On Generalized Simple Singular AP-Injective Rings

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ABSTRACT

A ring R is said to be generalized right simple singular AP-injective, if for any maximal essential right ideal M of R and for any $b\in M$, bR/bM is AP-injective. We shall study the characterization and properties of this class of rings. Some interesting results on these rings are obtained. In particular, conditions under which generalized simple singular AP-injective rings are weakly regular rings, and Von Neumann regular rings.

Key word: AP-injective Rings, weakly continuous rings, socle of R, Von Neumann regular rings.

حول الحلقات البسيطة المنفردة المعممة وغامرة من النمط –AP

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الملخص

يقال للحلقة R بأنها حلقة بسيطة منفردة معممة وغامرة من النمط – AP إذا كان كل مثالي أعظمي أساسي أيمن M في R ولكل b∈M فان bR/bM غامر من النمط – AP. قمنا بدراسة مميزات وخواص هذا الصنف من الحلقات. بصورة عامة, ما هي الشروط للحلقة البسيطة المنفردة المعممة والغامرة من النمط –AP لكي تكون حلقة منتظمة بضعف وحلقة منتظمة حسب مفهوم فون نيومان.

الكلمات المفتاحية : الحلقات الغامرة من النمط – AP, الحلقات المستمرة بضعف, السكوكل لـ R , حلقات فون نيومان.

1. Introduction:

Throughout this paper, R is an associative ring with identity, and R-module is unital. For $a \in R$, r(a) and l(a) denote the right annihilator and the left annihilator of a, respectively. We write J(R), Y(R)(Z(R)), N(R) and Soc(R_R) for the Jacobson radical, the right (left) singular ideal, the set of nilpotent elements and right socle of R, respectively. $X \leq M$ denoted that X is a submodule of module M.

Recall that a ring R is called right **MC2-ring** if eRa=0 implies aRe=0, where a, $e^2 = e \in R$ and eR is minimal right ideal of R[8]. A ring R is **Von Neumann (weakly) regular** provided that for every $a\in R$ there exists $b\in R$ ($b\in RaR$) such that a=aba (a=ab resp.). Recall that a ring R is right (left) **weakly continuous** if J(R)=Y(R) (J(R) = Z (R)), R / J(R) is regular and idempotent can be left module J(R)[5]. Clearly every regular ring is right (left) weakly continuous. A ring R is called **zero commutative** (briefly ZC-ring)if ab=0 implies ba=0, $a,b \in R$[1]. A right R-module M is **principally injective** (briefly P-injective), if for any principal right ideal aR of R and any right R-homomorphism of aR into M can be extended to one of R into M[11]. The ring R is called right P-injective if R_R is P-injective.

2. Generalized Simple Singular AP-injective Rings

Recall that a module M_R with $S=End(M_R)$ is said to be **almost principally injective** (briefly AP-injective), if for any $a\in R$, there exists an S-submodule X_a of M such that $l_M(r_R(a))=Ma\oplus X_a$ as left S-module[6]. AP-injectivety has been studied by many authors (see [9,10]). Actually, Zhao Yu-e [12] investigated some properties of rings whose simple singular right R-module is AP-injective. Now, we give a generalized AP-injective.

Definition 2.1:

A ring R is called a **generalized right (left) simple singular AP-injective**, if for any maximal essential right (left) ideal M of R, any $b\in M$, bR/bM (Rb/Mb) is AP-injective.

The following lemma which is due to Zhao Yu-e [12], plays a central role in several of our proofs

Lemma 2.2:

Suppose M is a right R-module with $S=End(M_R)$. If $l_M r_R(a)=M_a\oplus X_a$, where X_a is left S-submodule of M_R. Set $f: aR\rightarrow M$ is a right R-homomorphis, then $f(a)= ma+x$ with $m\in M$, $x\in X_a$.

Lemma 2.3:

If M is a maximal right ideal of R and $r(a) \subseteq M$ with $a \in M$, then

1- $aR \neq aM$
2- $R/M \cong aR/aM$.

Proof:

(1) If aR = aM, then a = ay for some y in M, which implies that $1-y \in r(a) \subseteq M$, whence $1 \in M$, contradicting $M \neq R$.
(2) From (1) $aR \neq aM$, then the right R- homomorphism $g:R/M\rightarrow aR/aM$ is defined by $g(r+M) = ar+aM$ for all $r\in R$ implies that $R/M \cong aR/aM$. ■

We start this section with the following results.

Proposition 2.4:

Let R be generalized right simple singular AP-injective ring, then

1- $J(R) \cap Y(R) = 0$
2- $Soc(R_R) \cap Y(R) = 0$

Proof :

(1) Let $a\in J(R) \cap Y(R)$. If $a \neq 0$, then $r(a)\neq R$ and $RaR + r(a)$ is an essential right ideal of R. We shall prove that $RaR +r(a) = R$. If not, there exists a maximal essential right ideal M containing $RaR +r(a)$. Since $r(a)\subseteq M$ and $a \in M$, then by Lemma 2.3 $R/M \cong aR/aM$. Therefore, R/M is AP-injective and $l_{R/M}r(a)= (R/M)a \oplus X_a$, $X_a\leq R/M$. Let $f:aR\rightarrow R/M$ defined by $f(ar) = r +M$ for all $r \in R$. Note that f is a well-defined and by Lemma 2.2 $1 + M = f(a) = ba + M +x$, $b \in R$, $x \in X_a$. Hence $1- ba + M = x \in R/M \cap X_a = 0$, so $1-ba \in M$**.** Since $a\in J(R)$, then $ba\in J(R)\subseteq M$ and hence $1\in M$, which is a contradiction. Therefore $J(R) \cap Y(R) = 0$.
(2) Let $k\in Soc(R_R) \cap Y(R)$. If $k\neq 0$, then kR is a minimal right ideal and r(k) is an essential right ideal of R. Since every minimal one –sided ideal of R is either nilpotent or direct summand of R [8]. Thus, if $(kR)^2 \neq 0$, then kR is a direct summand and hence r(k) is also direct summand which is a contradiction. If

$(kR)^2 = 0$, then $k^2 = 0$ and $k \in r(k)$. But r(k) is maximal essential right ideal of R. Therefore, by Lemma 2.3 $R/r(k) \cong k(r(k))$. Hence, R/r(k) is AP-injective, so there exists $c \in R$ and $x \in X_a$ as a proof (1) such that $1-ck \in r(k)$. Since, $ck \in RkR \subseteq r(k)$, then $1 \in r(k)$. This is also contradiction, therefore $Soc(R_R) \cap Y(R) = 0$. ∎

Following [7], for a prime ideal P of a ring R, we put $O_P = \{a \in P: ab=0 \text{ for some } b \in R \setminus P\}$.

In general, O_P not subset of a prime ideal P. as the following example shows.

Example [2]:

Let R be a ring of 2×2 matrices over a field F. Then, $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a prime ideal of R. Let $a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \in R$. Then $ab = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $b \in R \setminus P$. Thus, $a \in O_P$, but. $a \notin P$ ∎

Theorem 2.5:

Let P be a prime ideal of a generalized right simple singular AP-injective ring with $O_P \subseteq P$, then P is maximal.

Proof :

We claim that RaR + P = R for $a \in R/P$. if not, there exists a maximal ideal M of R containing RaR+ P. Moreover, M is a maximal right ideal of R. Suppose not, then there exists a maximal right ideal K of R such that $M \subseteq K$. If K is not essential in R. Then K is a direct summand of R, so we can write K=r(e) for some $0 \neq e = e^2 \in R$. Then, ea=0, since $e \notin P$, then $a \in O_P \subseteq P$. Therefore, K must be essential right ideal of R.

Now, suppose that aR=aK, then a=ac for some $c \in K$ that implies a(1-c)= 0. Since, $a \notin P$, then $1-c \in O_P \subseteq P \subseteq K$ which is a contradiction. If $aR \neq aK$, the right R-homomorphism g: $R/K \to aR/aK$ is defined by g(b+K)=ab+aK for all $b \in R$ which implies that $R/K \cong aR/ak$**.** Therefore, R/K is AP-injective . Let $f: aR \to R/K$ be defined by f(ar)= r+K for all $r \in R$. So by Lemma 2.2 f(a)= ca+K+x, $x \in X_a$. Hence, 1-ca+K=$x \in R/K \square X_a = 0$, so $1-ca \in K$ whence $1 \in K$. Therefore, M is a maximal essential right ideal of R. So by the same method in the above proof P is a maximal of R. ∎

Recall that R is called **2-Primal** if its prime radical P(R) concedes with the set N(R) [7]. Kim and Kwak [3] showed that if R is a 2-primal, then $O_P \subseteq P$ for each prime ideal of R.

Corollary 2.6:

Let R be 2-primal generalized right simple singular AP-injective ring, then every prime ideal of R is maximal. ∎

Proposition 2.7:

Let R be ZC-generalized simple singular AP-injective rings, then for any $a,b \in R$ with ab=0, then r(a) + r(b) = R.

Proof:

Suppose that ab=0 and $r(a) + r(b) \neq R$. Then, there exists a maximal right ideal M containing r(a) + r(b). If M not essential, then there exists $0 \neq e = e^2 \in R$ such that M=r(e). Since $b \in r(a) \subseteq M = r(e) = l(e)$, then be=0 which implies that $e \in r(b) \subseteq M = r(e)$, so that $e = e^2 = 0$ which is a contradiction. Therefore, M must be essential.

Since, $r(a) \subseteq M$ and $a \in M$, then by Lemma 2.3 $R/M \cong aR/aM$. Therefore R / M is AP-injective. Let $f: aR \to R/M$ is defined by f(ar) = r+M for all $r \in R$. Note that f is well-defined and by Lemma 2.2 1+ M = f(a) = ca + M +x, c $\in$ R, x $\in$ X_a. Hence, 1 – ca + M = x$\in$R/M $\cap$ X_a=0, so 1-ca $\in$ M. Since, a $\in$ r(b) and R is ZC- ring, then ca $\in$ r(b) $\subseteq$ M whence 1$\in$M which is a contradiction. Therefore, r(a) + r(b) = R. ■

3. The Connection between Generalized Simple Singular AP-injective and Other Rings

In this section, we give the connection between Von Neumann regular rings and generalized simple singular AP-injective rings.

Theorem 3.1:

Let R be right MC2-generalized right simple singular AP-injective, then R is right weakly regular ring.

Proof:

We will show that RaR + r(a) = R for any a $\in$ R. Suppose that there exists b $\in$ R such that RbR + r(b) $\neq$ R. Then, there exists a maximal right ideal M of R containing RbR + r(b). If M not essential, then M is a direct summand of R. So, we can write M=eR for some $0 \neq e = e^2 \in$R. Thus, (1-e)Rb= 0, since R is MC2 and (1-e)R is minimal, then bR(1-e) = 0. Hence, (1-e) $\in$ r(b) $\subseteq$ M, so 1 $\in$ M. It is a contradiction. Therefore, M must be essential right ideal of R.

Since, r(a) $\subseteq$ M and a$\in$ M, then by Lemma 2.3 R/ M $\cong$ aR / aM. Therefore, R / M is AP-injective. Let f:bR$\rightarrow$R/M defined by f(br) = r +M for all r $\in$ R. Note that f is well-defined and by Lemma 2.2**,** 1+M=f(b) = cb + M +x, c $\in$ R, x $\in$ X_b. Hence, 1 – cb + M = x $\in$ R/M $\cap$ X_b = 0, so 1-cb $\in$ M. Since, cb $\in$RbR $\subseteq$ M, then 1 $\in$ M which is a contradiction. Therefore, that RaR + r(a) = R for all a $\in$ R. Hence, R is a right weakly regular ring. ■

Now, we shall prove the main results of this section.

Theorem 3.2:

Let R be a ring, then the following statements are equivalent:

(1) R is Von Neumann regular.

(2) R is generalized right simple singular AP-injective right weakly continuous.

Proof :

(1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Suppose that Y(R) $\neq$ 0. Then, there exists a non-zero element a $\in$ Y(R) such that a^2 =0. We claim that Y(R) + r(a) =R. If not, there exists a maximal essential right ideal M containing Y(R) + r(a). Since, r(a) $\subseteq$ M and a$\in$ M , then by Lemma 2.3 R/ M $\cong$ aR / aM. Therefore, R/M is AP-injective and $l_{R/M}$r(a)=(R/M)a $\oplus$ X_a, X_a $\leq$R/M. Let f:aR$\rightarrow$R/M be defined by f(ar) = r +M for all r $\in$ R. Note that f is well-defined and by Lemma 2.2**,** 1 + M = f(a) = ba + M +x, b $\in$ R, x $\in$ X_a. Hence, 1 – ba + M = x $\in$ R/M $\cap$ X_a = 0, so 1-ba $\in$ M. Since, a $\in$ Y(R) = J(R) implies that ca $\in$ J(R) $\subseteq$ M and 1 $\in$ M, which is a contradiction. Therefore, Y(R) + r(a) =R. Thus, we can write 1= c + d, for some c$\in$Y(R) and d $\in$ r(a) . Thus, a=ca and so (1-c)a = 0. Since c $\in$ Y(R) = J(R), 1-c is invertible. Thus a=0 contradicting a $\neq$ 0. Therefore, Y(R)=0. ■

Lemma 3.3: [4]

For any a $\in$ Cent(R), if a=ara for some r $\in$ R, then there exists b$\in$Cent(R) such that a=aba (where Cent(R) is the center of R).

Theorem 3.4:

R is right non-singular generalized right simple singular AP-injective, then Cent(R) is Von Neumann regular ring.

Proof:

First, we have to prove Cent(R) is reduced . Let $0 \neq a \in$ Cent(R) and $a^2=0$ implies that $a \in r(a)$ .If r(a) is essential, then $a \in Y(R)=0$ implies that a=0 .We are done . If r(a) not essential ,there exists a non-zero right ideal I in R such that $r(a) \cap I=0$.Then, $Ia \subseteq I \cap r(a)$ [$a \in$ Cent(R)] but $I \cap r(a)=0$ implies that Ia=0 and we get $I \subseteq l(a)=r(a)$ so I=0 contradiction. Therefore, a=0 ,so Cent(R) is a reduced ring . Now, we shall show that aR+r(a)=R for any $a \in$ Cent(R) .If not ,there exists a maximal right ideal M of R such that $aR+r(a) \subseteq M$ observe that M is an essential right ideal of R. If not, then M is a direct summamd of R . So, we can write M=r(e) for some $0 \neq e=e^2 \in R$. Since, $a \in M$ and $a \in$ Cent(R), ae=ea=0. Thus, $e \in r(a) \subseteq M=r(e)$, whence e=0 . It is a contradiction. Therefore, M must be an essential right ideal of R.

Since, $r(a) \subseteq M$ and $a \in M$, then by Lemma 2.3 $R/M \cong aR/aM$. Therefore, R / M is AP-injective. Let $f: aR \rightarrow R/M$ defined by f(ar) =r +M for all $r \in R$. Note that f is well-defined and by Lemma 2.2, 1+M= f(a) = ca + M +x, $c \in R$, $x \in X_a$. Hence, $1 - ca + M = x \in R/M \cap X_a = 0$, so $1-ca \in M$ since, $a \in$ cent(R), then $ca = ac \in M$, and hence $1 \in M$. Therefore, aR + r(a) = R for all $a \in$ cent(R) and so we have a = ara for some $r \in R$. Applying Lemma 3.3, Cent(R) is Von Neumann regular ring. ■

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