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حول الحلقات الغامرة من النمط SSAGP

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الخلاصة

في هذا البحث نحن نختبر بعض خواص الحلقات التي يكون فيها كل مقاس بسيط منفرد ايمن من النمط AGP- (اختصارا تكتب غامرة من النمط AGP) برهنا ان Y(R)=0 عندما R حلقة من النمط SSAGP يمنى كذلك برهنا

1. لتكن R حلقة متممة مقيدة وغامرة من النمط SSAGP يمنى وكل مثالي أعظمي ايمن أساسي من النمط Gw . فان R حلقة منتظمة بقوة.
2. لتكن R حلقة غامرة من النمط SSAGP و r(e) مثالي من النمط Gw لكل عنصر متحايد $e \in R$. فان Z(R)=0.
3. لتكن R حلقة غامرة من النمط SSAGP، MERT و CM يمنى. فان R حلقة منتظمة بقوة أو شبه بسيطة ارتيرينية.

الكلمات المفتاحية: منتظمة، مختزلة، غامرة من النمط-P، غامرة من النمط-AGP.

On SSAGP-injective Rings

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ABSTRACT

In this paper, we investigate some properties of rings whose simple singular right R- modules are A Gp-injective (or SSAGP- injective for short). It is proved that: Y(R)=0 where R is a right SSAGP- injective rings. It is also proved that

1. Let R be a complement right bounded, SSAGP – injective rings and every maximal essential right ideal is Gw-ideal. Then R is strongly regular ring.
2. Let R be SSAGP – injective and r(e) is Gw-ideal for every idempotent element $e \in R$. Then Z(R)=0.
3. Let R be SSAGP – injective, MERT and right CM. Then R is either strongly regular or semi simple Artinian.

Keyword: regular, reduced, P-injective, AGP-injective.

1- Introduction:

Throughout this paper, R denotes as associative ring with identity and all modules are unital. The symbols J(R) and Y(R) (Z(R)) respectively for the Jacobson radical and right (left) singular ideal of R. As usual, R is a reduced ring, if N(R)=0 (N(R) the set of all nilpotent elements of (R). R is a right ERT (resp., MERT) ring if every essential (resp. maximal essential) right ideal of R is an ideal [1]. A ring R is abelian if every idempotent of R is central. R is regular if for every $a \in R$, there exists $b \in R$ such that $a = aba$. R is strongly regular if for every $a \in R$, there exists $b \in R$ such that $a=a^2b$. It is known that a ring R is strongly regular if and only if R is a reduced regular ring [2]. Following [3]. The ring R is a right weakly regular (resp., left weakly regular) if for every $a \in R$, $a = ac$ $(a = ca)$ for some $c \in RaR$ and R is weakly regular, if it is both left and right weakly regular. A regular ring is clearly weakly regular, but a weakly regular ring needs not to be regular (for example) [3].

It is known that all generalizations of injectivity have been discussed in many papers [4, 6, 5]. R is called a p-injective ring [2], if every right R-homomorphism from aR to R can be extended to endomorphism of R, where $a \in R$. In [1], p-injective rings were extended to Ap-injective rings and AGP-injective rings. A ring R is called Gp-injective, if for every $a \in R$ there exists $n \in Z^+$ such that $a^n \neq 0$ and $\ell r(a^n) = Ra^n$ [7].

R is called right AP-injective [1], if for any $a \in R$, there exists a left ideal X_a of R such that $\ell r(a) = Ra \oplus Xa$. A ring R is called right AGP-injective, if for any $0 \neq a \in R$, there exists a positive integer n and a left ideal Xa of R such that $a^n \neq 0$ and $\ell r(a^n) = Ra^n \oplus Xa$ [6].

Clearly, AP-injectivity and AGP-injectivity are the generalization of P-injectivity, and they have many properties [4] [6] [5].

2- Right AGP-injective Rings:

The following lemma which is due to Zhao Yu-e [5], plays a central role is several of our proofs.

Lemma 2-1:

Suppose M is a right R-module with S=End (M_R). If $\ell_M r_R(a) = Ma \oplus Xa$, where Xa is a left S-submodule of M_R. Set $f : aR \to M$ is a right R-homomorphism, then f(a)=ma+x with $m \in M$, $x \in Xa$.

Lemma 2-2: [6]

If R is a right AGP-injective ring, then J(R)=Y(R).

The following result extends Lemma (2.2).

Proposition 2-3:

If R is a semiprime, ERT is a right AGP-injective ring, then $Y(R) = J(R) = 0$.

Proof: If $Y(R) \neq 0$, there exists $0 \neq y \in Y(R)$ such that $y^2 = 0$. Since $r(y)$ is an ideal of R (R is ERT), $yR \subseteq r(y)$ implies that $RyR \subseteq r(y)$, $yRyR = 0$, whence $(yR)^2 = 0$. Since R is a semiprime, we have $y = 0$. This contradicts that $Y(R) = 0$. Thus $Y(R) = J(R) = 0$.

A ring R is called right CM ring [8], iff, for any maximal essential right ideal of R, every complement right subideal is an ideal of R.

Lemma 2-4: [8]

If R is a right CM ring and a right nonsingular, then R is either a semi simple Artinian or a reduced.

Lemma 2-5: [4]

If R is a reduced right AGP-injective, then R is strongly regular.

Now,we give the following proposition

Proposition 2-6:

If R is a semi prime, ERT is a right AGP-injective and CM ring, then R is either strongly regular or semi – simple Artinian.

Proof: By proposition (2.3) and Lemma (2.4), R is either a semi simple Artinian or it is reduced. If R is reduced, then R is strongly regular Lemma (2.5).

3- The Regularity of SSAGP-injective rings:

Definition 3-1:

A ring R is called SSAGP-injective ring, if every simple singular right R – module is AGP injective.

Following [9], a right ideal L of a ring R is a generalized weak ideal (Gw-ideal) if for all $a \in L$, there exists a positive integer n such that $Ra^n \subseteq L$.

Proposition 3-2:

Let R be a right SSAGP-injective and every maximal essential right ideal of R is Gw-ideal. Then $Y(R) = 0$.

Proof: Suppose that $Y(R) \neq 0$. Then there exists $0 \neq a \in Y(R)$ such that $a^2 = 0$. Hence r(a) is contained in a maximal essential right ideal of R. Thus R/M is AGP-injective, then $\ell_{R/M} r(a) = (R/M)a \oplus Xa$, $Xa \leq R/M$. Let $f : aR \to R/M$, be defined by $f(ar) = r + M$, for every $r \in R$. Since $r(a) \subseteq M$ is will define By Lemma (2-1), $1 + M = f(a) = ca + M$, $c \in R$, $x \in Xa$, $1 - ca + M = x \in R/M \bigcap Xa$, so $1 - ca \in M$. By hypothesis M is Gw-ideal, and $ac \in M$. So there exists $n \in Z^+$, such that $c(ac)^n \in M$. Since M is a right ideal, $(c - cac) \in M$, $c(ac)^{n-1} = (c - cac)(ac)^{n-1} + c(ac)^n \in M$, continuing in the process, we have $c(ac) \in M$, thus $c = (c - cac) + cac \in M$, so $ca \in M$, $1 \in M$, which is a contradiction. Thus $a = 0$ and so $Y(R) = 0$.

Following [8], a ring R is called a complement right (left) bounded if every non zero complement right (left) ideal of R contains a non zero ideal of R.

Lemma 3-3: [9]

Let R be a complement right (left) bounded and a right (left) non singular ring. Then R is reduced.

Theorem 3-4:

Let R b a complement right bounded ring and every maximal essential right ideal is Gw-ideal. Then R is strongly regular, if R is a SSAGP-injective ring.

Proof: Let $a \in R$. If $r(a) + aR$ is not essential, then there exists a non zero a complement right ideal L of R such that $(r(a) + aR) \bigcap L = 0$. Since R is a complement right bounded, there exists a non zero ideal I of R and $I \subseteq L$. Let $0 \neq x \in I$, then $ax \in I \bigcap aR = 0$. This implies that $x \in r(a) \bigcap I = 0$. This is a contradiction to $x \neq 0$. Therefore $r(a) + aR$ is an essential right ideal of R. If $r(a) + aR \neq R$, then there exists a maximal right ideal M of R such that $r(a) + aR \subseteq M$. Since $r(a) + aR$ is essential, M is essential. Then R/M is a simple singular right R-module, hence by hypothesis, it is AGP-injective. There exists a positive integer n such that $a^n \neq 0$ and $\ell_{R/M} r(a^n) = (R/M)a^n \oplus X_a^n$. Let $f : a^n R \to R/M$ be defined by $f(a^n r) = r + M$. By Proposition (3.2) and Lemma (3.3) we get R which is reduced, f is well defined R-homomorphism. Thus by Lemma (2.1),
$f(a^n) = ca^n + M + y$, $c \in R$, $y \in X$, and $f(a^n) = 1 + M$, and so $1 - ca^n + M = y \in R/M \bigcap X = 0$, $1 - ca^n \in M$. Suppose $ca^n \notin M$, then $M + ca^n R = R$

implying $x + ca^n r = 1$ for some $x \in M$, $r \in R$. Now, M is a Gw-ideal and $a^n rc \in M$, so there exists $k \in Z^+$ such that $c(a^n rc)^k \in M$. Then $(1-x)^{k+1} = (ca^n r)^{k+1} = c(a^n rc)^k a^n r \in M$. So that $1 \in M$ which is a contradiction. Hence $r(a) + aR = R$. So $z + ab = 1$ for some $z \in r(a)$ and $b \in R$, which yields $a = a^2 b$. This proves that R is strongly regular.

Lemma 3-5: [9]

The following conditions are equivalent to the ring R.

1. R is a belian.
2. $\ell(e)$ is Gw-ideal of R for every $e^2 = e \in R$.
3. $r(e)$ is Gw-ideal of R for every $e^2 = e \in R$.

Theorem 3-6:

Let R be a right SSAGP-injective and r(e) is a Gw-ideal of R for every idempotent element e of R. Then $Z(R) = 0$.

Proof: If $Z(R) \neq 0$, then there exists $0 \neq a \in Z(R)$ such that $a^2 = 0$. Suppose that $r(a) + RaR \neq R$ and M be a maximal right ideal of R such that $r(a) + RaR \subseteq M$. If M is not essential, then $M = r(e)$ for some $e^2 = e \in R$. Since $a \in M = r(e)$ then $ae = 0$ Lemma (3.5). Hence $e \in r(a) \subseteq M = r(e)$ and $e = e^2 = 0$, contradict $e \neq 0$. Thus M is essential. Since every simple singular right R-module is AGP-injective, then R/M is AGP-injective and $\ell r(a) = (R/M)a \oplus Xa$. Let $f : aR \to R/M$ be defined by $f(ab) = b + M$, $b \in R$, by Lemma (2.1), $1 + M = f(a) = ca + M + x$, $1 - ca + M = x \in (R/M) \cap X = 0$, thus $1 - ca \in M$, since $ca \in RaR \subseteq M$, $1 \in M$, which is a contradiction. Hence $RaR + r(a) = R$. This implies that $x + \sum y_i a z_i = 1$ for some $x \in r(a)$, $y_i, z_i \in R$. This yields $(1 - \sum y_i a z_i) = x \in r(a)$ and $a(1 - \sum y_i a z_i) = 0$ that is $a \in \ell(1 - \sum y_i a z_i)$. Now, $a \in Z(R)$, hence $\ell(\sum y_i a z_i)$ is an essential left of R. Therefore $\ell(1 - \sum y_i a z_i) \cap \ell(y_i a z_i) = 0$ that is $\ell(1 - \sum y_i a z_i) = 0$, whence $a = 0$. This is a contradiction to $a \neq 0$. Thus $Z(R) = 0$.

Theorem 3-7:

Let R be a right SSAGP-injective and r(e) is a Gw-ideal of R for every idempotent element $e \in R$. Then R is a right weakly regular ring.

Proof: We need only to prove $RaR + r(a) = R$ for every $a \in R$. If not, then there exists a maximal right ideal M of R containing $RaR + r(a)$. If M is not an essential, then $M = r(e)$ for any $e^2 = c \in R$. Proceeding as in the proof of Theorem (3.6), we get a contradiction. Thus M is an essential. So R/M is a simple singular right R-Module. Since R/M is AGP-injective there exists a positive integer n such that $a^n \neq 0$ and $\ell_{R/M} r(a^n) = (R/M)a^n \oplus Xa^n$, $Xa^n \leq R/M$. Let $f : a^n R \to R/M$ be defined by $f(a^n r) = r + M$, since $r(a) \subseteq M$, f is well defined R-homomorphism. So $f(a^n) = 1 + M$, thus $1 - ca^n + M = x \in (R/M) \cap X = 0$. So $1 - ca^n \in M$ and so $1 \in M$, which are contradiction. Therefore $RaR + r(a) = R$ for any $a \in R$. Hence R is a right weakly regular.

Lemma 3-8: [10]

Let R be a ring such that r(a) is a Gw-ideal for all $a \in R$, then R is a right weakly regular if and only if R is a left weakly regular.

From Theorem (3.7) and Lemma (3.8) we get:

Corollary 3-9:

Let R be a right SSAGP-injective and r(a) is a Gw-ideal of R for every $a \in R$. Then R is a weakly regular ring.

Theorem 3-10:

Let R be a right CM, MERT, and SSAGP -injective. Then R is either strongly regular or a semi simple Artinian.

Proof: Depending on proposition (3.2) and Lemma (2.4), R is either a semi simple Artinian or a reduced. If R is not asemi simple Artiain, then R is reduced. For any $0 \neq a \in R$, we will show that $aR + r(a) = R$. Suppose not, then there exists a maximal right ideal M of R containing $aR + r(a)$. If M is not an essential, then is it a direct summand of R. Thus $M = r(e)$ for some $e^2 = e \in R$. Note that $a \in r(e)$. It follows $ea = ae = 0$. This implies that $e \in r(e) \subseteq M = r(e)$ and $e = e^2 = 0$, which are contradiction. Since R/M is AGP-injective, there exists a positive integer n such that $a^n \neq 0$ and $\ell_{R/M} r_R(a^n) = (R/M)a^n \oplus Xa^n, Xa^n \leq R/M$. Proceeding as the proof of Theorem (3.7) we get, $1 - ca^n \in M$, $c \in R$. Since R is MERT and $ca^n \in M$. Thus $1 \in M$, which is a contradiction. Hence R is a strongly regular.

A ring R is said to be a biregular ring if, for any $a \in R$, RaR generated by a central idempotent [11].

Lemma 3-11: [11]

A ring R is a biregular if and only if $R = RaR \oplus r(a)$ for all $a \in R$.

Theorem 3-12:

Let R be a right CM-ring, SSAGP-injective and every maximal essential right ideal of R is a Gw-ideal. Then R is a biregular ring.

Proof: By Proposition (3.2), $Y(R) = 0$. Since R is a right non singular, right CM, by Lemma (2.4), R is either a semi simple Artinian or a reduced. We consider this case when R is a reduced ring. For any, set $E = RaR \oplus r(a)$. Since R is a reduced of $RaR \cap r(a) = 0$. Suppose that $E \neq R$, then there exists a maximal right ideal M of R such that $RaR + r(a) \subseteq M$. We shall prove that M is an essential if not, then M must be a direct summand, so $M = r(e)$, $e^2 = e \in R$. Proceeding as the proof of Theorem (3.6), we get a contradiction. Therefore M must be an essential thus R/M is AGP-injective, as in Theorem (3.7), we get a contradiction. Therefore $R = E = RaR \oplus r(a)$. So R is a biregular ring Lemma (3.11).

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