On Simple GP – Injective Modules

Mohammed Th. Youns
College of Engineering
University of Mosul

Najlaa T. Jassim
College of Basic Education
University of Mosul

Received on: 07/03/2012 **Accepted on: 19/04/2012**

ABSTRACT

In this paper, we study rings whose simple right R-module are GP-injective. We prove that ring whose simple right R-module is GP-injective it will be right $s\pi$-weakly regular ring. Also, proved that if R is N duo ring or R is NCI ring whose simple right R-module is GP-injective is S-weakly regular ring.

Keywords: Modules, weak regular rings, N duo ring , NCI ring.

حول مقاسات بسيطة من النمط – GP

محمد ذنون يونس **نجلاء طارق جاسم**
كلية الهندسة **كلية التربية الاساسية**
جامعة الموصل

تاريخ الاستلام: 2012/03/07 **تاريخ القبول: 2012/04/19**

الملخص

في هذا البحث درست الحلقات التي كل مقاس بسيط أيمن عليها يكون غامر من النمط -GP، لقد تم برهان تكون الحلقة منتظمة ضعيفة من النمط $s\pi$- اليمنى إذا كان كل مقاس بسيط أيمن عليها هو غامر من النمط -GP. كذلك تم برهان إذا كانت R هي حلقة N duo أو حلقة NCI والتي كل مقاس بسيط أيمن عليها هو غامر من النمط -GP فأن R هي حلقة منتظمة ضعيفة من النمط -S.

الكلمات المفتاحية : المقاسات ، حلقات منتظمة ضعيفة، حلقة N duo أو حلقة NCI.

1- Introduction

Throughout in this paper, R is associative ring with identity and all modules are unitary. For a subset X of R, the left(right) annihilator of X in R is denoted by l(X)(r(X)). If X={a}, we usually abbreviate it to l(a)(r(a)). We write J(R),N(R), $N^*(R)$, $P(R)$ for the Jacobson radical, the set of nilpotent elements, the nil radical (that means the sum of all nil ideals), prime radical (that means the intersection of all prime ideals)respectively. $N_2(R) = \{a \in R / a^2 = 0\}$. A ring R is called NI if $N^*(R) = N(R)$ [9]. A ring R is 2-primal if $N(R) = P(R)$[2]. A ring R is said to be semiprimitive if J(R)=0 [1]. An element a in the ring R said to be right (left) weakly regular if $a \in aRaR (a \in RaRa)$ [12].

A right R-module M is called Generalized Principally injective (briefly, GP-injective) if for any $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of $a^n R$ into M extends to one of R into M [8]. Right GP-injective modules are called right YJ-injective modules by several authors [16].

2. Some Properties of Rings whose Simple Right R-module are GP-injective.

We give a different prove that proved by Kim, *et. al. in* [8].

Theorem 2.1

Let R be a ring whose every simple right R-module is GP-injective. Then R is semiprime.

Proof:

We shall show that is no nilpotent ideal in R, if not, suppose there exists$0 \neq a \in R$ with $(aR)^2 = 0$, $aRaR = 0$, that means $RaR \subseteq r(a)$, there exists a maximal right ideal M of R containing r(a), R/M is GP-injective. Hence, there exists an appositive integer n=1 such that $a \neq 0$ and any R-homomorphism of aR into R/M extends to one of R into R/M, we define$f\colon aR \to R/M$ such that $f(ar) = r + M$ where $r \in R$. We have to show that f is well defined R-homomorphism, let $ax = ay$ where $x, y \in R$, $a(x-y) = 0$, $(x-y) \in r(a) \subseteq M$, x+M=y+M, f(ax)=x+M=y+M=f(ay), f(ax)=f(ay),so f is well defined right R-homomorphism, since R/M is GP-injective, there exists $b + M \in R/M$ such that 1+M=f(a)=(b+M)(a+M)=ba+M, 1+M=ba+M, $1 - ba \in M$, since$ba \in RaR \subseteq M$, we get that $1 \in M$, which is a contradiction. Therefore, a=0. This shows that R issemiprime.

We give a different prove that is proved by Xue *in* [16].

Proposition 2.2

Let R be a ring whose simple right R-module is GP-injective. Then, R is semiprimitive.

Proof:

We shall show that $J(R) = 0$,if not, there exists $0 \neq a \in J(R)$,then either $Ra^nR + r(a^n) = R$, or not, if not, there exists a maximal right ideal M of R containing $Ra^nR + r(a^n)$,R/M is GP-injective, there exists a positive integer n and $a^n \neq 0$ such that any R-homomorphism of a^nR into R/M extends to one of R into R/M,Let$f\colon a^nR \to R/M$ such that $f(a^nr) = r + M$, where $r \in R$, we have to show that f is well defined, let $a^nx = a^ny$ where $x, y \in R, a^n(x-y) = 0, (x-y) \in r(a^n) \in M$, then (x-y)+M=M, x+M=y+M, f(a^nx)=x+M=y+M=f(a^ny), f(a^nx)=f(a^ny),so f is well defined right R-homomorphism, since R/M is GP-injective, there exists $b + M \in R/M$ such that 1+M=f(a^n)=(b+M)(a^n+M)=ba^n+M, 1+M=ba^n+M,$1 - ba^n \in M$, since$ba^n \in Ra^nR \subseteq M$,we get that $1 \in M$, which is a contradiction. That means $Ra^nR + r(a^n) = R$, in particular there exists $y, z \in R$ and $v \in r(a^n)$ such that $ya^nz + v = 1$, $a^nya^nz + a^nv = a^n, a^nya^nz = a^n, a^n(1 - ya^nz) = 0$, since $a \in J(R)$,so $ya^nz \in J(R)$, $1 - ya^nz$ is invertible, there exists $u \in R$,such that $(1 - ya^nz)u = 1, a^n = a^n(1 - ya^nz)u = 0u = 0$,must $a^n = 0$ which is a contradiction with $a^n \neq 0$. Therefore, a=0, so, J(R)=0.This shows that R is semiprimitive.

Corollary 2.3

Let R be a ring whose simple right R-module is GP-injective. Then, $N^*(R) = 0$.

Proof:

We shall show that $N^*(R) = 0,$ since $N^*(R)$ is the large nil ideal of R, It is clearly that J(R) containing every nil ideal, so $N^*(R) \subseteq J(R)$,but $J(R) = 0$ by Proposition 2.3. Thisshows that $N^*(R) = 0$.

Theorem 2.4

Let R be a ring whose simple right R-module is GP-injective. Then, the set $N_2(R)$is right weakly regular.

Proof:

We shall show that $RbR + r(b) = R,$ for all $b \in N_2(R),$ if not,suppose there exists $0 \neq a \in N_2(R),$ such that $RaR + r(a) \neq R,$ then there exists a maximal right ideal M of R containing $RaR + r(a),$ R/M is GP-injective, there exists appositive integer n=1 such that $a \neq 0$ and any R-homomorphism of aR into R/M extends to one of R into R/M. Now, let $f: aR \to R/M$ such that $f(ar) = r + M$where $r \in R$. Note that f is well defined right R-homomorphism, since R/M is GP-injective there exists $b + M \in R/M$ such that 1+M=f(a)=(b+M)(a+M)=ba+M, 1+M=ba+M, $1 - ba \in M$, since $ba \in RaR \subseteq M$, we get that $1 \in M,$ which is a contradiction.. Therefore,$RaR + r(a) = R$. In particular, there exists $y, z \in R$ and $v \in r(a)$ such that $yaz + v = 1, ayaz + av = a, ayaz = a,$ that is for all $a \in N_2(R)$. This shows that the set $N_2(R)$is right weakly regular.

Theorem 2.5

Let R be a ring without zero divisors whose simple right R-module is GP-injective. Then, R is a simple ring.

Proof:

We shall show that there is no two sided ideal of R, if not there exists a two sides ideal of R, RaR is a two sided ideal for some $0 \neq a \in R$,since $0 \neq a$,RaR$\neq 0,$ if $RaR \neq R$,there exists a maximal right ideal M of R containing $RaR,$ R/M is GP-injective, there exists a positive integer n and $a^n \neq 0$ such that any R-homomorphism of a^nR into R/M extends to one of R into R/M, we define $f: a^nR \to R/M$ such that $f(a^nr) = r + M,$ where $r \in R,$ let r_1,$r_2 \in R$ such that $a^n r_1 = a^n r_2$,$a^n(r_1 - r_2)$=0,$r_1 - r_2 \in r(a^n) = 0,$ since a is a non-zero divisor, so must $r_1 = r_2,$ $f(a^nr_1) = r_1 + M = r_2 + M = f(a^nr_2),$ so f is well defined R-homomorphism, since R/M is GP-injective, there exists $b + M \in R/M$ such that 1+M=f(a^n)=(b+M)(a^n+M) =ba^n+M, 1+M=ba^n+M, but b$a^n \in RaR \subseteq$M,$1 - ba^n \in M, 1 \in M,$ which is a contradiction. Therefore,$RaR = R,$ for all $a \in R,$ that means R not containing any two sided ideal of R. This shows that R a simple ring.

3. Rings Whose Simple Right R-module are GP-injective and it relation with other Rings.

In this section, we give different conditions to the ring whose simple right R-module is GP-injective to get the reduced, S-weakly regular, regular, strongly regular ring.

A ring R is said to be N duo if aR=Ra, for all $a \in N(R)$[15].

Theorem 3.1

Let R be N duo ring whose every simple right R-module is GP-injective. Then, R is a reduced ring.

Proof:

We shall show that N(R) =0, if not, there exists $0 \neq a \in N(R)$with $a^2 = 0$, if aR+r(a) $\neq R$, there exists a maximal right ideal M of R containing $aR + r(a)$,R/Mis GP-injective, there exists an appositive integer n=1 such that $a \neq 0$ and any R-homomorphism of aR into R/M that extends to one of R into R/M. Let $f: aR \to R/M$ such that $f(ar) = r + M$where, $r \in R$. Note that f is well defined right R-homomorphism, since R/M is GP-injective there exists $b + M \in R/M$ such that 1+M=f(a)=(b+M)(a+M)=ba+M, 1+M=b+M, $1 - ba \in M$, since R is N duo ring and $a \in N(R)$, we get aR=Ra, $ba \in Ra = aR \subseteq M$, so1 $\in M$, which is a contradiction. Therefore,$aR + r(a) = R$. In particular, there exists $z \in R$ and $v \in r(a)$ such that $az + v = 1, a^2z + av = 0 = a$,for all $a \in N(R)$,This shows that $R$is a reduced ring.

Call a ring R NCI if N(R) is containing a non-zero ideal of R whenever $N(R) \neq 0$. Clearly, NI ring is NCI [5].

Theorem 3.2

Let R be an NCI ring whose simple right R-module is GP-injective. then R is a reduced ring.

Proof:

We shall show that N(R) =0, if not, $0 \neq N(R)$, since R is an NCI ring, so N(R) is containing a non-zero ideal I, but I is nil ideal, It is clearly that J(R) containing every nil ideal, so I$\subseteq J(R) = 0$, form proposition 2.4, I=0, that is mean N(R) mustbe an ideal, similarly $N(R) \subseteq J(R) = 0$, N(R)=0.This is shows that $R$is reduced ring.

Theorem 3.3

Let R be aring whose simple right R-module is GP-injective. Then, the following conditions are equivalent:

1- *R is reduced ring.*
2- *R is N duo ring.*
3- *R is 2-priaml ring.*
4- *R is NI ring.*
5- *R is NCI ring.*

Proof:

$1 \longrightarrow 2$, is clear and $2 \longrightarrow 1$, by Theorem 3.2

$1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$, is clearand $5 \longrightarrow 1$, by Theorem 3.1

Call a ring R S-weakly regular ring if $a \in aRa^2R$, for all $a \in R$ [14].

Theorem 3.4

Let R be aring whose simple right R-module is GP-injective. Then, R is S-weakly regular ring. If satisfies one of the following conditions.

1- R is a reduced ring.
2- R is N duo ring.
3- R is 2-priaml ring.

4- R is NI ring.
5- R is NCI ring.

Proof:

We shall prove that R is S-weakly regular when R is reduced, and the proof of the other condition that is clearly form Theorem 3.3.

We shall show that $Rd^2R + r(d) = R,$ for all $d \in R,$if not, there exists $0 \neq a \in R,$such that$Ra^2R + r(a) \neq R,$there exists a maximal right ideal M of R containing $Ra^2R + r(a),$ R/M is GP-injective, there exists an appositive integer n such that $a^n \neq 0$ and any R-homomorphism of a^nR into R/M extends to one of R into R/M. Let $f: a^nR \to R/M$ such that $f(a^n r) = r + M$where,$r \in R.$ let $a^n x = a^n y$where $x, y \in R, a^n(x - y) = 0, (x - y) \in r(a^n) = r(a) \subseteq M,$ since R is a reduced ring, then (x-y)+M=M, x+M=y+M, f(a^nx)=x+M=y+M=f(a^ny), f(a^nx)=f(a^ny),so f is well defined right R-homomorphism, since R/M is GP-injective there exists $b + M \in R/M$ such that 1+M=f(a^n)=(b+M)(a^n+M)=ba^n+M, 1+M=b+M. Now ba^n $\in Ra^2R$ ⊆M, it is true when $n \geq 2,$ $1 - ba^n \in M, 1 \in M,$ which is a contradiction. Therefore, $Ra^2R + r(a^n) = R,$ for $n \geq 2.$ Now, when n=1, $f: aR \to R/M, f(ar) = r + M,$ 1+M=f(a)=(b+M)(a+M)=ba+M, 1+M=ba+M, by multiply a+M in the left side and b+M in the right side, we have ba+M=b^2a^2+M, since $b^2a^2 \in Ra^2R$ ⊆M, then $ba - b^2a^2 \in M$, we get that $ba \in M$, since 1+M=ba+M, 1-ba$\in M, 1 \in M,$ which is a contradiction. Therefore, $Ra^2R + r(a) = R$, for all a$\in R$. In particular, there exists y,$z \in R$ and $v \in r(a)$ such that y$a^2z + v = 1, aya^2z + av = aya^2z = a.$ This shows that R is an S-weakly regular ring.

An element a of ring R is said to be a right regular element if the right annihilator ideal is zero (r(a)=0) [6]. A ring R is said to be MERT if and only if every maximal essential right ideal of R is an ideal [14]. A ring R is called Kasch if every simple right R-module embeds in R, equivalently, for every maximal right ideal M of R is a right annihilator of R [3]. Call a ring R a right SF-ring if each simple right R-module is flat [13]. A ring R is said to be regular if $a \in aRa,$ for all $a \in R[11]$.

Lemma 3.5 [6]

Let R be a semiprime ring with maximum condition on left and right annihilators. Then, every essential right ideal contains a regular element.

Lemma 3.6 [13]

R/I is right flat R-module if and only if for each x in R there is some y in R such that x=yx.

Lemma 3.7 [7]

Let R be a MERT ring, then the following conditions are equivalent:
1- *R is regular.*
2- *R is right SF.*

Theorem 3.8

Let R be a MERT ring whose every simple right module is GP-injective and satisfies maximum condition on left and right annihilators. Then R is Kasch ring and right SF-ring, hence R is regular ring.

Proof :

We shall prove that every maximal right ideal is direct summand, if not, suppose that M a maximal right ideal of R which is not a direct summand of R, then M is a maximal essential right ideal of R, by Theorem 2.1 and Lemma 3.5, we have M containing a non-zero divisor a, R/M is GP-injective, there exists a positive integer n and $a^n \neq 0$ such that any R-homomorphism of a^nR into R/M extends to one of R into R/M, Let $f: a^nR \to R/M$ such that $f(a^nr) = r + M$, where $r \in R$, since a is a non-zero divisor f is well defined right R-homomorphism, since R/M is GP-injective, there exists $b + M \in R/M$ such that 1+M=f(a^n)=ba^n+M, $1 - ba^n \in M$, since $a \in M$, M is essential right ideal and R is MERT ring, M is an ideal, so $ba^n \in M$, we get that $1 \in M$, which is a contradiction. Therefore, M a direct summand. This shows that every maximal right ideal of R is a direct summand.

There exists J right ideal for any maximal right ideal M such that $M \oplus J = R$, in particular there exists $m \in M \; and \; j \in J$ such that m+j=1, so for all $d \in M$, jd=0, then$M \subseteq r(j)$, but M is a maximal right ideal, we have M=r(j), for every maximal right ideal, This shows that R is a right kasch ring.

Also, md=d for all $d \in M$, from Lemma 3.6, we get that R/M is flat right R-module and that is for all M maximal right ideal of R. This shows that R is a right SF-ring.

Since, R is MERT and right SF-ring, by using Lemma 3.7, we get that R is a regular ring.

Theorem 3.9

Let R be N duo ring and MERT whose simple right R-module is GP-injective. Then, R is a strongly regular ring.

Proof:

We shall show that $dR + r(d) = R$, for all $d \in R$. If not then there exists $aR + r(a) \neq R$, for some $a \in R$, there exists a maximal right ideal M of R containing $aR + r(a)$. M is either essential or direct summand, if M is not essential, then M=r(e) for some $0 \neq e = e^2 \in R$, by Theorem 3.1, R is a reduced ring, $a \in r(e) = l(e), ae = 0$, $e \in r(a) \subseteq r(e)$, e$\in r(e), e^2 = 0$, but $e = e^2$, hence e=0, and M==r(e)=r(0)=R, M=R, which is a contradiction. Therefore, M is an essential right ideal of R. Thus, R/M is GP-injective, there exists a positive integer n such that$a^n \neq 0$ and any R-homomorphism of a^nR into R/M extends to one of R into R/M, Let $f: a^nR \to R/M$be defined by$f(a^nr) = r + M$, where $r \in R$.Note that f is well defined right R-homomorphism, because R is a reduced ring. since R/M is GP-injective, there exists $b + M \in R/M$ such that 1+M=f(a^n)=ba^n+M, $1 - ba^n \in M$, since $a \in M$, M is an essential right ideal and R is MERT ring, M is an ideal, so $ba^n \in M$, we get that $1 \in M$, which is also contradiction. Therefore, $aR + r(a) = R$, for all $a \in R$. This shows that R is a strongly regular ring.

Finally, we give the following important result.

In [16] Xue proved , if every simple left R-module is GP-injective, then for any nonzero $a \in R$, there exists a positive integer n=n(a) such that $a^n \neq 0$ and $RaR + l(a^n) = R$ [Proposition 2].

In the above proof, we have $RaR + l(a^n) = R$, and $Ra^nR + l(a^n) = R$, it is clear that proof$Ra^nR + l(a^n) = R$ leads us to $RaR + l(a^n) = R$ because $Ra^nR \subseteq RaR$, we give a new proof that strengthens the above $Ra^{2n}R + r(a^n) = R$, hence $Ra^{2n}R \subseteq Ra^nR \subseteq RaR$.

A ring R is said to be right (left) *$s\pi$-weakly regular ring* if, for every $a \in R$, there exists a positive integer n, depending on a such that $a^n \in a^nRa^{2n}R$ $(a^n \in Ra^{2n}Ra^n)$[10].

Theorem 3.10

Let R be a ring whose simple right R-module is GP-injective. Then R is right $s\pi$-weakly regular ring.

Proof:

Let $a \in R$,and a is not a nilpotent element, if $Ra^{2n}R + r(a^n) \neq R$, then there exists a maximal right ideal M of R containing $Ra^{2n}R + r(a^n)$, by hypothesis R/M is $GP - injective$, we define $f: a^nR \to R/M$ such that $f(a^nr) = r + M$, where $r \in R$, we show that f is well defined, let $a^nx = a^ny$where $x, y \in R$, $a^n(x - y) = 0$, $(x - y) \in r(a^n) \in M$,then (x-y)+M=M, x+M=y+M, f(a^nx)=x+M=y+M=f(a^ny), f(a^nx)=f(a^ny). So f is well defined right R-homomorphism, since R/M is GP-injective, there exists $b + M \in R/M$ and $a^n \neq 0$ such that 1+M=f(a^n)=(b+M)(a^n+M)=ba^n+M, 1+M=ba^n+M, $1 - ba^n \in M$, since1+M=ba^n+M, multiply by $a^n + M$ from left and b+M from right, we get ba^n+M=b^2a^{2n}+M, $ba^n - b^2a^{2n} \in M$, but $b^2a^{2n} \in Ra^{2n}R \subseteq M$, so $ba^n \in M$, but $1 - ba^n \in M$, we get that $1 \in M$, which is a contradiction. Therefore,$Ra^{2n}R + r(a^n) = R$. In particular, there exists $y, z \in R$ and $v \in r(a)$ such that $ya^{2n}z + v = 1, a^nya^{2n}z + a^nv = a^n, a^nya^{2n}z = a^n$, and that is for all $a \in R$ which is not nilpotent elements. When a is nilpotent element, there exists a positive integer m such that $a^m = 0$, so $a^m = a^mra^{2m}s$, for any $r, s \in R$.This shows that R is a right $s\pi$-$weakly\ regular\ ring$.

REFERENCES

[1] Anderson, F. W. and Fuller, K. R. (1992),“**Ring and Categories of Modules**”, Graduate Texts in Math. No. 13, Springer-Verlag, New Yourk.

[2] Birkenmeier, G. F. Heatherly, H. E. and Lee, E. K. (1993), Completely prime ideals and associated radicals, Proc. Biennial Ohio State-Denison Conference 1992, edited by S. K. Jain and S. T. Rizvi, World Scientific, Singapore, New Jersey –London-Hong Kong, pp. 102-129.

[3] Faith, C. (1982), Injective modules and injective quotient rings, Lecture Notes in Pure and Appl. Math.,Vol. 72.

[4] Gupta,V. (1984), A generalization of strongly regular rings,Acta. Math. Hung., Vol. 43, No. (1-2), pp. 57-61.

[5] Hwang, S.U. , Jeon, Y. C., and Park, K. S. (2007), On NCI rings, Bull. Korean Math. Soc., Vol. 44, No. 2, pp. 215-223.

[6] Johnson, R. E. and Levy, L. S. (1968), Regular element in semiprime rings, Proceedings of the American Mathematical Society, Vol. 19, No.4, pp. 961-963.

[7] Jule, Z. and Xianneng, D. (1993), Von Neumann regularity of SF-rings, Communications in Algebra, Vol. 21, No. 7, pp. 2445-2451.

[8] Kim, N. K. Nam, S. B. and Kim, J. Y. (1995) " On simple GP-injective modules" Comm. Algebra, Vol. 23, No. 14, pp. 5437-5444.

[9] Lee, Y. and Huh, C. (1998), A note on $\pi - regular$ rings, KyunGpook Math. J., Vol. 38, pp. 157-161.

[10] Mahmood, R. D. and Abdul-Jabbar, A. M., (2008), $s\pi - weakly$ regular ring, Raf. J. of Comp. Sc. and Math’s., Vol. 5, No. 2, p.p.39-46.

[11] Neumann, J. V. (1936), On regular rings, Princeton N. J., Vol. 22, pp. 707-713.

[12] Ramamurhi, V. S. (1973), Weakly regular ring,Canda. Math. Bull., Vol. 16, No. 3, pp.317-321.

[13] Ramamurthi, V. S. (1975),On the injectivity and flatness of certain cyclic modules, Proc. Amer. Math. Soc., Vol. 48, No.1, pp. 21-25.

[14] Yue Chi Ming, R. (1980),On V-rings and prime rings, J. of Algebra, Vol. 62, pp. 13-20.

[15] Wei, J. and Li, L. (2010), Nilpotent elements and reduced rings*, Turk. J. Math., Vol. 34, pp. 1-13.

[16] Xue, W. (1998),A note on YJ-injectivity, Riv. Mat. Univ. Parma, Vol. 6, No. 1, pp. 31-37.