Existence, Uniqueness and Stability Theorems for Certain Functional Fractional Initial Value Problem

Joseph G. Abulahad **Shayma A. Murad**
College of Education
University of Dohuk, Iraq

Received on: 21 / 10 / 2009 **Accepted on: 11 / 4 / 2010**

ABSTRACT

In this paper, we deal with non-linear functional fractional differential equation with initial condition in L_1 space. We will study the existence, uniqueness and stability of the solution of fractional differential equation.

Keywords: Fractional differential equation; Riemann Liouville Fractional derivative; Stability.

نظريات الوجود والوحدانية والاستقرارية لبعض مشكلات القيمة الأولية الجزئية الوظيفية

جوزيف عبد الاحد **شيماء مراد**

كلية التربية، جامعة دهوك

تاريخ استلام البحث: 2009/10/21 **تاريخ قبول البحث: 2010/04/11**

الملخص

في هذا البحث تطرقنا إلى المعادلات التفاضلية ذات الرتب الكسرية والغير الخطية مع شروط ابتدائية في الفضاء L_1 وقد تمت دراسة الوجود، والوحدانية والاستقرارية لحل المعادلات التفاضلية الكسرية.

الكلمات المفتاحية: المعادلة التفاضلية ذات رتب كسرية؛ مشتقة Riemann Liouville، الاستقرارية.

1. Introduction

The fractional differential equation with initial conditions

$$y^{(\alpha)}(x) = f(x, y(x)) \qquad n-1 < \alpha < n, n \geq 2, \alpha \in R$$

$$y^{(\alpha-i)}(0) = c_i \quad , c_i \in R \quad i = 1, 2, ..., n \quad , c_n = 0$$

where f is assumed to be continuous on R×R, has been studied by many authors(see [1,8]). El-Salam [6] discussed a nonlinear weighted Cauchy type problem of a fractional differ-integral equation of fractional order which has the form

$$D^{\alpha}u(t) = f(t, u(\varphi(t))) \qquad 0 < \alpha < 1 \tag{1}$$

$$t^{1-\alpha}u(t)\Big|_{t=0} = b \tag{2}$$

and proved some local and global existence , uniqueness and stability theorems for problem (1) and (2).El-Sayed[7]studied the existence theorem of Lp-solution of weighted Cauchy type problem of fractional differ-integral function equation and proved uniqueness and stability of the solution for equation(1)and (2).

The purpose of this work is to generalize the work of [7] to study the existence, uniqueness and stability of the solution of the initial value problem which has the form

$$y^{(\alpha)}(x) = f(x, y(\phi(x))) \qquad , n-1 < \alpha < n \ , n \geq 2, \alpha \in R \tag{3}$$

with initial condition

$$y^{(\alpha-1)}(0) = c_i \quad , c_i \in R \quad i = 1, 2, ..., n \quad , c_n = 0 \tag{4}$$

2. Preliminaries

Let $L_1(I)$ be the class of Lebesgue integrable functions on the interval I=[a,b] , where $0 \le a < b, < \infty$ and let $\Gamma(.)$ be the gamma function. Recall that the operator T is compact if it is continuous and maps bounded sets into relatively compact ones .The set of all compact operators from the subspace $U \subset X$ into the Banach space X is denoted by C(U,X).
Moreover, we set $B_r = \{u \in L_1(I) : \|u\| < r, r > 0\}$ and $\|u\|_1 = \int_0^1 e^{Nt} |u(t)| dt$.

Definition 1: [2]

Let f be a function which is defined almost everywhere (a.e) on [a,b]. for $\alpha > 0$, we define

$$\underset{a}{\overset{b}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t) dt$$

Provided that this integral exists, where Γ is gamma function.

Definition 2: [9]

For a function f defined on the interval [a,b], the αth Riemann-liouville fractional order derivative of f is defined by

$$(D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s, y(s)) ds$$

where n=$[\alpha]$+1 and $[\alpha]$ denotes the integer part of α .

Lemma 1: [1]

Let α, $M >$ 0. If f is continuous and $|f| \le M$ for all $x \in (a, b]$, then $\lim_{x \to a} {}_a^x D^{-\alpha} f = 0$.

Remark 1: [1]

Let the assumptions of the Lemma(1),be satisfied ,then we define $\underset{a}{\overset{a}{I}} f = 0$

Definition 3: [2]

If $\alpha \in R$ and f(x) is defined a.e on $a \le x \le b$ we define $f^{(\alpha)}(x) = \underset{a}{\overset{x}{I}}{}^{-\alpha} f$ for all

x∈[a,b], provided that $\underset{a}{\overset{x}{I}}{}^{-\alpha} f$ exists.

Lemma 2: [2]

Let $\alpha, \beta \in R, \beta > -1$. *If* $x > a$ *then*

$$\underset{a}{\overset{x}{I}}{}^{\alpha} \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} = \begin{cases} \dfrac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} & \alpha+\beta \ne negative \ \ integer \\ 0 & \alpha+\beta = negative \ \ integer \end{cases}$$

Lemma 3: [2]

If $\alpha > 0$; n is the smallest integer greater than α, $f(x)$ is in $L(a, b)$ and ${}_a^xD^{\alpha-1}f$ exists and is absolutely continuous on $[a, b]$, then ${}_a^{a+}D^{\alpha-i}f = k_i$ exists for i = 1, 2, ..., n; ${}_a^xD^{\alpha}f$ exists a.e on $[a, b]$, is in $L(a,b)$ and

$$ {}_a^xD^{-\alpha}\,{}_a^tD^{\alpha}f = f(x) - \sum_{i=1}^{n}\frac{k_i(x-a)^{\alpha-i}}{\Gamma(\alpha-i+1)} \quad , \text{ a.e. on } a \le x \le b. $$

furthermore, the equality holds everywhere on (a, b], if, in addition, $f(x)$ is continuous on (a, b].

Lemma 4: [3]

The relation $\quad {}_a^xI^{\alpha}\,{}_a^tI^{\beta} f = {}_a^xI^{\alpha+\beta} f \quad$ holds if

(i) $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, and\ f(x)\ is\ in C^{(0)}\ on\ [a,b].$

(ii) $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)\le 0\ or\ \mathrm{Re}(\beta+m)>0, such\ that\ \beta \ne -m,\ and\ f(x)\ is\ in C^{(m)}\ on\ [a,b]$

(iii) $\mathrm{Re}(\alpha)\le 0, \mathrm{Re}(\alpha+n)>0\ , \mathrm{Re}(\beta)>0, and\ f(x)\ is\ in\ C^{(n)}\ on\ [a,b].$

(iv) $\mathrm{Re}(\alpha) \le 0, \mathrm{Re}(\beta) \le 0\ , \beta \ne -m, and\ f(x)\ is\ in C^{(m+n)}\ on\ [a,b].$

$when\ \beta = -m\ in (ii) and (iv), then$

$$ {}_a^xI^{\alpha}\,{}_a^tI^{-m} f = {}_a^xI^{\alpha-m} f - \sum_{p=0}^{m-1}\frac{(x-a)^{\alpha-m+p}}{\Gamma(\alpha+p-m+1)} f^{(p)}(a) $$

Theorem 1: (Roth Fixed Point Theorem) [5]

Let U be an open and bounded subset of a Banach space E, let $T \in C(\bar{U}, E)$. Then T has a fixed point if the following condition holds $T(\partial U) \subseteq \bar{U}$.

Theorem 2: (Nonlinear Alternative of Laray-Schauder Type) [5]

Let U be an open subset of convex set D in a Banach space E, assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) T has a fixed point in $\bar{U}$,or

(A2) There exists $\gamma \in (0,1)$ and $x \in \partial U$ such that $x = \gamma T x$.

Theorem 3: (Kolmogorov Compactness Criterion) [6]

Let $\Omega \subseteq L^p(0,1)$, $1 \le p \le \infty$. If

(i) Ω is bounded in $L^p(0,1)$ and

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p(0,1)$,where

$$ x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)ds\ . $$

3. Existence of Solution

We begin this section by proving the equivalence of the initial value problems (3) and (4) with the corresponding integral equation, **for proof see[1]**

$$y(x)=\sum_{i=1}^{n-1}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s,y(s))ds \tag{5}$$

First, we prove that y(x) satisfies the differential equation(3) almost every where, by definition (1) equation (5) can be written as

$$y(x)=\sum_{i=1}^{n-1}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\overset{x}{\underset{0}{I}}{}^{\alpha} f \tag{6}$$

Operating both sides of equation (6) by $\overset{x}{\underset{0}{I}}{}^{-\alpha}$, we get

$$\overset{x}{\underset{0}{I}}{}^{-\alpha} y(x)=\sum_{i=1}^{n-1}\overset{x}{\underset{0}{I}}{}^{-\alpha}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\overset{x}{\underset{0}{I}}{}^{-\alpha}\overset{x}{\underset{0}{I}}{}^{\alpha} f \tag{7}$$

From lemma (2), we have

$$\sum_{i=1}^{n-1}\overset{x}{\underset{0}{I}}{}^{-\alpha}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}=0$$

and by lemma(4)(iii), we have

$$\overset{x}{\underset{0}{I}}{}^{-\alpha}\overset{x}{\underset{0}{I}}{}^{\alpha} f = f(x,y(x)) \text{ a.e for each } x\in I$$

Here equation (6) ,becomes

$$\overset{x}{\underset{0}{I}}{}^{\alpha} y = f(x,y(x)) \text{ a.e}$$

Finally using definition (3), we obtain

$$y^{(\alpha)}(x)=f(x,y(x))$$

Next, we prove that y(x) satisfies the initial condition (4) operating by $\overset{x}{\underset{0}{I}}{}^{i-\alpha}$ on both sides of equation (5) ,we have

$$\overset{x}{\underset{0}{I}}{}^{i-\alpha} y(x)=\sum_{i=1}^{n-1}\overset{x}{\underset{0}{I}}{}^{i-\alpha}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\overset{x}{\underset{0}{I}}{}^{i-\alpha}\overset{x}{\underset{0}{I}}{}^{\alpha} f$$

By lemma (2), we have

$$\sum_{i=1}^{n-1}\overset{x}{\underset{0}{I}}{}^{i-\alpha}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}=\begin{cases}\displaystyle\sum_{p=1}^{i}\frac{c_p x^{i-\alpha}}{\Gamma(i-p+1)} & i=1,2,3...,n-1\\[2ex] \displaystyle\sum_{p=1}^{n-1}\frac{c_p x^{n-p}}{\Gamma(n-p+1)} & i=n\end{cases} \tag{8}$$

Now using (8), remark(1) and definition (3), we have

$$y^{(\alpha-i)}(0)=\begin{cases} c_i & i=1,2,3...,n-1 \\ 0 & i=n \end{cases}$$

Thus y(x) satisfies the initial condition (4). Now define the operator T as

$$Tg(x)=\sum_{i=1}^{n-1}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t,g(\phi(t)))dt \qquad t\in(0,1) \tag{9}$$

It is necessary to find a fixed point of the operator T and here we present the main results to prove some local and global existence theorem for equations (3)-(4) in the space $L_1(0,1)$.

Let us state the following assumptions:

(i) $f:(0,1)\times R\to R$ be a function with the following properties

a- for each $x\in(0,1), f(x,.)$ is continuous .

b- for each $y\in(0,1), f(.,y)$ is measurable.

c-There exists two function b(x) such that $|f(x,y)|\le b(x)|y|$ for each $x\in(0,1)$, $y\in R$,where $b(x)$ is measurable and bounded.

(ii) $\phi:(0,1)\to(0,1)$ is non-decreasing and there is a constant $M>0$ such that $\phi'\ge M$ a.e on (0,1).

Theorem 4:

Let assumption (i) and (ii) are satisfied. If $\sup|b(x)|\le M\,\Gamma(1+\alpha)$. Then the initial value problems (3)-(4) has a solution $y\in B_r$ where

$$r\ge\frac{\displaystyle\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+1)}}{\left(1-\dfrac{\sup|b(x)|}{M\,\Gamma(1+\alpha)}\right)}.$$

Proof. Let y be an arbitrary element in B_r, $B_r=\{u\in L_1(I):\|u\|<r, r>0\}$. Then from the assumption (i) − (ii), we have

$$\|Ty\|=\int_0^1|Ty(x)|dx$$

from equation (9) we obtain

$$=\int_0^1\left|\sum_{i=1}^{n-1}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t,y(\phi(t)))dt\right|dx$$

$$\le\int_0^1\left|\sum_{i=1}^{n-1}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)}\right|dx+\frac{1}{\Gamma(\alpha)}\int_0^1\left|\int_0^x(x-t)^{\alpha-1}f(t,y(\phi(t)))dt\right|dx$$

$$=\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^x(x-t)^{\alpha-1}|f(t,y(\phi(t)))|dtdx$$

$$=\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha)}\int_0^1\int_t^1(x-t)^{\alpha-1}dx\,|f(t,y(\phi(t)))|dt$$

$$=\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha)}\int_0^1\frac{(1-t)^{\alpha}}{\alpha}|f(t,y(\phi(t)))|dt$$

For $t\in(0,1)$,then $(1-t)^{\alpha}<1$,we obtain

$$\le\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha+1)}\int_0^1|f(t,y(\phi(t)))|dt$$

by using condition i(c), we obtain

$$\le\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha+1)}\int_0^1|b(t)||y(\phi(t))|dt$$

$$\le\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha+1)}\sup|b(x)|\int_0^1|y(\phi(t))|dt$$

$$\le\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{1}{\Gamma(\alpha+1)}\sup|b(x)|\int_0^1\frac{|y(\phi(t))|}{M}dt\,|\phi'| \tag{10}$$

Let $x=\phi(t),dx=\phi'(t)dt$ and then (10) becomes

$$\|Ty\|\le\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{\sup|b(x)|}{M\,\Gamma(\alpha+1)}\int_{\phi(0)}^{\phi(1)}|y(x)|dx \tag{11}$$

The estimate (10) shows that the operator T maps the space $L_1(0,1)$ into it self. Let $y\in\partial B_r$, that is $\|y\|=r$, The inequality (11) gives

$$\|Ty\|\le\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{\sup|b(x)|}{M\,\Gamma(\alpha+1)}r\le r \tag{12}$$

Which implies that

$$\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+2)}+\frac{\sup|b(x)|}{M\,\Gamma(\alpha+1)}r\le r \tag{13}$$

Therefore

$$r\ge\frac{\sum_{i=1}^{n-1}\frac{c_i}{\Gamma(\alpha-i+1)}}{\left(1-\frac{\sup|b(x)|}{M\,\Gamma(1+\alpha)}\right)}.$$

By the hypothesis that $\sup|b(x)|\le M\,\Gamma(\alpha+1)$ we conclude that $r>0$. Furthermore

$$\|f\|=\int_0^1|f(t,y(\phi(t))|dt$$

$$\le \int_0^1 |b(t)||y(\phi(t))|dt$$

$$\le \frac{\sup|b(t)|}{M}\|y\| \tag{14}$$

The inequality (14) insures that $f \in L_1(0,1)$.

Further, since f is continuous function in y by the assumption i(a), and $\overset{x}{\underset{0}{I}}{}^{\alpha}$ maps $L_1(0,1)$ continuously into itself, $\overset{x}{\underset{0}{I}}{}^{\alpha} f(t, y(\phi(t)))$ is continuous in y, since y is an arbitrary element in B_r, then T maps B_r continuously into $L_1(0,1)$.

Now we claim that T is compact . Let Ω be a bounded subset of B_r, then $T(\Omega)$ is bounded in $L_1(0,1)$. Next, it remains to show that $(Ty)_h \to Ty$ in $L_1(0,1)$ as $h \to 0$, uniformly with respect to $Ty \in T(\Omega)$, consider

$$\|(Ty)_h - Ty\| = \int_0^1 |(Ty)_h(x) - (Ty)(x)|dx$$

$$= \int_0^1 \left| \frac{1}{h}\int_x^{x+h} (Ty)(s)ds - (Ty)(x) \right| dx$$

$$\le \int_0^1 \left(\frac{1}{h}\int_x^{x+h} |(Ty)(s) - (Ty)(x)|ds \right) dx$$

$$\le \int_0^1 \frac{1}{h}\int_x^{x+h} \left| \sum_{i=1}^{n-1} \frac{c_i s^{\alpha-i}}{\Gamma(\alpha-i+1)} - \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} \right| dsdx + \int_0^1 \frac{1}{h}\int_x^{x+h} |I^{\alpha} f(s, y(\phi(s))) - I^{\alpha} f(x, y(\phi(x)))| dsdx \tag{15}$$

Since $f \in L_1(0,1)$,then $I^{\alpha} f(.) \in L_1(0,1)$. Also $\sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} \in L_1(0,1)$ then from theorem (3) we have

$\frac{1}{h}\int_x^{x+h} \left| \sum_{i=1}^{n-1} \frac{c_i}{\Gamma(\alpha-i+1)} (s^{\alpha-i} - x^{\alpha-i}) \right| ds \to 0$ as $h \to 0$,and

$\int_0^1 \frac{1}{h}\int_x^{x+h} |I^{\alpha} f(s, y(\phi(s))) - I^{\alpha} f(x, y(\phi(x)))| dx \to 0$ as $h \to 0$ almost every where for $x \in (0,1)$, from theorem (3), $T(\Omega)$ is relatively compact and hence T is a compact operator. Let $U = B_r$ and $E = L_1(0,1)$, and from theorem (1), T has a fixed point. This complete the proof.

Theorem 5:

Let the condition (i) and (ii) of theorem (4) be satisfied.

(iii) assume that every solution $y(x) \in L_1(0,1)$ to the equation

$$y(x) = \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + \overset{x}{\underset{0}{I}}{}^{\alpha} f \quad \text{a.e on } (0,1), \ n-1 < \alpha < n, \alpha \in R, n \ge 2$$

Satisfies $\|y\| \neq r$ (r is arbitrary but fixed). Then the fractional initial value problem (3) and (4) has at least one solution $y(x) \in L_1(0,1)$.

Proof. Let y(x) be an arbitrary element in the open set $B_r = \{y : \|y\| < r, r > 0\}$. Then from (11), we have

$$\|Ty\| \le \sum_{i=1}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{\sup|b(x)|}{M\,\Gamma(\alpha+1)}\|y\| \tag{16}$$

The inequality (16) implies that the operator T maps B_r continuously into $L_1(0,1)$; Furthermore ,form i(c),we have

$$\|f\| \le \frac{\sup|b(x)|}{M}\|y\| \tag{17}$$

Using theorem (4) ,T maps B_r continuously into $L_1(0,1)$ and T is compact operators. Now set $U = B_r$ and $D = E = L_1(0,1)$, then by using the assumption (iii), the condition (A2) of theorem (2) does not hold. Therefore from theorem (2) T has a fixed point .This complete the proof.

Theorem 6: (Uniqueness of The Solution)

Let the assumption (ii) of theorem (4) be satisfied .Let the right hand side f(x, y) of the fractional differential equation satisfies the Lipschitz condition that is $|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$. Then the initial value problem (3)and (4)has a unique solution.

Proof. Let $y_1(x)$ and $y_2(x)$ be any two solution of the integral equation(5), then consider

$$y_2(x) - y_1(x) = \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha - i + 1)} + I_0^{x\,\alpha} f(s, y_2(\phi(x))) - \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha - i + 1)} - I_0^{x\,\alpha} f(s, y_1(\phi(x)))$$

or $y_2(x) - y_1(x) = I_0^{x\,\alpha} f(s, y_2(\phi(x))) - I_0^{x\,\alpha} f(s, y_1(\phi(x)))$

$$y_2(x) - y_1(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[f(s, y_2(\phi(t))) - f(s, y_1(\phi(t)))\right] dt$$

$$|y_2(x) - y_1(x)| \le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|f(s, y_2(\phi(t))) - f(s, y_1(\phi(t)))\right| dt \tag{18}$$

Now by using the Lipschitz condition, we have

$$|y_2(x) - y_1(x)| \le \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|y_2(\phi(t)) - y_1(\phi(t))\right| dt \tag{19}$$

integrating both sides from 0 to 1 with respect to x ,we get

$$\int_0^1 |y_2(x)-y_1(x)|dx \le \frac{L}{\Gamma(\alpha)}\int_0^1\int_0^x (x-t)^{\alpha-1}|y_2(\phi(t))-y_1(\phi(t))|dtdx$$

or $\|y_2(x)-y_1(x)\| \le \frac{L}{\Gamma(\alpha)}\int_0^1\int_0^x (x-t)^{\alpha-1}|y_2(\phi(t))-y_1(\phi(t))|dtdx$

$$= \frac{L}{\Gamma(\alpha)}\int_0^1\int_t^1 (x-t)^{\alpha-1}dx\,|y_2(\phi(t))-y_1(\phi(t))|dt$$

$$= \frac{L}{\Gamma(\alpha)}\int_0^1 \frac{(1-t)^{\alpha}}{\alpha}|y_2(\phi(t))-y_1(\phi(t))|dt$$

For $t \in (0,1)$, then $(1-t)^{\alpha} < 1$, we obtain

$$\le \frac{L}{\Gamma(\alpha+1)}\int_0^1 |y_2(\phi(t))-y_1(\phi(t))|dt$$

$$\le \frac{L}{M\,\Gamma(\alpha+1)}\int_0^1 |y_2(\phi(t))-y_1(\phi(t))|\ |\phi'|dt$$

$$\le \frac{L}{M\,\Gamma(\alpha+1)}\int_{\phi(0)}^{\phi(1)} |y_2(x)-y_1(x)|dx$$

$$\|y_2-y_1\| \le \frac{L}{M\,\Gamma(\alpha+1)}\|y_2(x)-y_1(x)\|$$

Choose M such that $M\,\Gamma(\alpha+1) > L$ equation (19) gives

$$\|y_2-y_1\| < \|y_2-y_1\|$$

Contradiction, therefore $y_2 = y_1$. This complete the proof.

Theorem 7: (Stability of The Solution)

Suppose that the assumption of theorem (4)be satisfied, then the solution of the initial value problems (3) and (4) is uniformly stable.

Proof. Any solution of the fractional initial value problems (3) and (4) is given by

$$y(x) = \sum_{i=1}^{n-1}\frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t, y(\phi(t)))dt$$

Let $\bar{y}(x)$ be any other solution of (3)and(4),that

$$\bar{y}(x) = \sum_{i=1}^{n-1}\frac{\bar{c}_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t, \bar{y}(\phi(t)))dt$$

Now $y(x)-\bar{y}(x) = \sum_{i=1}^{n-1}\frac{(c_i-\bar{c}_i)x^{\alpha-i}}{\Gamma(\alpha-i+1)} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[f(t, y(\phi(t))) - f(t, \bar{y}(\phi(t)))\right]dt$

$$\left|y(x)-\bar{y}(x)\right| \le \sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left|f(t,y(\phi(t)))-f(t,\bar{y}(\phi(t)))\right|dt$$

Using the Lipschitz condition with Lipschitz constant L, we have

$$\left|y(x)-\bar{y}(x)\right| \le \sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{L}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt$$

following the technique of [6] ,we have

$$e^{-Nx}\left|y(x)-\bar{y}(x)\right| \le \sum_{i=1}^{n-1}\frac{e^{-Nx}\left|c_i-\bar{c}_i\right|x^{\alpha-i}}{\Gamma(\alpha-i+1)}+\frac{e^{-Nx}L}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt$$

Now

$$\int_0^1 e^{-Nx}\left|y(x)-\bar{y}(x)\right| \le \sum_{i=1}^{n-1}\int_0^1\frac{e^{-Nx}\left|c_i-\bar{c}_i\right|x^{\alpha-i}}{\Gamma(\alpha-i+1)}ds+\frac{L}{\Gamma(\alpha)}\int_0^1 e^{-Nx}\int_0^x (x-t)^{\alpha-1}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dtdx \qquad (20)$$

Let $s=Nx \to ds=Ndx$, the inequality (20) thus gives

$$\left\|y(x)-\bar{y}(x)\right\|_1 \le \sum_{i=1}^{n-1}\int_0^N\frac{\left|c_i-\bar{c}_i\right|}{\Gamma(\alpha-i+1)}\frac{e^{-s}s^{\alpha-i}}{N^{\alpha-i}N}ds+\frac{L}{\Gamma(\alpha)}\int_0^1 e^{-Nx}\int_0^x (x-t)^{\alpha-1}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dtdx$$

$$\le \sum_{i=1}^{n-1}\int_0^\infty\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}\Gamma(\alpha-i+1)}e^{-s}s^{\alpha-i}ds+\frac{L}{\Gamma(\alpha)}\int_0^1 e^{-Nx}\int_0^x (x-t)^{\alpha-1}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dtdx$$

from definition of the gamma function, we have

$$\left\|y-\bar{y}\right\|_1 \le \sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}}+\frac{L}{\Gamma(\alpha)}\int_0^1 e^{-Nx}\int_0^x (x-t)^{\alpha-1}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dtdx$$

$$=\sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}}+\frac{L}{\Gamma(\alpha)}\int_0^1\int_t^1 e^{-Nx}(x-t)^{\alpha-1}dx\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt$$

$$=\sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}}+\frac{L}{\Gamma(\alpha)}\int_0^1\int_t^1 e^{-N(x-t)}(x-t)^{\alpha-1}dxe^{-Nt}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt \qquad (21)$$

Now Let $z=N(x-t)\to dz=Ndx \to dx=\dfrac{dz}{N}$.

Thus equation(21), becomes

$$\left\|y-\bar{y}\right\|_1 \le \sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}}+\frac{L}{\Gamma(\alpha)}\int_0^1\int_0^{N(1-t)} e^{-z}\frac{z^{\alpha-1}}{N^{\alpha-1}N}dz\ e^{-Nt}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt$$

$$=\sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}}+\frac{L}{\Gamma(\alpha)N^{\alpha}}\int_0^1\int_0^{N(1-t)} e^{-z}z^{\alpha-1}dz\ e^{-Nt}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt$$

$$\le \sum_{i=1}^{n-1}\frac{\left|c_i-\bar{c}_i\right|}{N^{\alpha-i+1}}+\frac{L}{N^{\alpha}}\int_0^1 e^{-Nt}\left|y(\phi(t))-\bar{y}(\phi(t))\right|dt$$

$$\leq \sum_{i=1}^{n-1} \frac{|c_i - \bar{c}_i|}{N^{\alpha-i+1}} + \frac{L}{MN^{\alpha}} \int_0^1 e^{-Nt} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt \, |\phi'|$$

Let $s = \phi(t) \rightarrow ds = \phi'(t)dt$ and $t = \phi^{-1}(s)$, then we have

$$= \sum_{i=1}^{n-1} \frac{|c_i - \bar{c}_i|}{N^{\alpha-i+1}} + \frac{L}{MN^{\alpha}} \int_{\phi(0)}^{\phi(1)} e^{-N\phi^{-1}(s)} \left| y(s) - \bar{y}(s) \right| ds$$

$$\leq \sum_{i=1}^{n-1} \frac{|c_i - \bar{c}_i|}{N^{\alpha-i+1}} + \frac{L}{MN^{\alpha}} \int_{\phi(0)}^{\phi(1)} e^{-Ns} \left| y(s) - \bar{y}(s) \right| ds \, , \quad s < t$$

$\leq \sum_{i=1}^{n-1} \frac{|c_i - \bar{c}_i|}{N^{\alpha-i+1}} + \frac{L}{MN^{\alpha}} \|y - \bar{y}\|_1$ Therefore $\|y - \bar{y}\|_1 \leq \sum_{i=1}^{n-1} \frac{|c_i - \bar{c}_i|}{N^{\alpha-i+1}} + \frac{L}{MN^{\alpha}} \|y - \bar{y}\|_1$

or $\|y - \bar{y}\|_1 \leq \left(\frac{MN^{\alpha}}{MN^{\alpha} - L} \right) \sum_{i=1}^{n-1} \frac{|c_i - \bar{c}_i|}{N^{\alpha-i+1}}$

$$\|y - \bar{y}\|_1 \leq \sum_{i=1}^{n-1} \frac{MN^{\alpha} |c_i - \bar{c}_i|}{MN^{2\alpha-i+1} - LN^{\alpha-i+1}}$$

Then, If $\sum_{i=1}^{n-1} |c_i - \bar{c}_i| < \delta(t)$, then $\|y - \bar{y}\| < \varepsilon$

Therefore the solution of the initial value problem is uniformly stable.

Conclusion

We proved the existence of the solution for certain fractional differential equation by using Roth fixed point theorem in L_1 space, then we use the Laray-Schauder theorem we proved that the fractional differential equation has at least one solution then we prove the uniqueness theorem by using the Lipschitz condition. Also we discussed the stability of the solution and we proved that the solution is uniformly stable.

REFERENCES

[1] Alshamani J.G. (1979), “Some Existence and Stability for Differential Equation of Non-integer Order through Fixed Point”, Msc. Thesis, Mousel university, Iraq.

[2] Barrett J.H. (1954), “Differential Equations of Non-integer Order”, Canada. J. Math, (6), pp. 529-541.

[3] Bassam A.M . (1964), “Some Existence Theorems on Differential Equations Of Generalized Order”, (presented to mathematical association of America Texas section), April, 10.

[4] Deimling K. (1985), “Nonlinear Functional Analysis”, Springer Verlag.

[5] Dugundji, J.G. (1982), “A Fixed Point Theory”, Monografie Mathematyczne, PWN, Warsaw.

[6] El-Salam, A. and El-Sayed, A.M. (2007), “Weighted Cauchy Type Problem of a Fractional Differ-integral Equation”, Electronical Journal of Qualitative Theory of Differential Equations. No(30), pp.1-9.

[7] El-Sayed, A.M. and El-Salam, A. (2008), “Lp Solution of Weighted Cauchy Type Problem of a Fractional Differ-integral Functional Equation”, International Journal of Nonlinear Science, Vol (5), No(3), pp.281-288.

[8] Kilbas, A. A., Srivastava Hari M. and Trujillo Juan J. (2006), ”Theory and Applications of Fractional Differential Equations”, North Holland, Mathematics Studies, 204, Elsevier Science B .V., Amsterdam.