Existence Results of Fractional Mixed Volterra -Fredholm Integrodifferential Equations with Integral Boundary Conditions

Sohaib T. Hasan
College of Education
University of Mosul, Iraq

Rabeea M. Hani
College of Basic Education
University of Mosul , Iraq

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ABSTRACT

In this paper, we give an existence results for the fractional mixed Volterra -Fredholm integrodifferential equation with Integral Boundary Conditions. First, we use the Banach contraction principle to prove the existence and the uniqueness of the solution for the boundary value problem. Second, we study the existence of the solutions for the boundary value problem. using Krasnoselskii's fixed point theorem.

Keywords: Volterra -Fredholm equation, Integral Boundary Conditions, Banach contraction principle.

نتائج الوجود للمعادلات التفاضلية التكاملية من نوع فولتيرا – فريدهولم ومن رتبة كسرية مع شروط حدودية تكاملية

ربيع هاني
كلية التربية الأساسية، جامعة الموصل

صهيب حسن
كلية التربية، جامعة الموصل

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الملخص

في هذا البحث أعطينا بعض نتائج الوجود للمعادلات التفاضلية التكاملية من نوع فولتيرا – فريدهولم ومن رتبة كسرية مع شروط حدودية تكاملية. إذ برهنا أولا وجود ووحدانية الحل باستخدام مبدأ باناخ للتطبيقات الانكماشية، ثم برهنا بعد ذلك وجود الحلول للمعادلة باستخدام مبرهنة كراسنوسيلسكي للنقطة الثابتة.

الكلمات المفتاحية: معادلات فولتيرا – فريدهولم، شروط حدودية تكاملية، مبدأ باناخ للتطبيقات الانكماشية.

1. Introduction

We consider the following fractional integrodifferential boundary value problem with integral boundary condition

$$
\begin{aligned}
{}^cD^\alpha y(t) &= f\left(t, y(t), \int_0^T k_1(t,s,y(s))ds, \int_0^t k_2(t,s,y(s))ds\right) \\
y(0) &= \int_0^T g(s,y(s))ds \\
y(T) &= \int_0^T h(s,y(s))ds \qquad (1.1)
\end{aligned}
$$

where ${}^CD^\alpha$ is the standard Caputo derivative, and $1 < \alpha < 2$ and $t \in J = [0,T]$, $y \in C(J,R)$ the Banach space with norm: $\|y\| = max_{t\in J}|y(t)|$ and the functions $f: J \times R \times R \times R \to R$, $k_1: J \times J \times R \to R$, $k_2: J \times J \times R \to R$, $g: J \times J \to R$ and $h: J \times J \to R$ are continuous functions. Here, For brevity let

$$K_1 y(t) = \int_0^T k_1(t,s,y(s))ds \qquad ,K_2 y(t) = \int_0^t k_2(t,s,y(s))ds$$

The problem of existence and uniqueness of solution for fractional differential equations have been considered by many authors; see for example [1], [2], [3], [7], [8], [9], [11], [14]. An integrodifferential equation is an equation which involves both integrals and derivatives of an unknown function. The existence and uniqueness problems of fractional nonlinear differential and integrodifferential equations as a basic theoretical part of some applications are investigated by many authors (see for examples

[1], [13], and [14]). It arises in many fields like electronic, fluid dynamics, biological models, and chemical kinetics. A well-known example is the equations of basic electric circuit analysis. In recent years, the theory of various integrodifferential equations in Banach spaces has been studied deeply due to their important values in sciences and technologies, and many significant results have been established; see for example [10], [12] .

In [4] the authors studied the non-fractional mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions using Leray - Schauder theorem, and In [15] the authors studied the fractional differential equations with integral boundary conditions By means of the famous Banach contraction mapping principle.

we study in this paper the existence of the solution of the boundary value problem for fractional integrodifferential equations (in the case of $1 < \alpha < 2$) with integral boundary conditions in Banach spaces by using Banach and Krasnoselskii's fixed point theorems.

For the sake of clarity, we list the necessary definitions from fractional calculus theory here. These definitions can be found in the recent literature.

Definition 2.1 [5] Let $\alpha > 0$, for a function $y : (0, +\infty) \to R$. The the fractional integral of order α of y is defined by

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds$$

provided the integral exists.

Definition 2.2 The Caputo derivative of a function $y: (0, +\infty) \to R$ is given by

$${}^{c}D^{\alpha} y(t) = I^{n-\alpha}\left(D^{n} y(t)\right) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t - s)^{n-\alpha-1} y^{(n)}(s) ds$$

provided the right side is point wise defined on $(0, +\infty)$,where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α.

The properties of the above operators can be found in [6] and the general theory of fractional differential equations can be found in [5]. Γ denotes the Gamma function:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} \ t^{\alpha-1} \ dt$$

The Gamma function satisfies the following basic properties:

(1) For $any\ n \in R$

$\Gamma(n + 1) = n\Gamma(n)$ and if $n \in Z\ then\ \Gamma(n) = (n - 1)!$

(2) For any $1 < \alpha \in R$, then

$$\frac{\alpha + 1}{\Gamma(\alpha + 1)} = \frac{\alpha + 1}{\alpha\,\Gamma(\alpha)} < \frac{2}{\Gamma(\alpha)}$$

From Definition 2.1, we can obtain the following lemma.

Lemma 2.1 Let $0 < n - 1 < \alpha < n$. If we assume $y \in C^n(0,1)$, the fractional differential equation

$$ {}^cD^{\alpha}y(t) = 0 $$

has a unique solution

$$ y(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k $$

Lemma 2.2 The function $y \in C^2[O,T]$ is a solution of boundary value problem (1.1), if and only if y is a solution of the following fractional integral equation:

$$ y(t) = \int_0^T \left(1 - \frac{t}{T}\right) g\big(s, y(s)\big) + \frac{t}{T} h\big(s, y(s)\big) ds $$
$$ - \frac{t}{T\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s)) ds $$
$$ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s)) ds \qquad (1.2) $$

That is, every solution of (1.2) is also a solution of (1.1) .

Proof . By integrating the equation ${}^cD^{\alpha}y(t) = f(t, y(t), K_1 y(t), K_2 y(t))$, to the order α we have

$$ y(t) = y(0) + y'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s)) ds $$

Then

$$ y(T) = y(0) + y'(0)T + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s)) ds $$

By the boundary conditions

$$ y(0) = \int_0^T g\big(s, y(s)\big) ds $$

$$ y(T) = y(0) + y'(0)T + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s)) ds $$
$$ = \int_0^T h\big(s, y(s)\big) ds $$

we have

$$y'(0) = \frac{1}{T}[\int_0^T h(s, y(s)) - g(s, y(s))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds]$$

Thus

$$y(t) = \int_0^T g(s, y(s))ds + \frac{t}{T}[\int_0^T (h(s, y(s)) - g(s, y(s)))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds]$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds$$

$$y(t) = \int_0^T \left(1-\frac{t}{T}\right) g(s, y(s)) + \frac{t}{T} h(s, y(s))ds$$

$$-\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds$$

Therefore, the proof is completed. ■

To proceed, we need the following assumptions:

(H1) There exists a positive constant L such that

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$$

For all $t \in J$ and $x_1, y_1, z_1, x_2, y_2, z_2 \in R$

(H2) There exist positive constants N_1, N_2 such that

$$|k_1(t, s, y_1) - k_1(t, s, y_2)| \leq N_1|y_1 - y_2|$$

and

$$|k_2(t, s, y_1) - k_2(t, s, y_2)| \leq N_2|y_1 - y_2|$$

For all $y_1, y_2 \in R$, Let $N = max\{N_1, N_2\}$

(H3) There exist positive constants M_1, M_2 such that

$$|g(t,y_1) - g(t,y_2)| \le M_1|y_1 - y_2|$$

and

$$|h(t,y_1) - h(t,y_2)| \le M_2|y_1 - y_2|$$

For all $t \in J$ and $y_1,y_2 \in R$, Let $M = max\{M_1,M_2\}$

Lemma 2.3. If **(H2)** is satisfied, then for each $y_1,y_2 \in C(J,R)$ the estimates

$$\|K_1y_1 - K_1y_2\| \le N\|y_1 - y_2\|\,T$$

$$\|K_2y_1 - K_2y_2\| \le N\|y_1 - y_2\|\,T$$

are satisfied.

Proof . By **(H2)** we have

$$|K_1y_1(t) - K_1y_2(t)| = \left|\int_0^t k_1(t,s,y(s))ds - \int_0^t k_1(t,s,y(s))ds\right|$$

$$\le \int_0^t |k_1(t,s,y(s)) - k_1(t,s,y(s))|\,ds$$

$$\le \int_0^t N_1\|y_1 - y_2\|\,ds = N_1\|y_1 - y_2\|t \le N_1\|y_1 - y_2\|T$$

$$\therefore \|K_1y_1 - K_1y_2\| \le N\|y_1 - y_2\|\,T$$

Similarly, for the other estimate, we use assumption **(H2)** ,to get

$$\|K_2y_1 - K_2y_2\| \le N_2\|y_1 - y_2\|\,T \qquad \blacksquare$$

3. Main Results

In this section, we give the existence and uniqueness of the solutions for problem (1.1).

Theorem 3.1 If **(H1)** , **(H2)** and **(H3)** are satisfied, and

$$q\Gamma(\alpha+1) \ge MT\Gamma(\alpha+1) + (2L\,T^{\alpha} + 4LNT^{\alpha+1}) \quad \text{for some } 0 < q < 1$$

then the fractional integrodifferential equation (1.1) has a unique solution.

Proof. We use the Banach contraction principle to prove the existence and uniqueness of the solution to (1.1).

Let $E = C^1[O,T]$ with the norm $\|y\| = max_{t\in[0,T]}|y(t)|$. $(E,\|\cdot\|)$ is a Banach space. Consider the operator $F: E \to E$ defined by

$$(Fy)(t) = \int_0^T \left(1 - \frac{t}{T}\right)g(s,y(s)) + \frac{t}{T}h(s,y(s))ds$$

$$- \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y(s),K_1y(s),K_2y(s))ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ f(s,y(s),K_1y(s),K_2y(s))ds$$

It is clear that y is the solution of the boundary value problem (1.1) if and only if y is a fixed point of F. The mapping $F\colon\ E \to E$ is a continuous and compact operator on E. In the following, we prove that F has a unique fixed point in E. First of all, for any $x, y \in E$, we can get that

$$|Fy(t) - Fx(t)|$$

$$= \left|\int_0^T \left(1-\frac{t}{T}\right)(g(s,y(s)) - g(s,x(s)) + \frac{t}{T}h(s,y(s)) - h(s,x(s))ds\right.$$

$$-\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\ f(s,y(s),K_1y(s),K_2y(s)) - f(s,y(s),K_1x(s),K_2x(s))ds$$

$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ f(s,y(s),K_1y(s),K_2y(s)) - f(s,x(s),K_1x(s),K_2x(s))ds\right|$$

$$\leq \int_0^T \left(1-\frac{t}{T}\right)|g(s,y(s)) - g(s,x(s))| + \frac{t}{T}|h(s,y(s)) - h(s,x(s))|\,ds$$

$$+\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y(s),K_1y(s),K_2y(s)) - f(s,x(s),K_1x(s),K_2x(s))|\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s),K_1y(s),K_2y(s)) - f(s,x(s),K_1x(s),K_2x(s))|\,ds$$

$$\leq \left(1-\frac{t}{T}\right)\|y-x\|\int_0^T M_1 ds + \frac{t}{T}\|y-x\|\int_0^T M_2\, ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} L(\|y-x\| + N_1\|y-x\|T + N_2\|y-x\|\,T)\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L(\|y-x\| + N_1\|y-x\|T + N_2\|y-x\|\,T)\,ds$$

$$\leq \left(1-\frac{t}{T}\right)\|y-x\|MT+\frac{t}{T}\|y-x\|MT$$

$$+\frac{L(\|y-x\|+N_1\|y-x\|T+N_2\|y-x\|\,T)}{\Gamma(\alpha)}\left(\int_0^T (T-s)^{\alpha-1}ds+\int_0^t (t-s)^{\alpha-1}ds\right)$$

$$=\|y-x\|MT+\frac{L(\|y-x\|+N_1\|y-x\|T+N_2\|y-x\|\,T)}{\Gamma(\alpha)}\left(\frac{T^\alpha}{\alpha}+\frac{t^\alpha}{\alpha}\right)$$

$$\leq \left(MT+\frac{2L\,T^\alpha+4LNT^{\alpha+1}}{\Gamma(\alpha+1)}\right)\|y-x\| \quad \leq q\|y-x\|$$

Then $\|Fy-Fx\|\leq q\|y-x\|$

which implies that F is a contraction mapping.

By means of the Banach contraction mapping principle, F has a unique fixed point which is a unique solution of the boundary value problems (1.1).■

Our second result uses the following Krasnoselskii fixed point theorem.

Theorem 3.2. Let B be a closed convex and nonempty subsets of Banach space X**.** Let $\mathcal{L}$ and $\mathcal{N}$ be two operators such that

(1) $\mathcal{L}x+\mathcal{N}y\in B$ whenever $x,y\in B$.
(2) $\mathcal{L}$ is a contraction mapping.
(3) $\mathcal{N}$ is compact and continuous.

Then there exists $z\in B$ such that $z=\mathcal{L}z+\mathcal{N}z$.

Suppose that

(H4) There exist positive constants $\tilde{M}_1,\tilde{M}_2,\tilde{M}_3$ such that

$\|f(t,x,y,z)\|\leq \tilde{M}_1$

$\|h(t,y)\|<\tilde{M}_2$,$\|g(t,y)\|<\tilde{M}_3$

$for\ all\ x,y,z\in R\ and\ t\in J$, Let $\tilde{M}=max\{\tilde{M}_1,\tilde{M}_2,\tilde{M}_3\}$

Theorem 3.3. Assume that the conditions $(\mathbf{H}_1)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ are satisfied. Then the fractional integrodifferential equation (1.1) has at least one solution on J provided that $MT<1$

Proof. Let us choose

$$r\geq\left(T+\frac{2T^\alpha}{\Gamma(\alpha)}\right)\tilde{M}$$

Consider the disk $B_r=\{y\in C(J,R):\|y\|\leq r\}$
Define on B_r the operators $\mathcal{L}$ and $\mathcal{N}$ by

$$(\mathcal{L}x)(t) = \int_0^T \left(1 - \frac{t}{T}\right) g\big(s, x(s)\big) + \frac{t}{T} h\big(s, x(s)\big) ds$$

and

$$(\mathcal{N}y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds$$
$$- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds$$

Step1. For any $x; y \in B_r$ then $\mathcal{L}x + \mathcal{N}y \in B_r$

In fact,

$$|(\mathcal{L}x)(t) + (\mathcal{N}y)(t)| = \Bigg| \int_0^T \left(1 - \frac{t}{T}\right) g\big(s, x(s)\big) + \frac{t}{T} h\big(s, x(s)\big) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds$$
$$- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), K_1 y(s), K_2 y(s))ds \Bigg|$$
$$\le \int_0^T \bar{M}\ ds + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^T \bar{M}\ ds + \int_0^t \bar{M}\ ds \right) \le \left(T + \frac{2T^{\alpha}}{\Gamma(\alpha)}\right) \bar{M} \le r$$

$\therefore \|\mathcal{L}x + \mathcal{N}y\| \le r$, Hence, we deduce that $\mathcal{L}x + \mathcal{N}y \in B_r$

Step 2. We show that $\mathcal{L}$ is a contraction mapping.

For any $t \in J$ and $x, y \in C(J, R)$ we have

$$|\mathcal{L}x(t) - \mathcal{L}y(t)| = \Bigg| \int_0^T \left(1 - \frac{t}{T}\right) g\big(s, x(s)\big) + \frac{t}{T} h\big(s, x(s)\big) ds$$
$$- \int_0^T \left(1 - \frac{t}{T}\right) g\big(s, y(s)\big) + \frac{t}{T} h\big(s, y(s)\big) ds \Bigg|$$
$$\le \int_0^T \left(1 - \frac{t}{T}\right) M_1 \|x - y\| + \frac{t}{T} M_2 \|x - y\|\ ds \le MT\|x - y\|$$

Then

$$\|\mathcal{L}x - \mathcal{L}y\| \le MT\|x - y\|$$

Since $MT < 1$, then $\mathcal{L}$ is a contraction mapping.

Step 3. Now , we have to prove that $\mathcal{N}$ is continuous and compact.

For this purpose, we assume that $y_n \to y$ in *C(J,R)*. It comes from the continuity of k_1 and k_2 that

$$k_1(t,s,y_n(s)) \to k_1(t,s,y(s)) \;\; and \; \|k_1(t,s,y_n(s)) - k_1(t,s,y(s))\| \le 2K_1^*$$

,

$$k_2(t,s,y_n(s)) \to k_2(t,s,y(s)) \;\; and \; \|k_2(t,s,y_n(s)) - k_2(t,s,y(s))\| \le 2K_2^*$$

By the dominated convergence theorem,

$$\int_0^T k_1(t,s,y_n(s))ds \to \int_0^T k_1(t,s,y(s))ds \quad , \int_0^t k_2(t,s,y_n(s)) \to \int_0^t k_2(t,s,y(s))ds$$

as $n \to \infty$ Then by **(H1)** we have

$$f(s,y_n(s),K_1y_n(s),K_2y_n(s)) \to f(s,y(s),(K_1y(s),K_2y(s)) \; as \; n \to \infty , s \in J$$

$$\|f(s,y_n(s),K_1y_n(s),K_2y_n(s))\| \le \widetilde{M}$$

By the dominated convergence theorem we have

$$|(\mathcal{N}y_n)(t) - (\mathcal{N}y)(t)| =$$

$$\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y_n(s),K_1y_n(s),K_2y_n(s))ds$$

$$-\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y_n(s),K_1y_n(s),K_2y_n(s))$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s),K_1y(s),K_2y(s))ds$$

$$+\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y(s),K_1y(s),K_2y(s))ds\Bigg|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |f(s,y_n(s),K_1y_n(s),K_2y_n(s)) - f(s,y(s),K_1y(s),K_2y(s))| \, ds$$

$$+\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} |f(s,y_n(s),K_1y_n(s),K_2y_n(s)) - f(s,y(s),K_1y(s),K_2y(s))| \, ds$$

$$\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \int_0^t |f(s, y_n(s), K_1 y_n(s), K_2 y_n(s)) - f(s, y(s), K_1 y(s), K_2 y(s))| \, ds$$

$$+ \frac{T^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |f(s, y_n(s), K_1 y_n(s), K_2 y_n(s)) - f(s, y(s), K_1 y(s), K_2 y(s))| \, ds$$

$$\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |f(s, y_n(s), K_1 y_n(s), K_2 y_n(s)) - f(s, y(s), K_1 y(s), K_2 y(s))| \, ds$$

$\longrightarrow 0$ as $n \to \infty$

which implies that $\mathcal{N}$ is continuous.

To prove that $\mathcal{N}$ is a compact operator, we have observed that $\mathcal{N}$ is a composition of two operators, that is, $N = U \circ V$ where

$$(V\, y)(t) = f\big(t, y(t), (K_1 y)(t), (k_2 y)(t)\big) \,, \qquad for\ all\ t \in J$$

and

$$(Uy)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\, y(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1}\, y(s) ds \,, \quad for\ all\ t \epsilon J$$

Since for the same reason as $\mathcal{N}$ the operator V is also continuous, it suffices to prove that V is uniformly bounded and U is compact to prove that $\mathcal{N}$ is compact.

Let $y \in B_r$. then

$$\|(V\, y)\| = \big\|f\big(t, y, (K_1 y), (k_2 y)\big)\big\| \leq \tilde{M}$$

from which we deduce that V is uniformly bounded on B_r.

$$|(Uy)(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\, y(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1}\, y(s) ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\, r\ ds + \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1}\, r\, ds \leq \frac{2T^{\alpha} r}{\Gamma(\alpha + 1)}$$

Then

$$\|Uy\| \leq \frac{2T^{\alpha} r}{\Gamma(\alpha + 1)}$$

on the other hand, for $0 < s < t_2 < t_1 < T$,

$$|(Uy)(t_1) - (Uy)(t_2)|$$
$$= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} y(s) ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} y(s)\, ds + \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} y(s) ds \right|$$
$$= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} y(s) ds \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_2} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \, \|y(s)\| ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t_1} (t_1 - s)^{\alpha-1} \|y(s)\| ds$$
$$\leq \frac{r}{\Gamma(\alpha)} |\, 2(t_1 - t_2)^{\alpha} + t_2{}^{\alpha} - t_1{}^{\alpha}|$$

which does not depend on y. So UB_r is relatively compact. By the Arzela-Ascoli Theorem, U is compact. Thus, we have proved that $\mathcal{N}$ is continuous and compact, $\mathcal{L}$ is a contraction mapping and $\mathcal{L}x + \mathcal{N}y \in B_r$ if $x, y \in B_r$. Hence, the Krasnoselskii theorem lead us to conclude that the boundary value problems (1.1) has at least one solution on J. ■

4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional BVP

$$^cD^{\alpha} y(t) = \frac{e^{-t}}{2(19 + e^t)} \left(\int_0^1 2se^{-y(s)^2} ds + \int_0^t 2(t - s) \frac{|y(s)|}{|y(s)| + 1} ds) \right)$$

$$y(0) = \int_0^1 \frac{(1-s)^2}{2} \cos y(s)\, ds$$

$$y(1) = \int_0^1 \frac{s^2}{2}\, sin\, y(s)\, ds \tag{4.1}$$

Here,

$$f(t, x, y, z) = \frac{e^{-t}}{2(19 + e^t)} [y + z] \quad ,$$

And $k_1(t, s, y) = 2se^{-y^2}$, $k_2(t, s, y) = 2(t - s) \frac{|y|}{|y|+1}$

And $g(t, y) = \frac{(1-s)^2}{2} \cos y$, $h(t, y) = \frac{s^2}{2} \sin y$

For all $t \in J = [0,1]$ and $t, x, y, z \in R$,

Then we have :

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)| \le \frac{e^{-t}}{2(19+e^t)}(|y_1-y_2|+|z_1-z_2|)$$

$$\le \frac{e^0}{2(19+e^0)}(|y_1-y_2|+|z_1-z_2|)$$

$$\le \frac{1}{40}(|y_1-y_2|+|z_1-z_2|)$$

Then $L = \frac{1}{40}$, And

$$|f(t,x,y,z)| = \left|\frac{e^{-t}}{2(19+e^t)}\left[\int_0^1 2s\frac{y(s)}{y(s)+1}ds + \int_0^t 2(t-s)e^{-y^2(s)}ds\right]\right|$$

$$\le \frac{1}{40}\left[\int_0^1 2sds + \int_0^t 2(t-s)\,ds)\right] \le \frac{1}{40}$$

And

$$\left|\frac{\partial K_1(t,s,y)}{\partial y}\right| = \left|2(t-s)e^{-y^2}.(-2y)\right| \le 2\,,$$

$$\left|\frac{\partial K_2(t,s,y)}{\partial y}\right| = \left|2S\ \frac{-1}{(1+y)^2}\right| \le 2$$

Then $N_1 = N_2 = N = 2$

And

$$\left|\frac{\partial g(t,y)}{\partial y}\right| = \left|\frac{(1-t)^2}{2}.(-\sin y)\right| \le \frac{1}{2}$$

$$\left|\frac{\partial h(t,y)}{\partial y}\right| = \left|\frac{s^2}{2}\cos y\right| \le \frac{1}{2}$$

Then $M_1 = M_2 = M = \frac{1}{2}$

Then (H_1) , (H_2) and (H_3) are satisfied with $L = \frac{1}{40}$, $N = 2$, $M = \frac{1}{2}$.

Hence for all $\alpha \in (1,2) \implies 1! < \Gamma(\alpha+1) < 2!$

Therefore

$$MT + \frac{2L\,T^{\alpha} + 4LNT^{\alpha+1}}{\Gamma(\alpha+1)} = \frac{1}{2} + \frac{2\left(\frac{1}{40}\right)1^{\alpha} + 4\left(\frac{1}{40}\right)(2)1^{\alpha+1}}{\Gamma(\alpha+1)} \le 0.75 < 1$$

Then by Theorem 3.1 the fractional BVP (4.1) has a unique solution on $[0,1]$. ∎

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