solution from (n+2)th plane to (n+3)th plane by replacing $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$ with explicit finite difference approximation at (n+2)th plane then, $\dfrac{\partial^2 u}{\partial z^2}$ by an implicit finite difference approximation at the (n+3)th plane .

Then, eq.(1) in parabolic model becomes .

$$\frac{u_{i,j,k}^{n+3} - u_{i,j,k}^{n+2}}{\Delta t} = \frac{u_{i-1,j,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i+1,j,k}^{n+2}}{(\Delta x)^2} + \frac{u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}}{(\Delta y)^2}$$

$$+ \frac{u_{i,j,k-1}^{n+3} - 2u_{i,j,k}^{n+3} + u_{i,j,k+1}^{n+3}}{(\Delta z)^2} \qquad \dots(12)$$

Multiply eq.(12) by Δt when $\Delta x = \Delta y = \Delta z$ then, we have for a square region and we get

$$u_{i,j,k}^{n+3} - u_{i,j,k}^{n+2} = \frac{\Delta t}{(\Delta x)^2}\left(u_{i-1,j,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i+1,j,k}^{n+2} + u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}\right.$$

$$\left. + u_{i,j,k-1}^{n+3} - 2u_{i,j,k}^{n+3} + u_{i,j,k+1}^{n+3}\right)$$

Let $r = \dfrac{\Delta t}{(\Delta x)^2}$ we get

$$u_{i,j,k}^{n+3} - u_{i,j,k}^{n+2} = r\left(u_{i-1,j,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i+1,j,k}^{n+2}\right) + r\left(u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}\right)$$

$$+ r\left(u_{i,j,k-1}^{n+3} - 2u_{i,j,k}^{n+3} + u_{i,j,k+1}^{n+3}\right)$$

And

$$u_{i,j,k}^{n+3} - r\left(u_{i,j,k-1}^{n+3} - 2u_{i,j,k}^{n+3} + u_{i,j,k+1}^{n+3}\right) = u_{i,j,k}^{n+2} + r\left(u_{i-1,j,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i+1,j,k}^{n+2}\right)$$

$$+ r\left(u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i+1,j,k}^{n+2}\right)$$

We simplify and rearrange the above equation and we get

$$-ru_{i,j,k-1}^{n+3} + (1+2r)u_{i,j,k}^{n+3} - ru_{i,j,k+1}^{n+3} = (1-4r)u_{i,j,k}^{n+2} + r\left(u_{i-1,j,k}^{n+2} + u_{i+1,j,k}^{n+2}\right)$$

$$+ r\left(u_{i,j-1,k}^{n+2} + u_{i,j+1,k}^{n+2}\right) \qquad \dots(13)$$

Expressed from the above equations (9), (11) and (13) by the system AX = B

$$\begin{bmatrix} 1+2r & -r & \dots & \dots & & & & 0 & 0 \\ -r & 1+2r & -r & \dots & \dots & \dots & & 0 & 0 & 0 \\ 0 & -r & 1+2r & -r & & & & & & \vdots \\ 0 & 0 & -r & 1+2r & -r & & & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \dots & \dots & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \dots & \dots & \dots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & -r & 1+2r & -r & 0 \\ 0 & 0 & & & \dots & \dots & \dots & -r & 1+2r & -r \\ 0 & 0 & 0 & & \dots & \dots & \dots & & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_{2,j,k}^{n+1} \\ u_{3,j,k}^{n+1} \\ u_{4,j,k}^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_{m1-2,j,k}^{n+1} \\ u_{m1-1,j,k}^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} (1-4r)u^n_{2,j,k} + r\left[u^n_{2,j-1,k} + u^n_{2,j+1,k} + u^n_{2,j,k-1} + u^n_{2,j,k+1}\right] \\ (1-4r)u^n_{3,j,k} + r\left[u^n_{3,j-1,k} + u^n_{3,j+1,k} + u^n_{3,j,k-1} + u^n_{3,j,k+1}\right] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ (1-4r)u^n_{m1-1,j,k} + r\left[u^n_{m1-1,j-1,k} + u^n_{m1-1,j+1,k} + u^n_{m1-1,j,k-1} + u^n_{m1-1,j,k+1}\right] \end{bmatrix}$$

Similarly, applying the above procedure with remainder of equations

These systems are of a tridiagonal linear system of equations and can be solved by the Gauss elimination.

4. Numerical Stability

There two methods, we used here one including the effect of boundary are conditions and the other excluding the effect of boundary conditions which are used to investigate stability. Both methods are attributed to John von Neumann. These approaches are Fourier and matrix methods. Fourier method, the primary observation in the Fourier method is that the numerical scheme is linear and therefore it will have solution in the form $u(x,t) = \lambda^t e^{i\alpha x}$.

Thus, numerical scheme is stable provided $|\lambda| < 1$ and unstable whenever $|\lambda| > 1$ [14].

4.1 Stability Analysis of ADE Method

The Von-Neumann method has been used to study the stability analysis of Parabolic model in three dimensions.

We can apply this method by substituting the solution in finite difference method at the time t by $u^n_{i,j,k} = \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z}$, when $\alpha, \beta, \gamma > 0$ and $m = \sqrt{-1}$ [5,8].

To apply von-Neumann on eq.(1) we have to linearlize the problem and from finite difference scheme for eq.(1)

$$u^{n+1}_{i,j,k} = (1-6r)u^n_{i,j,k} + r\left(u^n_{i-1,j,k} + u^n_{i+1,j,k} + u^n_{i,j-1,k} + u^n_{i,j+1,k} + u^n_{i,j,k-1} + u^n_{i,j,k+1}\right) \qquad \text{…(14)}$$

Where $\Delta x = \Delta y = \Delta z$ and $r = \dfrac{\Delta t}{(\Delta x)^2}$

We assume $u^n_{i,j,k} = \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z}$

Substituting in eq.(14) then, we have

$$\psi(t+\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z} = (1-6r)\psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z} + r\left(\psi(t)e^{m\alpha(x-\Delta x)}e^{m\beta y}e^{m\gamma z} + \psi(t)e^{m\alpha(x+\Delta x)}e^{m\beta y}e^{m\gamma z}\right.$$
$$+ \psi(t)e^{m\alpha x}e^{m\beta(y-\Delta y)}e^{m\gamma z} + \psi(t)e^{m\alpha x}e^{m\beta(y+\Delta y)}e^{m\gamma z} + \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma(z-\Delta z)} + \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma(z+\Delta z)}\left.\right)$$

Or

$$\frac{\psi(t+\Delta t)}{\psi(t)} = 1 + r\left(e^{-m\alpha\Delta x} + e^{m\alpha\Delta x} + e^{-m\beta\Delta y} + e^{m\beta\Delta y} + e^{-m\gamma\Delta z} + e^{m\gamma\Delta z} - 6\right)$$

$$\frac{\psi(t+\Delta t)}{\psi(t)} = \xi = 1 - 4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right) + \sin^2\left(\frac{\beta\Delta y}{2}\right) + \sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]$$

Where ξ is the amplification factor , for stable situation we need $|\xi| \leq 1$ and hence we have $-1 \leq 1 - 4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right) + \sin^2\left(\frac{\beta\Delta y}{2}\right) + \sin^2\left(\frac{\gamma\Delta z}{2}\right)\right] \leq 1$

Considering the left-side inequality (as the right-side inequality is always true), We have

$$-1 \leq 1 - 4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right) + \sin^2\left(\frac{\beta\Delta y}{2}\right) + \sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]$$

For some α, β and γ , $\sin^2\left(\frac{\alpha\Delta x}{2}\right), \sin^2\left(\frac{\beta\Delta y}{2}\right)$ and $\sin^2\left(\frac{\gamma\Delta z}{2}\right)$ are unity. Hence, we have $-1 \leq 1 - 4r(3)$, $r \leq \frac{1}{6}$ therefore $0 \leq r \leq \frac{1}{6}$

This is the condition for stability, in a square region $\Delta x = \Delta y = \Delta z$ when we use ADE method for eq.(1).

Thus the ADE method for eq.(1) is conditionally stable.

4.2 Stability Analysis of ADI Method

The ADI finite difference method from eq.(9)
Assuming
$$u_{i,j,k}^n = \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z} \qquad \qquad …(15)$$
And substitute (15) in (9) we have
$$-r\psi(t+\Delta t)e^{m\alpha(x-\Delta x)}e^{m\beta y}e^{m\gamma z} + (1+2r)\psi(t+\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z} - r\psi(t+\Delta t)e^{m\alpha(x+\Delta x)} =$$
$$(1-4r)\psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z} + r\left(\psi(t)e^{m\alpha x}e^{m\beta(y-\Delta y)}e^{m\gamma z} + \psi(t)e^{m\alpha x}e^{m\beta(y+\Delta y)}e^{m\gamma z}\right.$$
$$\left. + \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma(z-\Delta z)} + \psi(t)e^{m\alpha x}e^{m\beta y}e^{m\gamma(z+\Delta z)}\right)$$

Dividing by $e^{m\alpha x}e^{m\beta y}e^{m\gamma z}$ i.e.
$$\left[-re^{-m\alpha\Delta x} + (1+2r) - re^{m\alpha\Delta x}\right]\psi(t+\Delta t) = (1-4r)\psi(t) + \left[re^{-m\beta\Delta y} + re^{m\beta\Delta y} + re^{-m\gamma\Delta z}\right.$$
$$\left. + re^{m\gamma\Delta z}\right]\psi(t)$$

i.e.
$$\frac{\psi(t+\Delta t)}{\psi(t)}\left[-re^{-m\alpha\Delta x} + (1+2r) - re^{m\alpha\Delta x}\right] = (1-4r) + \left(re^{-m\beta\Delta y} + re^{m\beta\Delta y} + re^{-m\gamma\Delta z} + re^{m\gamma\Delta z}\right)$$

By using Euler formula
$$e^{m\alpha\Delta x} = \cos(\alpha\Delta x) + m\sin(\alpha\Delta x)$$
$$e^{-m\alpha\Delta x} = \cos(\alpha\Delta x) - m\sin(\alpha\Delta x)$$
$$e^{m\beta\Delta y} = \cos(\beta\Delta y) + m\sin(\beta\Delta y)$$
$$e^{-m\beta\Delta y} = \cos(\beta\Delta y) - m\sin(\beta\Delta y)$$
$$e^{m\gamma\Delta z} = \cos(\gamma\Delta z) + m\sin(\gamma\Delta z)$$
$$e^{-m\gamma\Delta z} = \cos(\gamma\Delta z) - m\sin(\gamma\Delta z)$$
Substituting in the above eq. we get
$$\frac{\psi(t+\Delta t)}{\psi(t)}\left[1 + 2r - 2r\cos(\alpha\Delta x)\right] = 1 - 4r + 2r\cos(\beta\Delta y) + 2r\cos(\gamma\Delta z)$$

i.e.

$$\frac{\psi(t+\Delta t)}{\psi(t)}\left[1+2r-2r\left(1-2\sin^2\left(\frac{\alpha\Delta x}{2}\right)\right)\right]=$$

$$1-4r+2r\left(1-2\sin^2\left(\frac{\beta\Delta y}{2}\right)\right)+2r\left(1-2\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right)$$

$$\Longrightarrow\quad \frac{\psi(t+\Delta t)}{\psi(t)}\left[1+4r\sin^2\left(\frac{\alpha\Delta x}{2}\right)\right]=1-4r\left[\sin^2\left(\frac{\beta\Delta y}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]$$

$$\frac{\psi(t+\Delta t)}{\psi(t)}=\frac{1-4r\left[\sin^2\left(\frac{\beta\Delta y}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\alpha\Delta x}{2}\right)}$$

Then, $\xi_I=\dfrac{1-4r\left[\sin^2\left(\frac{\beta\Delta y}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\alpha\Delta x}{2}\right)}$

Also from eq.(11) we assuming $u_{i,j,k}^{n+2}=\psi(t+2\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z}$ and

$$u_{i,j,k}^{n+1}=\psi(t+\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z} \qquad\qquad …(16)$$

Substituting (16) in (11) we have

$$-r\psi(t+2\Delta t)e^{m\alpha x}e^{m\beta(y-\Delta y)}e^{m\gamma z}+(1+2r)\psi(t+2\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z}-r\psi(t+2\Delta t)$$

$$e^{m\alpha x}e^{m\beta(y+\Delta y)}e^{m\gamma z}=(1-4r)\psi(t+\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma z}+r\left[\psi(t+\Delta t)e^{m\alpha(x-\Delta x)}e^{m\beta y}e^{m\gamma z}\right.$$

$$\left.+\psi(t+\Delta t)e^{m\alpha(x+\Delta x)}e^{m\beta y}e^{m\gamma z}+\psi(t+\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma(z-\Delta z)}+\psi(t+\Delta t)e^{m\alpha x}e^{m\beta y}e^{m\gamma(z+\Delta z)}\right]$$

i.e.

$$-r\psi(t+2\Delta t)e^{-m\beta\Delta y}+(1+2r)\psi(t+2\Delta t)-r\psi(t+2\Delta t)e^{m\beta\Delta y}=(1-4r)\psi(t+\Delta t)$$

$$+r\left[\psi(t+\Delta t)e^{-m\alpha\Delta x}+\psi(t+\Delta t)e^{m\alpha\Delta x}+\psi(t+\Delta t)e^{-m\gamma\Delta z}+\psi(t+\Delta t)e^{m\gamma\Delta z}\right]$$

i.e.

$$\frac{\psi(t+2\Delta t)}{\psi(t+\Delta t)}\left[-re^{-m\beta\Delta y}+1+2r-re^{m\beta\Delta y}\right]=(1-4r)+r\left[e^{-m\alpha\Delta x}+e^{m\alpha\Delta x}+e^{-m\gamma\Delta z}+e^{m\gamma\Delta z}\right]$$

By using Euler formula as previously

$$\frac{\psi(t+2\Delta t)}{\psi(t+\Delta t)}\left[1+2r-2r\cos(\beta\Delta y)\right]=(1-4r)+2r\cos(\alpha\Delta x)+2r\cos(\gamma\Delta z)$$

i.e.

$$\frac{\psi(t+2\Delta t)}{\psi(t+\Delta t)}\left[1+2r-2r\left(1-2\sin^2\left(\frac{\beta\Delta y}{2}\right)\right)\right]=1-4r+2r\left(1-2\sin^2\left(\frac{\alpha\Delta x}{2}\right)\right)$$

$$+2r\left(1-2\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right)$$

$$\frac{\psi(t+2\Delta t)}{\psi(t+\Delta t)}\left[1+4r\sin^2\left(\frac{\beta\Delta y}{2}\right)\right]=1-4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]$$

i.e.

$$\frac{\psi(t+2\Delta t)}{\psi(t+\Delta t)} = \frac{1-4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\beta\Delta y}{2}\right)}$$

Then $\xi_{II} = \dfrac{1-4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\beta\Delta y}{2}\right)}$

Similarity from eq. (13) we obtain

$$\xi_{III} = \frac{1-4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\beta\Delta y}{2}\right)\right]}{1+4r\sin^2\left(\frac{\gamma\Delta z}{2}\right)}$$

Thus, we found that the amplification factors are

$$\xi_I = \frac{1-4r\left[\sin^2\left(\frac{\beta\Delta y}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\alpha\Delta x}{2}\right)} \qquad \ldots(17)$$

$$\xi_{II} = \frac{1-4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\beta\Delta y}{2}\right)} \qquad \ldots(18)$$

$$\xi_{III} = \frac{1-4r\left[\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\beta\Delta y}{2}\right)\right]}{1+4r\sin^2\left(\frac{\gamma\Delta z}{2}\right)} \qquad \ldots(19)$$

Where ξ_I, ξ_{II} and ξ_{III} stand for the *I* plane, *II* plane and *III* plane. However, in either form unconditional stability is lost.

Furthermore, the combined three-levels have the form :

$\xi_{ADI} = \xi_I \cdot \xi_{II} \cdot \xi_{III} =$

$$\frac{\left[1-4r\left(\sin^2\left(\frac{\beta\Delta y}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right)\right]\left[1-4r\left(\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right)\right]\left[1-4r\left(\sin^2\left(\frac{\alpha\Delta x}{2}\right)+\sin^2\left(\frac{\beta\Delta y}{2}\right)\right)\right]}{\left[1+4r\sin^2\left(\frac{\alpha\Delta x}{2}\right)\right]\left[1+4r\sin^2\left(\frac{\beta\Delta y}{2}\right)\right]\left[1+4r\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]} \qquad \ldots(20)$$

     A careful analysis of (20) shows that there is a finite stability bound. Each individual equation is conditionally stable by itself, we have that stable provided

$\left|\xi_{ADI}\right|\le 1$ so $\left|\xi_{I}\cdot\xi_{II}\cdot\xi_{III}\right|\le 1$ this yields: $\left|\xi_{I}\right|\cdot\left|\xi_{II}\right|\cdot\left|\xi_{III}\right|\le 1$. Let us consider the cases $\left|\xi_{I}\right|\le 1$, $\left|\xi_{II}\right|\le 1$ and $\left|\xi_{III}\right|\le 1$. We will show the values of r which satisfy the condition $\left|\xi_{I}\right|\le 1$ in (17)

$$\left|\frac{1-4r\left[\sin^2\left(\frac{\beta\Delta y}{2}\right)+\sin^2\left(\frac{\gamma\Delta z}{2}\right)\right]}{1+4r\sin^2\left(\frac{\alpha\Delta x}{2}\right)}\right|\le 1$$

For some values α,β and γ we have $\sin^2\left(\frac{\alpha\Delta x}{2}\right),\sin^2\left(\frac{\beta\Delta y}{2}\right)$ and $\sin^2\left(\frac{\gamma\Delta z}{2}\right)$ are unity. Hence, we have $-1\le\frac{1-4r(2)}{1+4r}\le 1$.

Considering the left-side inequality (as the right-side inequality is always true), we have $-1\le\frac{1-8r}{1+4r}$ i.e. $-1-4r\le 1-8r$. We get $r\le\frac{1}{2}$ therefore, $0\le r_I\le\frac{1}{2}$.

Similarly, applying the above procedure with ξ_{II} in (18) and ξ_{III} in (19), we obtain that $0\le r_{II}\le\frac{1}{2}$ and $0\le r_{III}\le\frac{1}{2}$; this shows that the ADI method is conditionally stable in three-dimensional problem. Therefore, the combined three-levels are conditionally stable [6, 17].

5. Numerical Results

Example (1) [9]:

We consider the initial and boundary value problem as follows :

$$\begin{array}{ll}
\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2} & (0<x,y,z<1;t>0) \\[2mm]
u(x,y,z,0)=\sin(x+y+z), & (0\le x,y,z\le 1) \\[1mm]
u(0,y,z,t)=\exp(-3t)\sin(y+z), & \\
u(1,y,z,t)=\exp(-3t)\sin(1+y+z), & (0\le y,z\le 1;t\ge 0) \\
u(x,0,z,t)=\exp(-3t)\sin(x+z), & \\
u(x,1,z,t)=\exp(-3t)\sin(x+1+z), & (0\le x,z\le 1;t\ge 0) \\
u(x,y,0,t)=\exp(-3t)\sin(x+y), & (0\le x,y\le 1;t\ge 0) \\
u(x,y,1,t)=\exp(-3t)\sin(x+y+1), &
\end{array}\right\}\quad\ldots(21)$$

By using the numerical methods such as ADE method and ADI method of (21), we take the parameters $\Delta x=\Delta y=\Delta z=\frac{1}{10}$ and $\Delta t=r(\Delta x)^2$ for convenience using the exact solution of (21) $u(x,y,z,t)=\exp(-3t)\sin(x+y+z)$. Also, we compute the stability of each of the above methods and we conclude that the ADE method is conditionally stable where, $r\le\frac{1}{6}$ and ADI methods are also conditionally stable where, $r\le\frac{1}{2}$ is compared between them and with the exact solution.

Example (2) [16]:

We solve the following initial-boundary value problem :

PDE
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} \qquad (0 \le x, y, z \le \pi; t \ge 0)$$

$$u(0, y, z, t) = u(\pi, y, z, t) = 0,$$

BC $\quad u(x, 0, z, t) = u(x, \pi, z, t) = 0, \qquad (0 \le x, y, z \le \pi; t \ge 0)$ ...(22)

$$u(x, y, 0, t) = u(x, y, \pi, t) = 0,$$

IC $\quad u(x, y, z, 0) = 2\sin x \sin y \sin z \qquad (0 \le x, y, z \le \pi)$

We use the numerical methods such as ADE method and ADI methods of (22). We take the parameters $\Delta x = \Delta y = \Delta z = \dfrac{\pi}{10}$ and $\Delta t = r(\Delta x)^2$ for convenience by using the exact solution of (22) $u(x, y, z, t) = 2\exp(-3t)\sin x \sin y \sin z$. Also, we compute the stability of each of the above methods and we conclude that the ADE method is conditionally stable where, $r \le \dfrac{1}{6}$ and ADI methods are also conditionally stable where, $r \le \dfrac{1}{2}$ is compared between them and with the exact solution.

Table (1) with respect to example (1) contains the numerical solution of the Parabolic equation in three dimensions by using the above two methods with space step size $\Delta x = \Delta y = \Delta z = 0.1$ and time step size $\Delta t = r(\Delta x)^2$ where $r = \dfrac{1}{8}$. Also, we present comparison figures(3) for values of concentration u by the methods.

Table (2) with respect to example (1) contains the relative error comparison of ADE method with exact solution and ADI method with exact solution of the Parabolic equation in three dimensions at space step size $\Delta x = \Delta y = \Delta z = 0.1$ and time step size $\Delta t = r(\Delta x)^2$ where $r = \dfrac{1}{8}$.

Table (3) with respect to example (2) contains the numerical solution of the Parabolic equation in three dimensions by using the above two methods with space step size $\Delta x = \Delta y = \Delta z = \dfrac{\pi}{10}$ and time step size $\Delta t = r(\Delta x)^2$ where $r = \dfrac{1}{8}$. Also, we present comparison figures (4) for values of concentration u by the methods.

Table (4) with respect to example (2) contains the relative error comparison of ADE method with Exact solution and ADI method with Exact solution of the Parabolic equation in three dimensions at space step size $\Delta x = \Delta y = \Delta z = \dfrac{\pi}{10}$ and time step size $\Delta t = r(\Delta x)^2$ where $r = \dfrac{1}{8}$.

Table (1): is a comparison between the two methods ADE and ADI with the exact solution for the values of concentration u that are computed at space step size $\Delta x = \Delta y = \Delta z = 0.1$, $r = \dfrac{1}{8}$ and $\Delta t = r(\Delta x)^2$.

Point (i,j,k,n)	Exact Solution	ADE Method	ADI Method

(4,6,2,1)	0.783326909627483	0.783326909627483	0.783326909627483
(3,7,5,1)	0.932039085967226	0.932039085967226	0.932039085967226
(5,2,2,2)	0.562529029304166	0.562526828039469	0.562529548231699
(6,2,3,2)	0.714671043169683	0.714668246549321	0.714671648939347
(6,2,2,3)	0.639404127994110	0.639399674141941	0.639405199732211
(6,2,3,3)	0.711996045513164	0.711990787154908	0.711997319914961
(6,2,2,4)	0.637010852709908	0.637004824033134	0.637012521041193
(6,10,10,4)	0.737363041234928	0.737356115923518	0.737364954625562
(6,2,2,5)	0.634626535401321	0.634619187059868	0.634628924534946
(6,10,10,5)	0.734603107939522	0.734594692250449	0.734605835528647
(6,2,2,6)	0.632251142538846	0.632242661045841	0.632254410725571
(6,10,10,6)	0.731853505012428	0.731843817621965	0.731857219481380
(5,2,2,7)	0.552079876933901	0.552071453706244	0.552084926396277
(6,10,10,7)	0.729114193787308	0.729103400388248	0.729119086472935
(6,2,2,8)	0.627526996661259	0.627516644524798	0.627532557431839
(6,10,10,8)	0.726385135742553	0.726373366120191	0.726391403002175
(6,2,2,9)	0.625178177212767	0.625167035887518	0.625185154103224
(6,10,10,9)	0.723666292500741	0.723653651157792	0.723674125798434
(6,2,2,10)	0.622838149342695	0.622826293503129	0.622846711469224
(6,10,10,10)	0.720957625828092	0.720944198872030	0.720967204783124
(6,2,5,11)	0.810500155241008	0.810478226150170	0.810520922412845
(6,10,7,11)	0.875830205566404	0.875806720312609	0.875852427764604

Table (2): is a relative error comparison of the methods ADE with the exact solution and ADI with the exact solution at space size $\Delta x = \Delta y = \Delta z = 0.1$, $r = \dfrac{1}{8}$ and $\Delta t = r(\Delta x)^2$.

Point (*i,j,k,n*)	Error of ADE Method With Exact solution	Error of ADI Method With Exact solution
(4,6,2,1)	**0**	**0**
(3,7,5,1)	**0**	**0**
(5,2,2,2)	**3.913157512402389e-006**	**9.224902280013131e-007**
(6,2,3,2)	**3.913157512452229e-006**	**8.476202719776912e-007**
(6,2,2,3)	**6.965629362591201e-006**	**1.676151364665402e-006**
(6,2,3,3)	**7.385375648834013e-006**	**1.789900104721094e-006**
(6,2,2,4)	**9.464009520808871e-006**	**2.618999782815910e-006**
(6,10,10,4)	**9.391996917166388e-006**	**2.594909870172700e-006**

(6,2,2,5)	1.157900126010213e-005	3.764629260407600e-006
(6,10,10,5)	1.145610327807386e-005	3.713010598019793e-006
(6,2,2,6)	1.341475314782071e-005	5.169127432865513e-006
(6,10,10,6)	1.323678905153516e-005	5.075426880869180e-006
(5,2,2,7)	1.525726259845934e-005	9.146253261886166e-006
(6,10,10,7)	1.480344114991009e-005	6.710451763707669e-006
(6,2,2,8)	1.649671889199063e-005	8.861404545197716e-006
(6,10,10,8)	1.620300551689009e-005	8.628011936088742e-006
(6,2,2,9)	1.782103991384771e-005	1.115984324363079e-005
(6,10,10,9)	1.746847003942847e-005	1.082446118328699e-005
(6,2,2,10)	1.903518527073746e-005	1.374695261919981e-005
(6,10,10,10)	1.862377979151310e-005	1.328643277883371e-005
(6,2,5,11)	2.705624508090585e-004	2.562266238049262e-005
(6,10,7,11)	2.681484795339333e-004	2.537272414081325e-005

Point (i,j,k,n)	Exact Solution	ADE Method	ADI Method
(4,3,2,1)	0.293892626146237	0.293892626146237	0.293892626146237
(7,5,3,1)	1.063313510440050	1.063313510440050	1.063313510440050
(2,10,2,2)	0.056872642518016	0.056850621398073	0.056876808436968
(10,10,3,2)	0.108178194531369	0.108136307872132	0.108186118580100
(10,2,2,3)	0.054806204572748	0.054763770801568	0.054814233971712
(2,2,3,3)	0.104247595984634	0.104166882155451	0.104262868809048
(10,3,2,4)	0.100459813695858	0.100343164580931	0.100481891345830
(2,9,3,4)	0.191085920882486	0.190864041080742	0.191127915068227
(10,3,2,5)	0.096809658510437	0.096659806550511	0.096838026842775
(3,10,3,5)	0.184142913133320	0.183857877767385	0.184196872907972
(10,9,2,6)	0.093292129819009	0.093111655800397	0.093326303050237
(3,2,3,6)	0.177452175966845	0.177108893983998	0.177517177315329
(10,3,2,7)	0.089902408706757	0.089693749194099	0.089941928031700
(10,2,3,7)	0.089902408706757	0.089693749194099	0.089941928031700
(9,2,2,8)	0.086635851351631	0.086401305780018	0.086680283624917
(10,9,3,8)	0.164791181945494	0.164345049756992	0.164875697151579
(4,9,2,9)	0.218574376262447	0.217898236106089	0.218702493624445
(2,10,3,9)	0.083487982662447	0.083229720103876	0.083536919140181
(9,9,2,10)	0.153033534241174	0.152501067405843	0.153134451112478

(10,3,3,10)	0.153033534241174	0.152501067405843	0.153134451112478
(2,7,2,11)	0.125448145928949	0.124963255459567	0.125540066944623
(2,3,3,11)	0.147473140208941	0.146903117275181	0.147581199843718

Table (3): is a comparison between the two methods ADE and ADI with the exact solution for values of concentration u that are computed at space size $\Delta x = \Delta y = \Delta z = \dfrac{\pi}{10}$, $r = \dfrac{1}{8}$ and $\Delta t = r(\Delta x)^2$. Table (4): is a relative error comparison of the methods ADE with the exact solution and ADI with the exact solution at space size $\Delta x = \Delta y = \Delta z = \dfrac{\pi}{10}$, $r = \dfrac{1}{8}$ and $\Delta t = r(\Delta x)^2$.

Point (i,j,k,n)	Error of ADE Method With Exact solution	Error of ADI Method With Exact solution
(4,3,2,1)	0	0
(7,5,3,1)	0	0
(2,10,2,2)	3.872005760313238e-004	7.324996285312854e-005
(10,10,3,2)	3.872005760313093e-004	7.324996285326660e-005
(10,2,2,3)	7.742512277767201e-004	1.465052912632865e-004
(2,2,3,3)	7.742512277766800e-004	1.465052912631781e-004
(10,3,2,4)	1.161152013286873e-003	2.197659856237270e-004
(2,9,3,4)	1.161152013286785e-003	2.197659856234035e-004
(10,3,2,5)	1.547902990589807e-003	2.930320463268755e-004
(3,10,3,5)	1.547902990589854e-003	2.930320463266187e-004
(10,9,2,6)	1.934504217691750e-003	3.663034737665745e-004
(3,2,3,6)	1.934504217691776e-003	3.663034737660562e-004
(10,3,2,7)	2.320955752575711e-003	4.395802683355000e-004
(10,2,3,7)	2.320955752575557e-003	4.395802683355000e-004
(9,2,2,8)	2.707257653202280e-003	5.128624304269712e-004
(10,9,3,8)	2.707257653202869e-003	5.128624304265734e-004
(4,9,2,9)	3.093409977511271e-003	5.861499604335709e-004
(2,10,3,9)	3.093409977511389e-003	5.861499604335289e-004
(9,9,2,10)	3.479412783417774e-003	6.594428587494726e-004
(10,3,3,10)	3.479412783417955e-003	6.594428587498353e-004
(2,7,2,11)	3.865266128815142e-003	7.327411257671027e-004
(2,3,3,11)	3.865266128814974e-003	7.327411257673203e-004

(a)
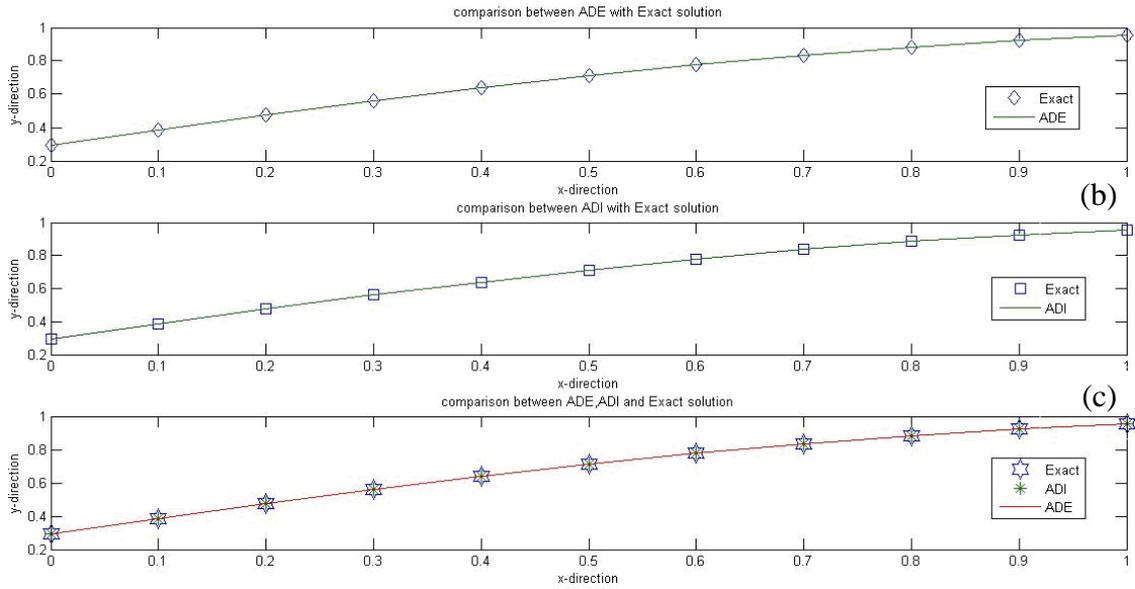
Figure (3). (a) The comparison between ADE method with exact Solution, (b) The comparison between ADI method with exact solution, (c) The comparison between ADE, ADI and exact solution, all for finding the concentration values *u(3,:,2,3)* at cubic *n=3*, level *k=2*, row *i=3* and for all columns *j* when $\Delta x = \Delta y = \Delta z = 0.1$, $\Delta t = r(\Delta x)^2$ and $r = \dfrac{1}{8}$.
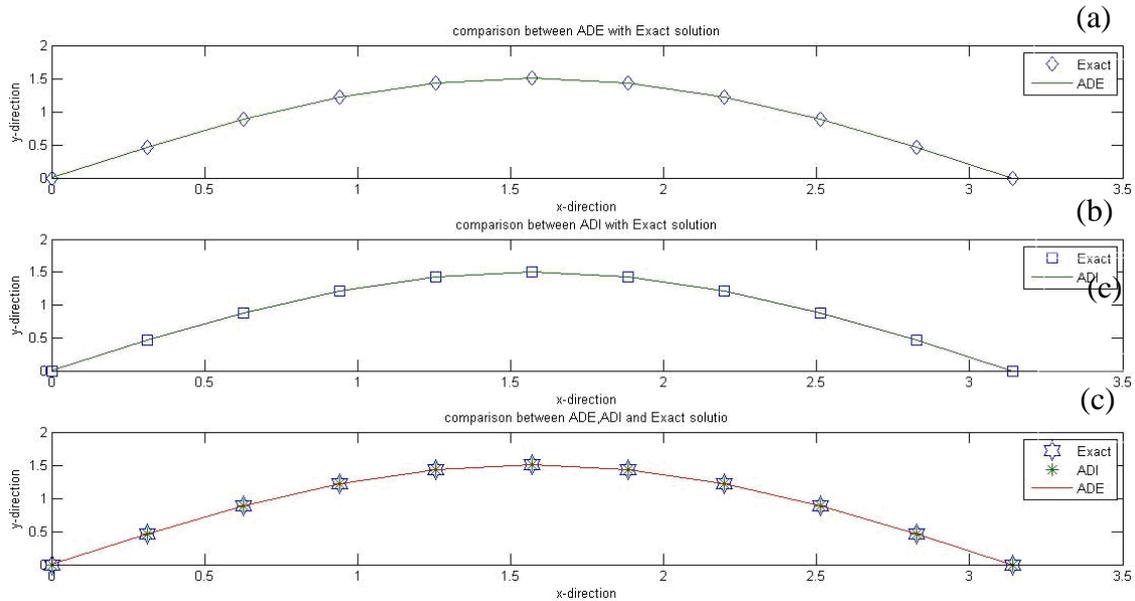
Figure (4). (a) The comparison between ADE method with exact Solution, (b) The comparison between ADI method with exact solution, (c) The comparison between ADE, ADI and exact solution, all for finding the concentration values *u(6,:,4,3)* at cubic *n=3*, level *k=4*, row *i=6* and for all columns *j* when $\Delta x = \Delta y = \Delta z = \dfrac{\pi}{10}$, $\Delta t = r(\Delta x)^2$ and $r = \dfrac{1}{8}$.

numerical solution by using ADE and ADI methods
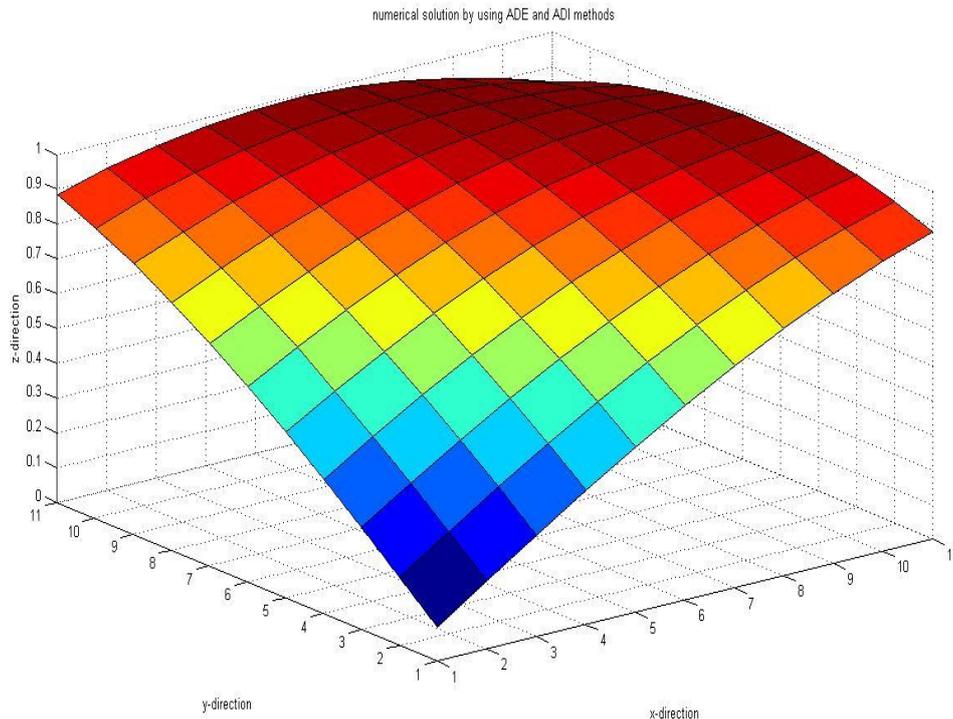
Figure (5). with respect to example (1) shows that the numerical solution by using ADE and ADI methods in 3-D figure of concentration values *u(:,:,2,3)* at cubic *n*=3, level *k*=2, for all rows *i* and for all columns *j* when $\Delta x = \Delta y = \Delta z = 0.1$, $\Delta t = r(\Delta x)^2$ and $r = \frac{1}{8}$.

numerical solution by using ADE and ADI methods
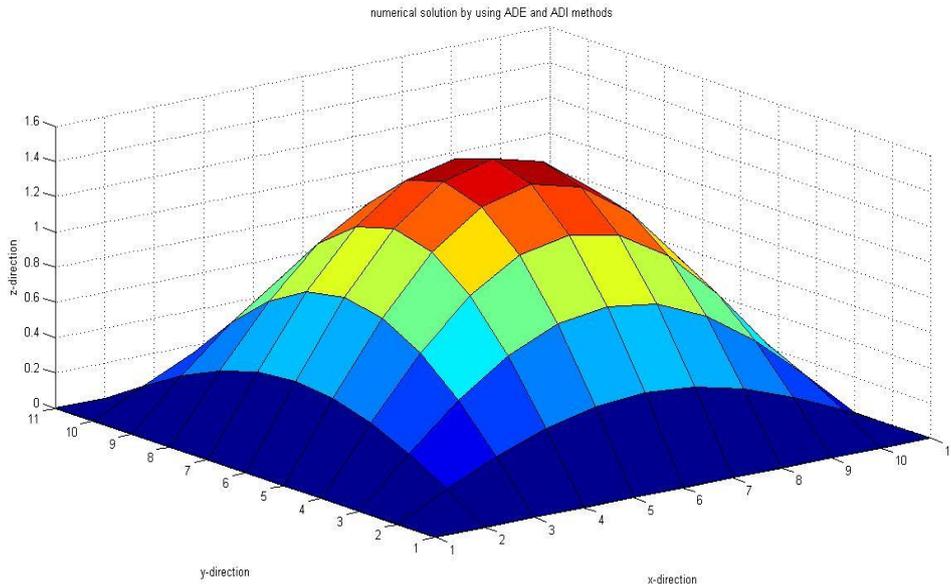
Figure (5). with respect to example (2) shows that the numerical solution by using ADE and ADI methods in 3-D figure of concentration values *u(:,:,4,3)* at cubic *n*=3 ,level *k*=4, for all rows *i* and for all columns *j* when $\Delta x = \Delta y = \Delta z = \frac{\pi}{10}$, $\Delta t = r(\Delta x)^2$ and $r = \frac{1}{8}$.

6. Conclusion

Through our study for numerical stability to the ADE method for PDEs in three-dimensional, we conclude that it is conditionally stable such as in two dimensions

equations, but the ADI method is lost the unconditionally stable that is in two dimensions. Also, we saw that from the numerical results the ADI method is better than the ADE method and its results are nearest to the exact solution compared with the results of ADE method.

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