The Detour Polynomials of Ladder Graphs

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ABSTRACT

The detour distance $D(u,v)$ between two distinct vertices u and v of a connected graph G is the length of a longest $u-v$ path in G. The detour index $dd(G)$ of G is defined by $\sum_{\{u,v\}} D(u,v)$, and the detour polynomial of G is $D(G;x)=\sum_{\{u,v\}} x^{D(u,v)}$. The detour indices and detour polynomials of some ladder graphs are obtained in this paper.

Keywords: Detour distance, Detour index, Detour polynomials, Ladder graphs.

متعددة حدود Detour لبيانات السلم

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الملخص

تعرف مسافة الالتفاف D(u,v) بين رأسين مختلفين u و v في بيان متصل G على أنها الطول لأطول درب بين u وv. ويعرف دليل الالتفاف dd(G) على أنه $\sum_{\{u,v\}} D(u,v)$ ، كما تعرف متعددة حدود الالتفاف للبيان G كالآتي $D(G;x)=\sum_{\{u,v\}} x^{D(u,v)}$.

تضمن هذا البحث إيجاد متعددة حدود الالتفاف ودليل الالتفاف لأنواع من البيانات المتصلة والتي هي بشكل سُلم (Ladder).

الكلمات المفتاحية: مسافة Detour ، دليل Detour ، متعددة حدود Detour ، بيانات السلم

1. Introduction

For the definitions of graph concepts and notations see the books [1] and [7].

The **detour distance** $D(u,v)$ between two distinct vertices u and v in a connected graph G is the maximum of the lengths of $u-v$ paths in G (See [2, 3, 4, 5, 6 and 8]). A $u-v$ path of length $D(u,v)$ is called ***u-v* detour**. As with standard distance, the detour distance D is a metric on the vertex set $V(G)$ of any connected graph G. That is

(1) $D(u,v)\geq 0$ for all vertices $u,\ v\in V(G)$,

(2) $D(u,v)=0$ if and only if $u=v$,

(3) $D(u,v)=D(v,u)$ for all vertices u and v of G, and

(4) $D(u,v)+D(v,w)\geq D(u,w)$ for all vertices u, v and w of G.

It is clear that $D(u,v)=1$ if and only if uv is a bridge of G, and $D(u,v)=p(G)-1$ if and only if G contains a hamiltonian $u-v$ path. Moreover,

$D(u,v) = d(u,v)$ for every two vertices u and v of G if and only if G is a tree. For other properties of the detour distance see [2 and 5].

The **detour eccentricity** $e_D(v)$ of a vertex v in a connected graph G is

$$e_D(v) = \max\{D(u,v) : u \in V(G)\} . \qquad \ldots(1.1)$$

The **detour radius** $rad_D(G)$ of a connected graph G is defined as

$$rad_D(G) = \min\{e_D(v) : v \in V(G)\} , \qquad \ldots(1.2)$$

while the **detour diameter** $diam_D(G)$ of G is

$$diam_D(G) = \max\{e_D(v) : v \in V(G)\} . \qquad \ldots(1.3)$$

In any connected graph G, the detour radius and detour diameter are related by the following inequalities[1]:

$$rad_D(G) \le diam_D(G) \le 2rad_D(G) . \qquad \ldots(1.4)$$

The **detour index** $dd(G)$ of a connected graph G is the Wiener index with respect to detour distance, that is

$$dd(G) = \sum_{u,\,v} D(u,v) , \qquad \ldots(1.5)$$

where the summation is taken over all unordered pairs of vertices u and v of G.

The **detour distance of a vertex** v, denoted by $d_D(v)$, is defined by

$$\mathrm{d_D(v)} = \sum_{\mathrm{u \in V(G)}} \mathrm{D(u,v)} \ . \qquad \ldots(1.6)$$

It is clear that

$$dd(G) = \frac{1}{2} \sum_{v \in V(G)} d_D(v) . \qquad \ldots(1.7)$$

This index has recently received some attention in the chemical literature [9], because $dd(G)$ certainly carries some interesting structural information for cyclic compounds.

We introduce distance polynomial based on detour distance of a connected graph G defined by

$$D(G;x) = \sum_{\{u,v\}} x^{D(u,v)} , \qquad \ldots(1.8)$$

where the summation is taken over all unordered pairs u,v of distinct vertices of G. Such polynomial of G will be called the **detour polynomial** (or **detour distance polynomial**) of G. It is clear that

$$dd(G) = \frac{d}{dx} D(G;x)\Big|_{x=1} . \qquad \ldots(1.9)$$

Moreover, one easily notice that

$$D(G;x) = \sum_{k=1}^{\delta_D} C_D(G,k) x^k , \qquad \ldots(1.10)$$

where $\delta_D = diam_D(G)$, and $C_D(G,k)$ is the number of unordered pairs of distinct vertices u, v such that $D(u,v) = k$. The **detour polynomial of a vertex** v of G is defined as

$$D(v,G;x) = \sum_{\substack{u \in V(G) \\ u \ne v}} x^{D(v,u)} . \qquad \ldots(1.11)$$

It is clear that

$$D(G;x)=\frac{1}{2}\sum_{v\in V(G)} D(v,G;x),$$

and

$$D(v,G;x)=\sum_{k\geq 1}^{e_D(v)} C_D(v,G;k)x^k ,$$

in which $C_D(v,G;k)$ is the number of vertices $u(\neq v)$ such that $D(u,v)=k$ in G.

In this paper, we find detour polynomials and detour indices for a special class of graphs called ladders, namely $P_n \times K_2$ and Möbius ladder.

2. The Detour Polynomial of a Ladder L_n:

A ladder L_n is the graph $P_n \times K_2$, where P_n is the n-path, $n\geq 3$. It is clear that $p(L_n)=2n,\ q(L_n)=3n-2$ and $diam(L_n)=n$. Since L_n is a hamiltonian graph, then $diam_D(L_n)=2n-1$.

The graph L_n is shown in Fig. 2.1 with the vertices labeled u_1, v_1, u_2, v_2, …, u_n, v_n.

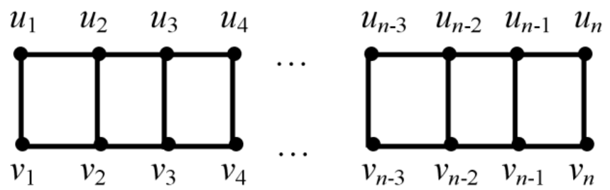

Fig. 2.1 The ladder L_n, $n\geq 4$

The following theorem determines the detour polynomial for L_n, $n\geq 4$.

Theorem 2.1:

For $n\geq 4$, we have

$$D(L_n;x)=(n^2-n+2)x^{2n-1}+(n^2-3n+4)x^{2n-2}+2(x^2+x+1)\sum_{i=2}^{\lfloor \frac{n}{2}\rfloor} x^{2n-2i} + \begin{cases} 0, & \text{for even } n, \\ 2x^{n+1}+x^n, & \text{for odd } n. \end{cases} \quad \ldots(2.1)$$

Proof: (I)

First assume n is even. From Fig.2.1, we find

$$D(u_1,u_j)=\begin{cases} 2n-1, & \text{for even } j\geq 2, \\ 2n-2, & \text{for odd } j\geq 3. \end{cases}$$

Also

$$D(u_1,v_j)=\begin{cases} 2n-1, & \text{for odd } j\geq 1, \\ 2n-2, & \text{for even } j\geq 2. \end{cases}$$

Therefore, by the symmetry of L_n, we obtain

$$D(w,L_n;x)=nx^{2n-1}+(n-1)x^{2n-2},\ w\in\{u_1,\ v_1,\ u_n,\ v_n\}. \quad \ldots(2.2)$$

Now, for $i=2,3,\ldots,\frac{n}{2};\ i\neq j$ and $j\in\{1,\ 2,\ \ldots,\ n\}$, we have

$$D(u_i,u_j)=\begin{cases}2n-1\,, & if\ |j-i|\ is\ odd\\ 2n-2\,, & if\ |j-i|\ is\ even\end{cases}$$

Also, for $i=2,3,...,\frac{n}{2}$ and $j\in\{1,\ 2,\ ...,\ n\}-\{i,\ i-1,\ i+1\}$ we have

$$D(u_i,v_j)=\begin{cases}2n-1, & \text{if } |j-i| \text{ is even,}\\ 2n-2, & \text{if } |j-i| \text{is odd.}\end{cases}$$

Finally, for $i=2,3,...,\frac{n}{2}$ and $j=i+1$ or i *or* $i-1$, we have

$$D(u_i,v_j)=2n-\begin{cases}2i, & \text{if } j=i+1,\\ 2i\text{-}1, & \text{if } j=i,\\ 2i\text{-}2, & \text{if } j=i-1.\end{cases}$$

Therefore, for $i=2,3,...,\frac{n}{2}$, we have

$$D(u_i,L_n;x)=(n-1)x^{2n-1}+(n-3)x^{2n-2}+x^{2n-2i}+x^{2n-2i+1}+x^{2n-2i+2}\,. \qquad \ldots(2.3)$$

It is clear from the Fig. 2.1, that (2.3) holds for v_i, u_{n+1-i} and v_{n+1-i}, where $i=2,3,...,\frac{n}{2}$. Thus, for **even** $n\geq 4$, we have from (2.2) and (2.3)

$$D(L_n;x)=\frac{1}{2}\sum_{w\in V(L_n)}D(w,L_n;x)$$

$$=\frac{1}{2}\Big\{4(nx^{2n-1}+(n-1)x^{2n-2})+4\sum_{i=2}^{\frac{n}{2}}\Big[(n-1)x^{2n-1}+(n-3)x^{2n-2}+x^{2n-2i}+x^{2n-2i+1}+x^{2n-2i+2}\Big]\Big\}$$

$$=(n^2-n+2)x^{2n-1}+(n^2-3n+4)x^{2n-2}+2(x^2+x+1)\sum_{i=2}^{\frac{n}{2}}x^{2n-2i},\ \text{for even } n. \qquad \ldots(2.4)$$

(**II**) If n is odd, then using the steps used in proving even case, we get (2.2), and (2.3) for $i=2$ to $i=\frac{n-1}{2}$. Then, we add $2D(u_{\frac{n+1}{2}},L_n;x)$ inside the brackets { }, where

$$D(u_{\frac{n+1}{2}},L_n;x)=(n-1)x^{2n-1}+(n-3)x^{2n-2}+2x^{n+1}+x^n\,.$$

This completes the proof of the theorem. ■

For L_2 and L_3, we obtain by direct calculation:

$$D(L_2;x)=4x^3+2x^2\,,$$

$$D(L_3;x)=8x^5+6x^4+x^3\,.$$

Corollary 2.2:

For $n\geq 2$, we have

$$dd(L_n)=\begin{cases}4n^3-\frac{13}{2}n^2+7n-4, & \text{for even } n,\\ 4n^3-\frac{13}{2}n^2+7n-\frac{7}{2}, & \text{for odd } n.\end{cases} \qquad \ldots(2.5)$$

Proof:

Taking the derivative of $D(L_n;x)$ with respect to x at $x=1$, we get

$$dd(L_n)=(n^2-n+2)(2n-1)+(n^2-3n+4)(2n-2)+6\sum_{i=2}^{\left\lfloor \frac{n}{2}\right\rfloor}(1)$$

$$+6\sum_{i=2}^{\left\lfloor \frac{n}{2}\right\rfloor}(2n-2i)+\begin{cases}0, & \text{for even } n,\\ 3n+2, & \text{for odd } n.\end{cases}$$

$$=4n^3-11n^2+19n-10+6(2n+1)(\left\lfloor \frac{n}{2}\right\rfloor-1)-12\sum_{i=2}^{\left\lfloor \frac{n}{2}\right\rfloor} i+\begin{cases}0, & \text{for even } n,\\ 3n+2, & \text{for odd } n.\end{cases}$$

$$=4n^3-11n^2+19n-10+\begin{cases}3(2n+1)(n-2)-6\left(\frac{n}{2}+2\right)\left(\frac{n}{2}-1\right), & \text{for even } n,\\ 3(2n+1)(n-3)-\frac{3}{2}(n^2-9)+3n+2, & \text{for odd } n.\end{cases}$$

Simplifying the expression we get (2.5). ■

Remark:

We notice that the polynomial $D(L_n;x)$ is of degree $2n-1$, and has n zeros, with nonzero coefficients a_i of the terms $a_i x^i$, $i=2n-1,...,n$.

3. The Detour Polynomial of a Möbius Ladder:

A Möbius ladder, denoted ML_n is a ladder L_n with the two edges u_1v_n and v_1u_n as shown in Fig.3.1. It is clear that $p(ML_n)=2n$, $q(ML_n)=3n$, $diam(ML_n)=\left\lceil \frac{n}{2}\right\rceil$ and $diam_D(ML_n)=2n-1$

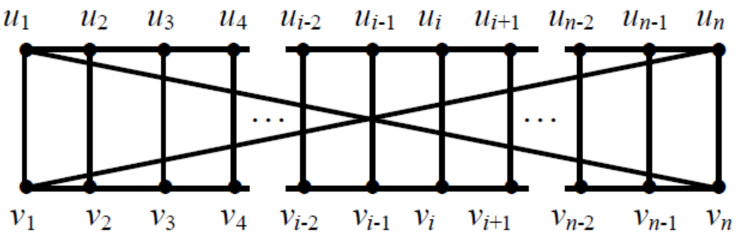

Fig.3.1 A Möbius ladder ML_n, $n\geq 5$

The graph ML_n is a cubic hamiltonian graph and it can be redrawn as shown in Fig.3.2 from which we see that all its vertices have the same detour polynomial. Thus

$D(ML_n;x)=nD(u_1,ML_n;$ …(3.1)

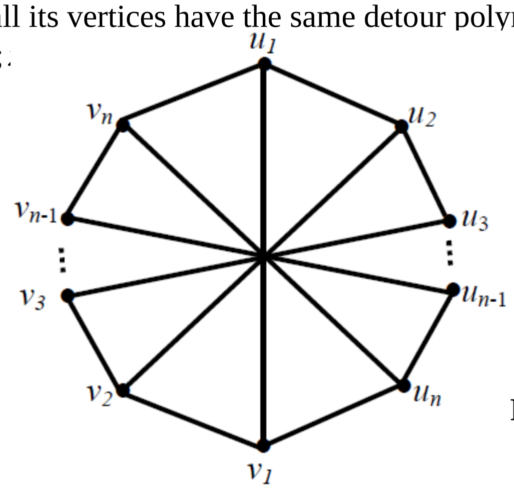

Fig.3.2 ML_n, $n\geq 5$

The detour polynomial of the Möbius ladder is obtained in the next theorem.

Theorem 3.1:

For $n \ge 2$,

$$D(ML_n;x) = \begin{cases} n(2n-1)x^{2n-1}, & \text{for even } n, \\ n^2x^{2n-1} + n(n-1)x^{2n-2}, & \text{for odd } n. \end{cases} \qquad \ldots(3.2)$$

Proof:

Assume that $n \ge 5$. We shall consider two cases for n.

(**I**) n **is even**.

If i is even, $i \ge 2$, then there is a hamiltonian $u_1 - u_i$ path in ML_n, namely:

$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, u_{i-1}, v_{i-1}, v_i, v_{i+1}, \ldots, v_n, u_n, u_{n-1}, \ldots, u_{i+1}, u_i$ (See Fig.3.1). Thus

$$D(u_1,u_i) = 2n-1, \qquad i = 2,4,\ldots,n. \qquad \ldots(3.3)$$

If i is odd, $i \ge 3$, there is also a hamiltonian $u_1 - u_i$ path in ML_n, namely:

$u_1, u_2, \ldots, u_{i-1}, v_{i-1}, v_{i-2}, \ldots, v_2, v_1, u_n, v_n, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_{n-3}, \ldots, v_i, u_i$.

Therefore

$$D(u_1,u_i) = 2n-1, \text{ for } i = 3,5,\ldots,n-1. \qquad \ldots(3.4)$$

Now, we determine $D(u_1,v_i)$. If i is even, $i \ge 2$, then there is a hamiltonian $u_1 - v_i$ path, namely:

$u_1, u_2, \ldots, u_{i-1}, v_{i-1}, v_{i-2}, v_{i-3}, \ldots, v_2, v_1, u_n, v_n, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_{n-3}, \ldots, u_i, v_i$.

Thus

$$D(u_1,v_i) = 2n-1, \text{ for } i = 2,4,\ldots,n. \qquad \ldots(3.5)$$

If i is **odd**, $i \ge 1$, then there is a hamiltonian $u_1 - v_i$ path, namely (this is for $i \ge 3$, for $i = 1$, it is clear):

$u_1, v_1, v_2, u_2, u_3, \ldots, v_{i-1}, u_{i-1}, u_i, u_{i+1}, \ldots, u_n, v_n, v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v_i$.

Thus

$$D(u_1,v_i) = 2n-1, \text{ for } i = 1,3,5,\ldots,n-1. \qquad \ldots(3.6)$$

Hence, for every vertex $w(\ne u_1)$ of ML_n, we have $D(u_1,w) = 2n-1$.

Thus, from (3.1), we obtain (3.2) for even n.

(*II*) n **is odd.**

Suppose that i is **even**, then it is clear from Fig.3.1, that

$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, u_{i-1}, v_{i-1}, v_i, v_{i+1}, \ldots, v_n, u_n, u_{n-1}, u_{n-2}, \ldots, u_i$,

is a hamiltonian $u_1 - u_i$ path for even i. Thus

$$D(u_1,u_i) = 2n-1, \text{ for } i = 2,4,\ldots,n-1. \qquad \ldots(3.7)$$

If i is **odd**, then

$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, u_{i-2}, v_{i-2}, v_{i-1}, v_i, \ldots, v_n, u_n, u_{n-1}, u_{n-2}, \ldots, u_i$ (which does not contain u_{i-1}) is a $u_1 - u_i$ detour of length $2n-2$, for odd i. Thus

$$D(u_1,u_i) = 2n-2, \text{ for } i = 1,3,5,\ldots,n. \qquad \ldots(3.8)$$

To find $D(u_1,v_i)$, first assume i is **even**, then

$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, v_{i-2}, u_{i-2}, u_{i-1}, u_i, \ldots, u_n, v_n, v_{n-1}, v_{n-2}, \ldots, v_i$ (which does not contain v_{i-1}) is a $u_1 - v_i$ detour of length $2n-2$ for even i. Thus

$D(u_1, v_i) = 2n-2,$ for $i = 2, 4, ..., n-1$. …(3.9)

Now, let i be **odd**, then there is a hamiltonian $u_1 - v_i$ path

$u_1,\ v_1, v_2,\ u_2,\ u_3, v_3, \ldots, v_{i-1},\ u_{i-1},\ u_i, u_{i+1}, \ldots,\ u_n, v_n,\ v_{n-1}, v_{n-2}, \ldots, v_i$.

Thus

$D(u_1, v_i) = 2n-1,$ for $i = 1, 3, 5, ..., n$. …(3.10)

From (3.7) and (3.10) we get n pairs of vertices (u_1, u_i) and (u_1, v_i) of detour distance $2n-1$; and from (3.8) and (3.9), we get $(n-1)$ pairs of vertices (u_1, u_i) and (u_1, v_i) of detour distance $2n-2$. Thus, from (3.1) we obtain (3.2) for odd n.

By direct calculation one may easily obtain:

$D(ML_2; x) = 6x^3$,

$D(ML_3; x) = 9x^5 + 6x^4$,

$D(ML_4; x) = 28x^7$,

$D(ML_5; x) = 25x^9 + 20x^8$,

which are the same results obtained from (3.2). Thus, the Theorem 3.1 holds for all values of $n \geq 2$. ■

From Theorem 3.1 and using (1.9) we get $dd(ML_n)$ as given in the next corollary:

Corollary 3.2:

For $n \geq 2$, the detour index of ML_n is

$$dd(ML_n) = \begin{cases} n(2n-1)^2, & \text{for even } n, \\ 4n^3 - 5n^2 + 2n, & \text{for odd } n. \end{cases}$$ ■

The following corollary is a useful graph theoretical result.

Corollary 3.3:

The Möbius ladder ML_n is hamiltonian-connected if and only if n is even.

A connected graph G of order p is called **saturated** (with respect to detour distance) [9] if $dd(G) = \frac{1}{2} p(p-1)^2$; that is if and only if G is a hamiltonian-connected graph. Thus ML_n is saturated if n is even. ■

The **density** [9] of a (p, q) connected graph G is defined as $den(G) = \frac{q}{p}$. One may show that the density of every saturated graph G is not less that $\frac{3}{2}$. Thus, from corollary 3.3, ML_n for even n, is saturated with minimum density $\frac{3}{2}$.

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