A Study on the Conditions of Oscillation of Solutions of Second Order Impulsive Delay Differential Equations

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ABSTRACT

Oscillation conditions of second order impulsive delay differential equations with impulses are investigated, some sufficient conditions for all solutions to be oscillatory are obtained. Also, two examples are given to illustrate the applicability of the results obtained.

Keywords: oscillation, impulsive differential equations, non oscillatory.

دراسة حول شروط تذبذب حلول المعادلات التفاضلية المتباطئة من الرتبة الثانية

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الملخص

تم في هذا البحث دراسة شروط التذبذب لحلول المعادلات التفاضلية المتباطئة النبضية, ولقد حصلنا على بعض الشروط الكافية التي تجعل جميع الحلول متذبذبة. وأيضاً أعطينا مثالين لتوضيح قابلية التطبيق للنتائج التي حصلنا عليها.

الكلمات المفتاحية: التذبذب ، المعادلات التفاضلية المتباطئة ، غير التذبذبية.

1. Introduction:

Many evolution processes in nature are characterized by the fact that at certain moments of time they experience an abrupt change of state. This has been the main reason for the development of the theory of impulsive ordinary differential equations. The impulsive differential equations are therefore a new branch of the theory of ordinary differential equations. The investigation of these equations was rather slow as compared to ordinary differential equations without impulse. This is due to the great difficulties caused by the specific properties of the impulsive equations such as beating, bifurcation, merging, and loss of property of autonomy of the solutions. Despite these difficulties, the theory of differential equations with impulses is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential equations. Moreover, such equations represent a natural framework for mathematical modeling of several real world phenomena.

In this paper, we are concerned with the problem of oscillation of solutions of second order impulsive delay differential equations.

Consider the following system

$$\left.\begin{array}{c} x' = f(t,x), \qquad t \neq \theta_i\,, \quad i = 1,2,\dots \\ \Delta x|_{t=\theta_i} = I_i(x) \\ x(t_0+) = x_0, \end{array}\right\} \qquad (1.1)$$

where $f(t,x)$ is a real-valued function defined on $D = R_+ \times \Omega$, D is a domain in R^2 ,$\{I_i(x)\}$ is a sequence of real numbers for $x \in \Omega$,and $\{\theta_i\}$ is a sequence of real number which satisfies $\theta_i < \theta_{i+1}$ and $lim_{i\to\infty}\theta_i = \infty$.

The solution of (1.1) is such a piecewise continuous function that has discontinuities of the first kind at $t = \theta_i$ satisfying the jumps condition, that is

$$\Delta x|_{t=\theta_i} = x(\theta_i +) - x(\theta_i -) = I_i\big(x(\theta_i -)\big).$$

assumes that $f(t,x) \in C(D)$ and $I_i(x) \in C(\Omega)$.

(1.1) Definitions

Definition 1.1 [3] A real-valued function $x(t)$ is called a solution of (1.1) on $[t_0, t_0+T)$, $T > 0$ if

(i) $x(t_0+) = x_0$ and $(t, x(t)) \in D\ for\ t \in [t_0, t_0+T)$,

(ii) $x(t)$ is continuously differentiable and satisfies (1.1) on every subinterval of $[t_0, t_0+T)$ not containing $t = \theta_i$,

(iii) $x(t+) = x(t) + I_i(x(t))\ for\ t = \theta_i \in [t_0, t_0+T)$ at which $x(t)$ is assumed to be left continuous , i.e., $x(\theta_i -) = x(\theta_i)$

Definition 1.2 [5] A nontrivial function x(t), which may be a solution of an impulsive differential equations (IDE), is called oscillatory if there exists a sequence $\{t_n\}$ such that $lim_{n\to\infty} t_n = \infty\ and\ x(t_n)x(t_n+) \le 0$. Otherwise, x(t) is said to be nonoscillatory. A nonoscillatory function is either eventually positive or negative, i.e., there exists t_1 such that $x(t) \ne 0$ for all $t > t_1$.

A differential equation is called oscillatory if every solution of the equation is oscillatory and nonoscillatory if it has at least one nonoscillatory solution.

Next, consider

$$[r(t)x'(t)]' + a(t)f(x(g(t)) = 0 \qquad (1.2)$$

Where the following conditions hold:

a) $r \in C^1(0,\infty), r(t) > 0;$

b) $a \in C(0,\infty), a(t) \ge 0;$

c) $g \in C^1(0,\infty), g(t) \le t, g'(t) \ge 0, lim_{t\to\infty} g(t) = \infty;$

d) $f \in C(-\infty,\infty) \cap C^1(-\infty,0) \cap C^1(0,\infty), yf(y) > 0, f'^{(y)} \ge 0, for\ y \ne 0.$

(1.2) Helping Results

Theorem1.1 ([13]) Suppose that $\int^{\infty} \frac{dt}{r(t)} = \infty$, and that there exist two positive functions $\rho(t) \in C^2(0,\infty)$ and $\phi(y) \in C^1(0,\infty)$ with the following properties:

$$\rho'(t) \ge 0, \quad (r(t)\rho'(t))' \le 0, \quad \phi'(y) \ge 0,$$

$$\int_{\mp\delta}^{\mp\infty} \frac{dy}{f(y)\phi(y)} < \infty \quad for\ some \quad \delta > 0$$

$$\int^{\infty} \frac{\rho(g(t))a(t)}{\phi(R_T(g(t)))} dt = \infty \quad for\ any \qquad T > 0$$

$where\ R_T(t) = \int_T^t \frac{ds}{r(s)}.\ Then, equation\ (1.2)\ is\ oscillatory.$

Theorem 1.2 ([13]) Suppose that $\int^{\infty} \frac{dt}{r(t)} < \infty, \int_{\mp 0}^{\mp\delta} \frac{dy}{f(y)} < \infty\ for\ some\ \delta > 0,$ and that there exists a positive function $\sigma(t) \in C^2(0,\infty)$ with the following properties:

$$\sigma'(t) \le 0, \qquad (r(t)\sigma'(t))' \ge 0$$

$$\int^{\infty} \frac{dt}{\sigma(t)r(t)} = \infty$$

$$\int^{\infty} \sigma(t)a(t)dt = \infty.$$

$Then, equation\ (1.1) is\ oscillatory.$

2. Main Results

Consider the following system which contains delay arguments and imposes impulse condition and obtain oscillation criteria for the corresponding impulsive differential equation

$$[r(t)x'(t)]' + a(t)f(x(g(t))) = 0\,, \quad t \neq \theta_k\,, k \in N$$
$$\Delta[r(t)x'(t)]\big|_{t=\theta_k} + b_k h\Big(x\big(g(t)\big)\Big) = 0, \qquad (2.1)$$
$$\Delta x(t)|_{t=\theta_k} = 0,$$

Assume that the following conditions are held:

a) $r \in C^1(0,\infty), r(t) > 0;$

b) $a \in C(0,\infty), a(t) \geq 0;$

c) $g \in C^1(0,\infty), g(t) \leq t, g'(t) \geq 0, lim_{t\to\infty} g(t) = \infty;$

d) $f \in C(-\infty,\infty) \cap C^1(-\infty,0) \cap C^1(0,\infty), yf(y) > 0, f'(y) \geq 0, for\ y \neq 0$

e) $\{b_k\}\ is\ positive\ sequence;\ k = 1,2,\ldots$

f) For a given $c_1 > 0$ there exists $c_2 > 0$ such that $\frac{h(x)}{f(x)} \geq c_2$ if $|x| \geq c_1$.

According to theorems 1.1 and 1.2, we get the following theorems.

Theorem 2.1 Let $\int^\infty \frac{dt}{r(t)} = \infty$. and there exist two positive functions $\rho(t) \in C^2(0,\infty)$ and $\phi(y) \in C^1(0,\infty)$ with the following properties:

$$\rho'(t) \geq 0, \quad \big(r(t)\rho'(t)\big)' \leq 0, \quad \phi'(y) \geq 0,$$
$$\int_{\mp\delta}^{\mp\infty} \frac{dy}{f(y)\phi(y)} < \infty \quad for\ some\ \ \delta > 0 \qquad (2.2)$$

Then if

$$\int^\infty \frac{\rho\big(g(t)\big)a(t)dt}{\phi\big(R_T\big(g(t)\big)\big)} + \sum_{t_0<\theta_k<\infty} \frac{\rho(g(\theta_k))b_k}{\phi\big(R_T\big(g(\theta_k)\big)\big)} = \infty \quad for\ any\ \ T > 0 \qquad (2.3)$$

$where\ R_T(t) = \int_T^t \frac{ds}{r(s)}.\ The\ equation\ (2.1)\ is\ oscillatory.$

Proof : Suppose there exists a nonoscillatory solution of (2.1). without losing the generality , assume that $x(g(t)) > 0$ for all sufficiently large t, $t > T$, from equation (2.1) , we have

$$[r(t)x'(t)]' = -a(t)f(x(g(t))) \leq 0,$$

$which\ implies\ that\ r(t)x'(t) is\ non-increasing\ whenever\ t \neq \theta_k\ then$

$$r(\theta_k+)x'(\theta_k^+) - r(\theta_k)x'(\theta_k) = -b_k x(g(\theta_k))$$
$$\Delta r(t)x'(t)\big|_{t=\theta_k} = -b_k x\big(g(\theta_k)\big) \leq 0$$

This implies that r(t)x'(t) is non-increasing for all $t > T$. From the assumption $\int^\infty \frac{dt}{r(t)} = \infty$, it follows that x'(t)$\geq 0$, i.e., x(t) is non-decreasing for $t \geq T$.

In fact, if x'(t*)<0 for some t*≥T, then r(t)x'(t)≤r(t*)x'(t*) for t≥t*, and an integration of the last inequality divided by r(t) gives

$$x(t) - x(t^*) \leq r(t^*)x'(t^*)\int_{t*}^t \frac{ds}{r(s)},$$

$which\ yields\ a\ contradiction\ in\ the\ limit\ as\ t \to \infty.\ \ let\ t_1\ be\ such\ that$

$g(t) > T$ for $t \geq t_1$. it is easy to see that there is a constant $A \geq 1$ such that

$$x(g(t)) \leq AR_T(g(t)) \qquad t \geq t_1 \qquad (2.4)$$

If we define

$$w(t) = \frac{\rho(g(t))r(t)x'(t)}{f(x(g(t)))\emptyset(R_T(g(t)))}$$

Then it follows that

$$w'(t) = \left[\frac{\rho(g(t))}{f(x(g(t)))\emptyset(R_T(g(t)))}\right]' r(t)x'(t) + \frac{\rho(g(t))}{f(x(g(t)))\emptyset(R_T(g(t)))}[r(t)x'(t)]', \quad t \neq \theta_k.$$

Using (2.1), we have

$$w'(t) = \left[\frac{\rho(g(t))}{f(x(g(t)))\emptyset(R_T(g(t)))}\right]' r(t)x'(t) + \frac{\rho(g(t))}{f(x(g(t)))\emptyset(R_T(g(t)))}\left(-a(t)f\left(x(g(t))\right)\right)$$

$, \quad t \neq \theta_k.$

Thus, we obtain

$$w'(t) = \left[\frac{\rho(g(t))}{f(x(g(t)))\emptyset(R_T(g(t)))}\right]' r(t)x'(t) - \frac{\rho(g(t))a(t)}{\emptyset(R_T(g(t)))}, \quad t \neq \theta_k.$$

$$\Delta w(t)|_{t=\theta_k} = \frac{\rho(g(t))}{f(x(g(t)))\emptyset(R_T(g(t)))}\left[-b_k h\left(x(g(\theta_k))\right)\right]. \qquad (2.5)$$

Clearly

$$w'(t) = \frac{\rho'(g(t))g'(t)r(t)x'(t)}{f(x(g(t)))\emptyset(R_T(g(t)))} - \frac{\rho(g(t))a(t)}{\emptyset(R_T(g(t)))} - \frac{\rho(g(t))r(t)x'(t)f'(x(g(t))g'(t)\emptyset(R_T(g(t)))}{[f(x(g(t)))\emptyset(R_T(g(t)))]^2} -$$

$$\frac{\rho(g(t))r(t)x'(t)f(x)\emptyset'(R_T(g(t)))\frac{1}{r(g(t))}g'(t)}{[f(x(g(t)))\emptyset(R_T(g(t)))]^2} \quad ,t \neq \theta_k.$$

Since ρ, f, x, g and $\emptyset$ are non-decreasing, the third and fourth terms of the right hand side are nonnegative and therefore,

$$w'(t) \leq \frac{\rho'(g(t))g'(t)r(t)x'(t)}{f(x(g(t)))\emptyset(R_T(g(t)))} - \frac{\rho(g(t))a(t)}{\emptyset(R_T(g(t)))}, \qquad t \neq \theta_k \quad (2.6)$$

In view of

$$\int_{t_1}^{t} w'(s)ds = w(t) - w(t_1) - \sum_{t_1<\theta_k<t} \Delta w$$

By integrating the inequality (2.6) on the interval $[t_1, t]$, we have

$$w(t) \leq w(t_1) + \int_{t_1}^{t} \frac{\rho'(g(s))g'(s)r(s)x'(s)}{f(x(g(s)))\emptyset(R_T(g(s)))}ds - \int_{t_1}^{t} \frac{\rho(g(s))a(s)}{\emptyset(R_T(g(s)))}ds -$$
$$- \sum_{t_1<\theta_k<t} \frac{b_k\rho(g(\theta_k))h(x(g(\theta_k))}{f(x(g(\theta_k)))\emptyset(R_T(g(\theta_k)))}$$

Since $x(t) \geq c_1$ for some c_1, and (f) holds, we easily get

$$w(t) \leq w(t_1) + \int_{t_1}^{t} \frac{\rho'(g(s))g'(s)r(s)x'(s)}{f(x(g(s)))\emptyset(R_T(g(s)))}ds - c^*\left[\int_{t_1}^{t} \frac{\rho(g(s))a(s)}{\emptyset(R_T(g(s)))}ds -\right.$$
$$\left.\sum_{t_1<\theta_k<t} \frac{b_k\rho(g(\theta_k))}{\emptyset(R_T(g(\theta_k)))}\right], \qquad (2.7)$$

Where $c^* = min\{1, c_2\}$.By using the inequalities $r(t)x'(t) \le r(g(t))x'(g(t))$ and $(r(t)\rho'(t))' \le 0$, (2.2) and (2.4), and applying Bonnet's theorem, the first integral on the right hand side of the above inequality is estimated as follows:

$$\int_{t_1}^{t} \frac{\rho'(g(s))g'(s)r(s)x'(s)}{f(x(g(s)))\emptyset(R_T(g(s)))} ds \le \int_{t_1}^{t} \frac{\rho'(g(s))g'(s)r(g(s))x'(g(s))}{f(x(g(s)))\emptyset(R_T(g(s)))} ds \le$$

$$r(g(t_1))\rho'(g(t_1)) \int_{t_1}^{t} \frac{g'(s)x'(g(s))}{f(x(g(s)))\emptyset(R_T(g(s)))} ds \le$$

$$Ar(g(t_1))\rho'(g(t_1)) \int_{x(t_1)/A}^{x(t)/A} \frac{dy}{f(y)\emptyset(y)} ds.$$

Thus ,the first integral on the right hand side of (2.7) remains bounded above as $t \to \infty$. Letting $t \to \infty$ in (2.7) we have

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \frac{\rho(g(t)r(t)x'(t)}{f(x(g(t)))\emptyset(R_T(g(t)))} = -\infty,$$

Which contradicts the fact that $x'(t) \ge 0 \; for \; t \ge t_1$. this completes the proof of the theorem.

Theorem 2.2 Let $\int^{\infty} \frac{dt}{r(t)} < \infty$ and $lim_{x\to 0} \frac{h(x)}{f(x)} \ne 0$. and that there exists a positive function $\sigma(t) \in C^2(0,\infty)$ with the following properties:

$$\sigma'(t) \le 0, \qquad (r(t)\sigma'(t))' \ge 0,$$

$$\int^{\infty} \frac{dt}{\sigma(t)r(t)} = \infty, \qquad (2.8)$$

$$\int_{\pm 0}^{\pm\delta} \frac{dy}{f(y)} < \infty \; for \; some \; \delta > 0 \qquad (2.9)$$

if

$$\int^{\infty} \sigma(t)a(t)dt + \sum_{t_0<\theta_k<\infty} \sigma(\theta_k)b_k = \infty, \qquad (2.10)$$

Then ,the equation (2.1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution such that $x(g(t)) > 0 \; for \; t \ge t_1$. It follows that $r(t)x'(t)$ is non-increasing for $t \ge t_1$ and so $x'(t)$ is eventually of constant sign.

Define

$$w(t) = \frac{\sigma(t)r(t)x'(t)}{f(x(g(t)))}.$$

Clearly,

$$w'(t) = \left[\frac{\sigma(t)}{f(x(g(t)))}\right]' r(t)x'(t) + \frac{\sigma(t)}{f(x(g(t)))}[r(t)x'(t)]', \quad t \ne \theta_k \qquad (2.11)$$

In view of (2.1), we obtain from (2.11) that

$$w'(t) = \left[\frac{\sigma(t)}{f(x(g(t)))}\right]' r(t)x'(t) + \frac{\sigma(t)}{f(x(g(t)))}\left(-a(t)f\left(x(g(t))\right)\right), \quad t \ne \theta_k.$$

Thus, we have

$$w'(t) = \left[\frac{\sigma(t)}{f(x(g(t)))}\right]' r(t)x'(t) - \sigma(t)a(t), \quad t \ne \theta_k$$

$$\Delta w(t)|_{t=\theta_k} = -b_k \frac{\sigma(\theta_k)h(x(g(\theta_k)))}{f(x(g(\theta_k)))} \qquad (2.12)$$

It follows that

$$w'(t)=\frac{r(t)x'(t)\sigma'(t)}{f(x(g(t)))}-\frac{\sigma(t)r(t)x'(t)[f(x(g(t)))]'}{[f(x(g(t)))]^2}-\sigma(t)a(t),t\neq\theta_k \quad (2.13)$$

Suppose that $x'(t)\geq 0$ it is clear that the second term on the right hand side of the above inequality is nonnegative and the first term is non-positive. Therefore,

$$w'(t)\leq -\sigma(t)a(t)$$

Integrating the above inequality, we get

$$w(t)\leq w(t_1)-\sum_{t_1<\theta_k<t}\Delta w-\int_{t_1}^{t}\sigma(s)a(s)ds$$

$$w(t)\leq w(t_1)-\left[\sum_{t_1<\theta_k<t}\frac{b_k\sigma(\theta_k)h(x(g(\theta_k)))}{f(x(g(\theta_k)))}+\int_{t_1}^{t}\sigma(s)a(s)ds\right].$$

Now, as $x(t)>0$ and $x'(t)\geq 0,$ we can make sure that there is a c_1 such that $x(t)>c_1$ $for\ all\ t\geq t_1$.By using this fact we have

$$w(t)\leq w(t_1)-c^*\left[\sum_{t_1<\theta_k<t}b_k\sigma(\theta_k)+\int_{t_1}^{t}\sigma(s)a(s)ds\right], \qquad (2.14)$$

Where $c^*=min\{1,c_2\}$ $,where$ $\frac{h(x)}{f(x)}\geq c_2$. Letting $t\to\infty$ in (2.14), we obtain a contradiction. So we must have $x'(t)\leq 0$. in this case, consider

$$w(t)=w(t_1)+\int_{t_1}^{t}\frac{r(s)x'(s)\sigma'(s)}{f(x(g(s)))}ds-\int_{t_1}^{t}\frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2}ds-\int_{t_1}^{t}\sigma(s)a(s)ds-$$
$$\sum_{t_1<\theta_k<t}\frac{b_k\sigma(\theta_k)h(x(g(\theta_k)))}{f(x(g(\theta_k)))},$$

It follows that if $lim_{x\to 0}\frac{h(x)}{f(x)}\neq 0$,then there exists c_3 such that $\frac{h}{f}>c_3$ so

$$w(t)=$$
$$w(t_1)+\int_{t_1}^{t}\frac{r(s)x'(s)\sigma'(s)}{f(x(g(s)))}ds-\int_{t_1}^{t}\frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2}ds-c^*\left[\int_{t_1}^{t}\sigma(s)a(s)ds-\right.$$
$$\left.\sum_{t_1<\theta_k<t}b_k\sigma(\theta_k)\right]. \qquad (2.15)$$

Where $c^*=min\{1,c_3,c_2\}$. the first integral on the right hand side of the above inequality is bounded from above in a similar way as in the previous theorem. In view of (2.15) we see that there exists a $t_2\geq t_1$ so that

$$w(t)+\int_{t_1}^{t}\frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2}ds\leq -1$$

or

$$w(t)+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds\leq -1$$

or

$$1+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds\leq -w(t) \qquad (2.16)$$

for $t\geq t_2$. multiplying both sides of (2.16) by

$$-\frac{[f(x(g(s)))]'}{f(x(g(s)))}\left\{1+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds\right\}^{-1}\geq 0$$

We get

$$-\frac{[f(x(g(s)))]'}{f(x(g(s)))} \leq w(t)\frac{[f(x(g(s)))]'}{f(x(g(s)))}\left\{1+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds\right\}^{-1}.$$

by Integrating from t_2 to t, we have

$$-\log f(x(g(t)))|_{t_2}^{t} \leq \log u(t)|_{t_2}^{t},$$

Where $u(t) = 1+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds$

Therefore,

$$\log\frac{f(x(g(t_2)))}{f(x(g(t)))} \leq \log\left\{1+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds\right\} \qquad (2.17)$$

From (2.16) we may write

$$\log\left\{1+\int_{t_1}^{t}\frac{[f(x(g(s)))]'}{f(x(g(s)))}w(s)ds\right\} \leq \log(-w(t)),$$

and from (2.17)

$$\log f\left(x(g(t_2))\right) - \log f\left(x(g(t))\right) \leq -\log w(t)$$

or

$$\log f\left(x(g(t_2))\right) \leq -\log[w(t)f(x(g(t)))],$$

or

$$\log f\left(x(g(t_2))\right) \leq -\log[r(t)\sigma(t)x'(t)],$$

or

$$f\left(x(g(t_2))\right) \leq -\sigma(t)r(t)x'(t),$$

or

$$-\frac{f(x(g(t_2)))}{\sigma(t)r(t)} \geq x'(t).$$

Integrating from t_2 to t ,we get

$$x(t)-x(t_2) \leq -f\left(x(g(t_2))\right)\int_{t_2}^{t}\frac{ds}{\sigma(s)r(s)},$$

Which is given $\lim_{t\to\infty} x(t) = -\infty$, which is contradiction. This proves the theorem.

3. Examples

Example 3.1 Consider the impulsive delay differential equation

$$\begin{aligned} &\left(\frac{1}{t}x'(t)\right)' + \frac{1}{t^2}x(t-T) = 0, \qquad t \neq i \\ &\Delta\left[\frac{1}{t}x'(t)\right]\Big|_{t=i} - \frac{1}{i^{\frac{1}{2}}}x(i-T) = 0, \qquad (3.1) \\ &\Delta x(t)|_{t=i} = 0 \end{aligned}$$

Here , we have

$a(t) = \frac{1}{t^2}$, $g(t) = t-T, f(y) = y, b_i = \frac{1}{i^{\frac{1}{2}}}$, $\theta_i = i, \emptyset(y) = y, \rho(t) = R_T(t),$

$r(t) = \frac{1}{i}, h(x) = x$

Clearly,
$$\int^{\infty} \frac{dt}{r(t)} = \infty,$$
$$\int^{\infty} \frac{dy}{y^2} < \infty,$$
$$\int^{\infty} \frac{1}{t^2} dt + \sum^{\infty} \frac{1}{i^{\frac{1}{2}}} = \infty.$$

Since all conditions of Theorem 2.1 are satisfied; (3.1) is oscillatory.
We note that if the equation is not subject to impulse condition, then since
$$\int^{\infty} \frac{1}{t^2} dt < \infty$$
The equation
$$\left(\frac{1}{t} x'(t)\right)' + \frac{1}{t^2} x(t-T) = 0$$
has a nonoscillatory solution by Theorem 1.1.

Example 3.2: Consider the impulsive delay differential equation
$$\left(t^2 x'(t)\right)' + \frac{1}{t} x(t-T) = 0, \qquad t \neq i$$
$$\Delta t^2 x'(t)|_{t=i} - \sqrt{i} x(i-T) = 0, \qquad (3.2)$$
$$\Delta x(t)|_{t=i} = 0$$
So that
$a(t) = \frac{1}{t}\,, g(t) = t - T, f(y) = y, b_i = \sqrt{i}\,, \theta_i = i, \sigma(t) = \frac{1}{t},\; r(t) = t^2, h(x) = x$
Clearly
$$\int^{\infty} \frac{dt}{r(t)} < \infty,$$
$$\int^{\infty} \frac{dt}{t} = \infty,$$
and
$$\int^{\infty} \frac{1}{t^2} dt + \sum^{\infty} \frac{1}{i^{\frac{1}{2}}} = \infty.$$
By Theorem 2.2, (3.2) is oscillatory.
We note that if the equation is not subject to impulse condition, then since
$$\int^{\infty} \frac{1}{t^2} dt < \infty$$
the equation
$$\left(t^2 x'(t)\right)' + \frac{1}{t} x(t-T) = 0$$
is nonoscillatory by Theorem 1.2.

Conclusions

In this paper, we are concerned with the problem of oscillation of solutions of impulsive delay differential equations. In view of the known results obtained for delay differential equations with impulses, we derived new oscillation criteria for delay differential equations with impulses. In particular, sufficient conditions are to be obtained under which all solutions of a certain impulsive differential equation oscillate. A definition of oscillation is given. The impulsive differential equations are adequate mathematical models for the description of evolution processes characterized by the combination of a continuous and jumps change of their state.

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