Some Algebraic and Analytical Properties of Special Matrices

Laith K. Shaakir **Akram S. Mohamed** **Nazar K. Hussein**
College of Computer Sciences and Mathematics
University of Tikrit

Received on: 02/02/2011 **Accepted on: 04/04/2011**

ABSTRACT

In this work we present a subset of $M_{n\times n}$ ($M_{n\times n}$ is the set of all $n\times n$ matrices) which we called the set of special matrices and denoted it by $S_{n\times n}$. We give some important properties of $S_{n\times n}$.

Keywords: Algebraic Properties, Analytical Properties, Special Matrices.

بعض الخصائص الجبرية والتحليلية للمصفوفات الخاصة

ليث شاكر **أكرم محمد** **نزار حسين**
كلية علوم الحاسوب والرياضيات، جامعة تكريت

تاريخ استلام البحث: 2011/02/02 **تاريخ قبول البحث: 2011/04/04**

الملخص

في هذا البحث قدمنا مجموعة جزئية من مجموعة المصفوفات M_{nxn} والتي رمزنا لها بالرمز S_{nxn} وأسميناها مجموعة المصفوفات الخاصة. ودرسنا بعض من خواصها.

الكلمات المفتاحية: الخصائص الجبرية، الخصائص التحليلية، المصفوفات الخاصة.

1. Introduction

Let $M_{n\times n}$ be the set of all $n\times n$ matrices that is $M_{n\times n}=\{[a_{ij}]: a_{ij}\in ¢, i, j=1,2,\cdots,n\}$. It is known that $M_{n\times n}$ is a vector space over ¢ with respect to the vector addition and scalar multiplication defined by $\lfloor a_{ij}\rfloor+\lfloor b_{ij}\rfloor=\lfloor a_{ij}+b_{ij}\rfloor$ and $c\lfloor a_{ij}\rfloor=\lfloor ca_{ij}\rfloor$ for every $\lfloor a_{ij}\rfloor,\lfloor b_{ij}\rfloor\in M_{n\times n}$ and $c\in ¢$. The matrix $A=\lfloor a_{ij}\rfloor\in M_{n\times n}$ is called special matrix if it can be written as

$$A=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}$$

We denote $S_{n\times n}$ to the set of all $n\times n$ special matrices i.e $S_{n\times n}=\{A=\lfloor a_{ij}\rfloor : A \quad is \quad special \quad matrix\}$. It is clear that $A+B\in S_{n\times n}$ and $\alpha A\in S_{n\times n}$, for every $A,B\in S_{n\times n}$ and α is scalar, Then $S_{n\times n}$ is a subspace of $M_{n\times n}$.

In this paper we study the special matrices, and we give some properties of $S_{n\times n}$.

2. Some Properties of Special Matrices

In this section we study some properties of special matrices. One can prove easily the following remark.

Remark (2.1):

1- The basis of $S_{n\times n}$ is

$$\left\{\begin{bmatrix}1&0&\cdots&0\\0&1&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&0&1\end{bmatrix},\begin{bmatrix}0&1&\cdots&0\\0&0&\ddots&\vdots\\\vdots&\ddots&\ddots&1\\1&\cdots&0&0\end{bmatrix},\cdots,\begin{bmatrix}0&0&\cdots&1\\1&0&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&1&0\end{bmatrix}\right\};$$

and the dimension of $S_{n\times n}$ is n.

2- The usual product of two matrices $A,B\in S_{n\times n}$ is also in $S_{n\times n}$, i.e $AB\in S_{n\times n}$ for every $A,B\in S_{n\times n}$.

Recall that the complex number λ is an eigenvalue of the matrix A, if there exists a non-zero vector X such that $AX=\lambda X$,then the vector X is called eigenvector for the matrix A with respect to the eigenvalue λ.

Definition (2.2) [1]:

Let λ be an eigenvalue of the matrix A, the multiplicity of λ is the number of linearly independent eigenvectors corresponding to the eigenvalue λ.

Remark (2.3) [1]:

Let A be an $n\times n$ matrix then

1- The matrix A has exactly n eigenvalues.
2- If the eigenvalues of A are distinct ,then the eigenvectors corresponding to these eigenvalues are linearly independent.

Lemma (2.4) [3]:

The equation $Z^n=1$ has n distinct non-zero roots in the field of complex numbers.

In the following theorem we find the eigenvalues and the eigenvectors for any special matrix.

Theorem (2.5):

Let A be a special matrix, i.e.,

$$A=\begin{bmatrix}a_1&a_2&\cdots&a_n\\a_n&a_1&\cdots&a_{n-1}\\a_{n-1}&a_n&\cdots&a_{n-2}\\\vdots&\vdots&\vdots&\vdots\\a_2&a_3&\cdots&a_1\end{bmatrix}.$$

Then, the eigenvalues of A are $\lambda_r=\sum_{i=1}^{n}a_i p_r^{i-1}$, $r=1,2,\cdots,n$ where $p_1,p_2,\cdots,p_n$ are the roots of the equation $Z^n=1$ also

$$Y_r = \begin{bmatrix} 1 \\ p_r \\ p_r^2 \\ \vdots \\ p_r^{n-1} \end{bmatrix}$$

is the eigenvector corresponding to the eigenvalue λ_r, $r = 1,2,\cdots,n$

Proof:

The eigenvalue λ and the eigen vector

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

for the matrix A satisfy the system $AY = \lambda Y$. This system can be written as follows

$$\sum_{k=1}^{n-m} a_k y_{k+m} + \sum_{k=n-m+1}^{n} a_k y_{k-(n-m)} = \lambda y_{m+1}, \quad m = 0,1,2,\cdots,n-1 .$$

Put $y_k = p^{k-1}$, $k = 1,2,,\cdots,n$ where p is a root of the equation $Z^n = 1$.

$$\sum_{k=1}^{n-m} a_k p^{k+m-1} + \sum_{k=n-m+1}^{n} a_k p^{k-(n-m)-1} = \lambda p^{m+1-1}, \quad m = 0,1,2,\cdots,n-1$$

$$\sum_{k=1}^{n-m} a_k p^m p^{k-1} + \sum_{k=n-m+1}^{n} a_k p^m p^{-n} p^{k-1} = \lambda p^m, \quad m = 0,1,2,\cdots,n-1$$

$$\sum_{k=1}^{n-m} a_k p^{k-1} + p^{-n} \sum_{k=n-m+1}^{n} a_k p^{k-1} = \lambda, \quad m = 0,1,2,\cdots,n-1$$

Since p is root of the equation $Z^n = 1$, then $p^{-n} = \dfrac{1}{p^n} = 1$ and hence $\lambda = \sum_{k=1}^{n} a_k P^{k-1}$ and since the equation $Z^n = 1$ have n roots say $p_1, p_2, \cdots, p_n$ then we have eigenvalues: $\lambda_r = \sum_{i=1}^{n} a_i p_r^{i-1}$ $\quad r = 1,2,\cdots,n$, and the eigenvector for λ_r is

$$Y_r = \begin{bmatrix} 1 \\ p_r \\ p_r^2 \\ \vdots \\ p_r^{n-1} \end{bmatrix}, \quad r = 1,2,\cdots,n$$

Corollary (2.6) :

Let A, B be special matrices. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A with respect to the eigenvectors $X_j = \left[1, p_j, p_j^2, \cdots, p_j^{n-1}\right]$, $j = 1,2,\cdots,n$, respectively and $\beta_1, \beta_2, \cdots, \beta_n$ are eigenvalues of B with respect to the eigenvectors

$X_j = [1, p_j, p_j^2, \cdots, p_j^{n-1}]$, $j = 1,2,\cdots,n$ respectively then the eigenvalues of the matrix AB are $\beta_1\lambda_1, \beta_2\lambda_2, \cdots, \beta_n\lambda_n$.

Proof:

We have $AX_j = \lambda_j X_j$ and $BX_j = \beta_j X_j$ where $j = 1,2,\cdots,n$. Therefore

$(AB)X_j = A\beta_j X_j = \beta_j AX_j = \beta_j\lambda_j X_j$, $j = 1,2,\cdots,n$.

Then, $\{\beta_j\lambda_j\}$, $j = 1,2,\cdots,n$ are eigenvalues of the matrix AB.

Corollary (2.7):

The eigen vectors $X_j = [1, p_j, p_j^2, \cdots, p_j^{n-1}]$, $j = 1,2,\cdots,n$ in Theorem(2.5) are linearly independent .

Proof:

Since the roots $\{p_j\}$, $j = 1,2,\cdots,n$ of the equation $Z^n = 1$ are distinct then the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_n \\ p_1^2 & p_2^2 & \cdots & p_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ p_1^{n-1} & p_2^{n-1} & \cdots & p_n^{n-1} \end{bmatrix}$$

is non-zero [1] (this determinant is called Vandermonde determinant), therefore the vector X_j, $j = 1,2,\cdots,n$ are linearly independent.

Recall that if A and B are two $n \times n$ matrices then A is similar to B if there exists an invertible matrix P such that $A = P^{-1}BP$.

Definition (2.8) [1]:

The matrix A is diagonalizable if A is similar to a diagonal matrix.

Theorem (2.9) [1]:

The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors , in this case A is similar to a diagonal matrix D with the elements of main diagonal are the eigenvalues of A and $P^{-1}AP = D$ where the columns of P are the n linearly independent eigenvectors of A.

The following results follows from Corollary(2.7) and Theorem(2.9).

Theorem (2.10):

If A is a special matrix then A is similar to the diagonal matrix D with the elements of main diagonal are the eigenvalues of A.

Corollary (2.11):

If A and B are special matrices, then $AB = BA$.

Proof:

Since A and B are special matrices, then $A = PD_1P^{-1}$ and $B = PD_2P^{-1}$ where the columns of the matrix P are the eigenvectors $X_j = [1, p_j, p_j^2, \cdots, p_j^{n-1}]$, $j = 1,2,\cdots,n$. Therefore,

$AB = PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1} = PD_2D_1P^{-1} = PD_2P^{-1}PD_1P^{-1} = BA$.

Corollary (2.12):

Let A be a special matrix, then a matrix A is invertible if and only if the zero number is not eigenvalue of A.

Proof:

From Theorem (2.10) , A is diagonalizable, i.e $A = PDP^{-1}$ where

$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$, $\{\lambda_i\}$ are eigenvalues of A,

Now if $\lambda_i \neq 0$ for each i then D is invertible where

$$D^{-1} = \begin{bmatrix} 1/\lambda_1 & & & 0 \\ & 1/\lambda_2 & & \\ & & \ddots & \\ 0 & & & 1/\lambda_n \end{bmatrix}$$

Therefore, $A^{-1} = PD^{-1}P^{-1}$.

Conversely, since $A = PDP^{-1}$ then $D = P^{-1}AP$, where

$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$, $\{\lambda_i\}$ are eigenvalues of A. Since A is invertible then D is also invertible, where $D^{-1} = P^{-1}A^{-1}P$ and hence $\lambda_i \neq 0$ for each *i*.

Recall that the rank of the matrix A is the number of the linearly independent rows in A.

Theorem (2.13) [1]:

Let A be $n \times n$ matrix . Then A is invertible if and only if the rank of A is equal to n.

The following results follows from Corollary(2.12) and Theorem(2.13)

Remark(2.14):

Let A be an $n \times n$ special matrix then the rank of A is equal to n if and only if zero is not eigenvalue of A.

Lemma (2.15): If

$$A=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}$$

is a special matrix, then A has at least one non-zero eigenvalue.

Proof:

The eigenvalues of A are $\lambda_j = a_1 + a_2 p_j + a_3 p_j^2 + \cdots + a_n p_j^{n-1}$, $j = 1,2,\cdots,n$ where p_j, $j = 1,2,\cdots,n$ are the roots of the equation $Z^n = 1$, by Theorem(2.6). If the eigenvalues are all zero then the polynomial $p(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$ has n distinct roots $p_1, p_2, \cdots, p_n$ this contradicts the fact that every polynomial of degree $n-1$ has exactly ($n-1$) roots. Thus, A has at least one non zero eigenvalue.

Recall that a matrix A is nilpotent if there exists apositive integer n such that $A^n = 0$.

The following theorem shows that the set of all special matrices $S_{n\times n}$ does not contain a nilpotent element except the zero matrix.

Theorem (2.16):

If A is a non-zero special matrix, then A is not nilpotent.

Proof:

Suppose that A is nilpotent, then there exists a positive integer number n such that $A^n = 0$. Since A is diagonalizable, then $A = PDP^{-1}$, where

$$D=\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

λ_i is an eigenvalue of A for each i, therefore $\left(PDP^{-1}\right)^n = 0$ that is $PDP^{-1}PDP^{-1}\cdots PDP^{-1} = 0$ this implies that $PD^nP^{-1} = 0$, that is the eigenvalues of A are zero's, this is contradiction to Lemma (2.15).

Suppose that A is a special matrix , we define the center of A as follows: $Z(A) = \{X \in M_{n\times n} : AX = XA\}$. It is clear that if $A = \alpha I$ where I is the identity matrix and α is a scalar, then $Z(A) = M_{n\times n}$.

The following remark follows from Corollary (2.11).

Remark (2.17) :

The set of all special matrices $S_{n\times n}$ is a subset of $Z(A)$, if A is a special matrix.

Lemma (2.18) [1,4] :

If A is an invertible $n \times n$ matrix, then the system $AX = b$ has a unique solution.

We prove the following theorem.

Theorem (2.19):

Let A be a special matrix, if the eigenvalues of A are distinct, then $Z(A) = S_{n\times n}$.

Proof:

From Remark (2.17), we obtain $S_{n\times n} \subseteq Z(A)$. Now ,we prove that $Z(A) \subseteq S_{n\times n}$. Let $X \in Z(A)$, where

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

satisfy $AX = XA$ therefore $AX(x_i) = XA(x_i)$ for every eigenvector x_i, $i = 1,2,\cdots,n$, that is $AX(x_i) = X(\lambda x_i) = \lambda_i X(x_i)$, this show that either $X(x_i) = 0$ or $X(x_i)$ is eigenvector for the eigenvalue λ_i. Since λ_i are distinct, then $X(x_i) = \alpha_i x_i$, $i = 1,2,\cdots,n$, where α_i is constant, thus we have n system

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ p_i \\ \vdots \\ p_i^{n-1} \end{bmatrix} = \alpha_i \begin{bmatrix} 1 \\ p_i \\ \vdots \\ p_i^{n-1} \end{bmatrix}, \; i = 1,2,\cdots,n \qquad \ldots(1)$$

Step1: Take the first equations of system (1) we have

$$x_{11} + x_{12} p_1 + x_{13} p_1^2 + \cdots + x_{1n} p_1^{n-1} = \alpha_1$$
$$x_{11} + x_{12} p_2 + x_{13} p_2^2 + \cdots + x_{1n} p_2^{n-1} = \alpha_2$$
$$\vdots$$
$$x_{11} + x_{12} p_n + x_{13} p_n^2 + \cdots + x_{1n} p_n^{n-1} = \alpha_n$$

We can write this system as

$$\begin{bmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{n-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & p_n & p_n^2 & \cdots & p_n^{n-1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad \ldots(2)$$

Step(2): Take the second equations of system (1) we have

$$x_{21} + x_{22} p_1 + x_{23} p_1^2 + \cdots + x_{2n} p_1^{n-1} = \alpha_1 p_1$$
$$x_{21} + x_{22} p_2 + x_{23} p_2^2 + \cdots + x_{2n} p_2^{n-1} = \alpha_2 p_2$$
$$\vdots$$
$$x_{21} + x_{22} p_n + x_{23} p_n^2 + \cdots + x_{2n} p_n^{n-1} = \alpha_n p_n$$

Multiply the first equation of this system by (p_1^{n-1}) and the second equation by (p_2^{n-1}) and so on and use the fact $p_i^n = 1$, $i = 1,2,\cdots,n$ we have,

$$x_{22} + x_{23}p_1 + x_{24}p_1^2 + \cdots + x_{2n}p_1^{n-2} + x_{21}p_1^{n-1} = \alpha_1$$
$$x_{22} + x_{23}p_2 + x_{24}p_2^2 + \cdots + x_{2n}p_2^{n-2} + x_{21}p_2^{n-1} = \alpha_2$$
$$\vdots$$
$$x_{22} + x_{23}p_n + x_{24}p_n^2 + \cdots + x_{2n}p_n^{n-2} + x_{21}p_n^{n-1} = \alpha_n$$

That is

$$\begin{bmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{n-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & p_n & p_n^2 & \cdots & p_n^{n-1} \end{bmatrix} \begin{bmatrix} x_{22} \\ x_{23} \\ \vdots \\ x_{21} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \ldots(3)$$

We continue these process until we have the system

$$\begin{bmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{n-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & p_n & p_n^2 & \cdots & p_n^{n-1} \end{bmatrix} \begin{bmatrix} x_{nn} \\ x_{n1} \\ \vdots \\ x_{nn-1} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \ldots(n+1)$$

Since

$$\begin{bmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{n-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & p_n & p_n^2 & \cdots & p_n^{n-1} \end{bmatrix}$$

is an invertible matrix, then by Lemma(2.18) , the systems $2,.\cdots,n+1$ have unique solution that is

$$\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix} = \begin{bmatrix} x_{22} \\ x_{23} \\ \vdots \\ x_{21} \end{bmatrix} = \cdots = \begin{bmatrix} x_{nn} \\ x_{n1} \\ \vdots \\ x_{nn-1} \end{bmatrix}, \text{ thus } \begin{array}{l} x_{11} = x_{22} = \cdots = x_{nn} \\ x_{12} = x_{23} = \cdots = x_{n1} \\ \quad\vdots \\ x_{1n} = x_{21} = \cdots = x_{nn-1} \end{array}$$

that is the matrix $X \in S_{n\times n}$ so that $Z(A) \subseteq S_{n\times n}$ and hence $Z(A) = S_{n\times n}$.

The following example shows that if the eigenvalues of the matrix $A \in S_{n\times n}$ are not distinct, then $Z(A) \neq S_{n\times n}$.

Example (2.20):

Let

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and take $X = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. It is clear that $AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and

$XA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and hence $XA = XA$, this implies that $X \in Z(A)$, but $X \notin S_{n\times n}$, so that $Z(A) \neq S_{n\times n}$.

Let $D_{n\times n}$ be the set of all diagonal matrix and define $T : S_{n\times n} \to D_{n\times n}$ as follows $T(A) = P^{-1}AP$ where the columns of P is the eigenvectors of A. Then we give the following result.

Theorem (2.21):

The mapping T is linear transformation which is one-to-one, onto and $T(AB) = T(A)T(B)$, $\forall A, B \in S_{n\times n}$.

Proof :

$T(A_1 + A_2) = P^{-1}(A_1 + A_2)P = P^{-1}(A_1)P + P^{-1}(A_2)P = T(A_1) + T(A_2)$

$T(\alpha A) = P^{-1}(\alpha A)P = \alpha P^{-1}AP = \alpha T(A)$, $\forall A \in S_{n\times n}$ and α is constant. Thus, T is linear transformation

Now, let $T(A_1) = T(A_2)$, that is $P^{-1}A_1P = P^{-1}A_2P$ so that $A_1 = A_2$ this implies that T is one-to-one

Now, we prove that T is onto

Let $D = \alpha I$, where α is scalar then take $A = D$ and $T(A) = D$, suppose $D \neq \alpha I$,

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

If $p_1, p_2, \cdots, p_n$ are the root of the equation $Z^n = 1$,then the system

$$\begin{bmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^{n-1} \\ 1 & p_2 & p_2^2 & \cdots & p_2^{n-1} \\ \vdots & & \vdots & \cdots & \vdots \\ 1 & p_n & p_n^2 & \cdots & p_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

has only one solution say $(c_0, c_1, c_2, \cdots, c_{n-1})$. Take,

$$A = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & & \vdots & \cdots & \vdots \\ c_1 & & & \cdots & c_0 \end{bmatrix} \in S_{n\times n}$$

it is clear from Theorem(2.6) that the eigenvalues of A are $\lambda_1, \lambda_2, \cdots, \lambda_n$ therefore $T(A) = P^{-1}AP = D \in D_{n\times n}$. Thus T is onto.

It is remained now to prove that $T(AB) = T(A)T(B), \forall A, B \in S_{nxn}$

$T(AB) = P^{-1}ABP = P^{-1}APP^{-1}BP = T(A)T(B)$ for every $A, B \in S_{n\times n}$.

Remark (2.22):

We can prove easily that, the inverse mapping $T^{-1}: D_{n\times n} \to S_{n\times n}$ which is defined by $T^{-1}(D) = PDP^{-1}$, for every $D \in D_{n\times n}$ is linear transformation .

We end this section by the following theorem

Theorem (2.23):

Let $p(x)$ be a polynomial of degree n and $A \in S_{n\times n}$. The eigenvalues of A are roots of $p(x)$ if and only if the matrix A is a roots of the polynomial matrix $p(X)$.

Proof:

Suppose that the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix A are the roots of the polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, i.e. $p(\lambda_i) = 0$, $i = 1,2,\cdots,n$ so that

$$\begin{bmatrix} p(\lambda_1) & & & 0 \\ & p(\lambda_2) & & \\ & & \ddots & \\ 0 & & & p(\lambda_n) \end{bmatrix} = 0 .$$

Then,

$$\begin{bmatrix} c_0 + c_1\lambda_1 + c_2\lambda_1^2 + \cdots + c_n\lambda_1^n & & & 0 \\ & c_0 + c_1\lambda_2 + c_2\lambda_2^2 + \cdots + c_n\lambda_2^n & & \\ & & \ddots & \\ 0 & & & c_0 + c_1\lambda_n + c_2\lambda_n^2 + \cdots + c_n\lambda_n^n \end{bmatrix} = 0$$

hence $c_0 I + c_1 D + c_2 D^2 + \cdots + c_n D^n = 0$ where I is the identity matrix and

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

therefore $P(c_0 I + c_1 D + c_2 D^2 + \cdots + c_n D^n)P^{-1} = 0$, where the column of p are the eigenvalues of the $\lambda_1, \lambda_2, \cdots, \lambda_n$. This implies

$c_0 PP^{-1} + c_1 PDP^{-1} + c_2 PD^2 P^{-1} + \cdots + c_n PD^n P^{-1} = 0$, that is

$c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n = 0$ thus A is a root of the polynomial matrix

$p(X) = c_0 I + c_1 X + c_2 X^2 + \cdots + c_n X^n$

Conversely, if A is a root of the polynomial

$p(X) = c_0 I + c_1 X + c_2 X^2 + \cdots + c_n X^n$

Then $p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n = 0$.

Therefore $c_0 PP^{-1} + c_1 PDP^{-1} + c_2 PD^2 P^{-1} + \cdots + c_n PD^n P^{-1} = 0$.

Hence, $P(c_0 I + c_1 D + c_2 D^2 + \cdots + c_n D^n)P^{-1} = 0$

Thus $c_0 I + c_1 D + c_2 D^2 + \cdots + c_n D^n = 0$.

Thus,

$$\begin{bmatrix} c_0+c_1\lambda_1+c_2\lambda_1^2+\cdots+c_n\lambda_1^n & & & 0 \\ & c_0+c_1\lambda_2+c_2\lambda_2^2+\cdots+c_n\lambda_2^n & & \\ & & \ddots & \\ 0 & & & c_0+c_1\lambda_n+c_2\lambda_n^2+\cdots+c_n\lambda_n^n \end{bmatrix}=0$$

this implies $p(\lambda_i)=c_0+c_1\lambda_i+c_2\lambda_i^2+\cdots+c_n\lambda_i^n=0$, $i=1,2,\cdots,n$ that is $\lambda_1,\lambda_2,\cdots,\lambda_n$ are roots of the polynomial $p(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n$.

3. Analytical Properties of Special Matrices

Recall that the vector space H is called Hilbert space if it is complete inner product space, The spaces R^n and C^n are Hilbert spaces ;Since $M_{n\times n}(R)$ equivalent R^{n^2} and $M_{n\times n}(C)$ equivalent C^{n^2}, then $M_{n\times n}(R)$ and $M_{n\times n}(C)$ are Hilbert spaces.In section two we see that $S_{n\times n}$ is a subspace of $M_{n\times n}$ also in this section we prove that $S_{n\times n}$ is Hilbert space. Finally, we show that $S_{n\times n}$ is Banach algebra.

Proposition (3.1):

The space $M_{n\times n}$ is inner product space where $\langle [a_{ij}],[b_{ij}]\rangle=\sum_{i,j}a_{ij}\bar{b}_{ij}$ for all $[a_{ij}],[b_{ij}]\in M_{n\times n}$, where $\bar{b}_{ij}$ is the complex conjugate of b_{ij}.

For the completeness we give the proof of the following theorem .

Theorem (3.2):

The space $M_{n\times n}$ is Hilbert space.

Proof:

We see in Proposition(3-1) that $M_{n\times n}$ is inner product space. It is remained to prove that $M_{n\times n}$ is complete, let $\{A^n\}$ be acauchy sequence in $M_{n\times n}$ that is for all $\varepsilon>0$, there exist positive integer k such that $\|A^n-A^m\|<\varepsilon$ for all $n,m>k$, therefore $|a_{ij}^n-a_{ij}^m|^2\le\sum_{i=1}^n\sum_{j=1}^n|a_{ij}^n-a_{ij}^m|^2=\|A^n-A^m\|^2<\varepsilon^2$ for all $n,m>k$ so that $|a_{ij}^n-a_{ij}^m|<\varepsilon$ for all $n,m>k$ and hence $\{a_{ij}^n\}$ is Cauchy sequence in the complex numbers ¢ for all i,j, since the space of complex numbers is complete then $\{a_{ij}^n\}$ is converge sequence say to a_{ij} .We claim that the sequence $\{A^m\}=\{[a_{ij}^m]\}$ is converge to $A=[a_{ij}]$ as $m\to\infty$, let $\varepsilon>0$, since $\{a_{ij}^m\}$ is converge to a_{ij} as $m\to\infty$ then there exist a positive integer number k_{ij} such that $|a_{ij}^m-a_{ij}|<\varepsilon/n$ for all $m>k_{ij}$, let $k=\max\{k_{ij}, i=1,2,\cdots,n, j=1,2,\cdots,n\}$.

Now, $\|A^m - A\|^2 = \sum_i \sum_j |a_{ij}^m - a_{ij}|^2 < \varepsilon^2$ for all $m > k$, hence $\|A^m - A\| < \varepsilon$ for all $m > k$, thus $\{A^m\}$ is converge sequence, this implies that $M_{n\times n}$ is complete inner product space and hence $M_{n\times n}$ is Hilbert space .

Corollary (3.3):

The space $S_{n\times n}$ is Hilbert space .

Proof:

If $\{A^n\} = \{[a_{ij}^n]\}$ is a cauchy sequence in $S_{n\times n}$ then $\{A^n\}$ is a cauchy sequence in $M_{n\times n}$, since $M_{n\times n}$ is complete then there exist $A = [a_{ij}] \in M_{n\times n}$ such that $\{A^n\} \to A$ (Theorem(3.2)), that is $\{a_{ij}^n\} \to a_{ij}$ for all i, j. Since $A^n = [a_{ij}^n] \in S_{n\times n}$ for all n then $A = [a_{ij}] \in S_{n\times n}$ and hence $S_{n\times n}$ is complete, this implies that $S_{n\times n}$ is Hilbert space.

Theorem (3. 4):

The space $S_{n\times n}$ is a Banach algebra.

Proof:

We must prove that $\|C\| \le \|A\|\|B\|$, for every $A, B \in S_{n\times n}$, where $C = AB$

Since

$$\|A\| = \sqrt{n(\sum_{j=1}^{n} |a_{1j}|^2)}\ ,\ \|B\| = \sqrt{n(\sum_{j=1}^{n} |b_{1j}|^2)}$$

then $\|A\|^2 \|B\|^2 = n^2 (\sum_{j=1}^{n} |a_{1j}|^2) \left(\sum_{j=1}^{n} |b_{1j}|^2 \right)$,

$$\|C\|^2 = n \sum_{j=1}^{n} |c_{1j}|^2 = n \left[\left| \sum_{k=1}^{n} a_{1k} b_{k1} \right|^2 + \left| \sum_{k=1}^{n} a_{1k} b_{k2} \right|^2 + \cdots + \left| \sum_{k=1}^{n} a_{1k} b_{kn} \right|^2 \right]$$

By Schwarz inequality we have

$$\|C\|^2 \le n \left[\sum_{k=1}^{n} |a_{1k}|^2 \sum_{k=1}^{n} |b_{k1}|^2 + \sum_{k=1}^{n} |a_{1k}|^2 \sum_{k=1}^{n} |b_{k2}|^2 + \cdots + \sum_{k=1}^{n} |a_{1k}|^2 \sum_{k=1}^{n} |b_{kn}|^2 \right]$$

$$= n \sum_{k=1}^{n} |a_{1k}|^2 \left[\sum_{k=1}^{n} |b_{k1}|^2 + \sum_{k=1}^{n} |b_{k2}|^2 + \cdots + \sum_{k=1}^{n} |b_{kn}|^2 \right] = \|A\|^2 \|B\|^2$$

Thus $\|C\| \le \|A\|\|B\|$, so that $S_{n\times n}$ is a Banach algebra

REFERENCES

[1] E.H. Connell, 2002, "Elements of Abstract and Linear Algebra", Florida U.S.A.

[2] Erwin Kreyszig, 1978, "Introductory Functional Analysis with Applications", John Wiley & Sons.

[3] Walter Rudin, ,1970 ,"Real and Complex Analysis", Mc GRAW-Hill.

[4] Kaare Brandt Petersen, Micheal Syskind Pedersen, 2008, "The Matrix Cookbook", http://matrix cook book.com.