Journal of AL-Qadisiyah for computer science and mathematics Vol.11 No.1 Year 2019
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

On Differential Sandwich Theorems of Multivalent Functions Defined by a Linear operar

Waggas Galib Atshan **Salwa Kalf Kazim**

Department of Mathematics , College of Computer Science and Information Technology , University of AL-Qadisiyah , Diwaniyah-Iraq

waggas.galib@qu.edu.iq , waggashnd@gmail.com

Recived : 24\12\2018 **Revised ://** **Accepted : 13\1\2019**

Available online : 28 /1/2019

DOI: 10.29304/jqcm.2019.11.1.480

Abstract:

The main object of the present paper is to derive some results for multivalent analytic functions defined by linear operator by using differential subordination and superordination

Keywords: Analytic functions, multivalent functions, Hadamard product, subordination, linear operators.

Mathematics Subject Classification: 30C45.

Journal of AL-Qadisiyah for computer science and mathematics Vol.11 No.1 Year 2019
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

1. Introduction

Let A_p denote the class of functions f of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} = \{1,2,\dots\};\ z\, U), \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

For two functions f and g are analytic in U, we say that the function f is subordinate to g in U, written $f \prec g$, if there exists Schwarz function w, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g(w(z)), z \in U$ If g is univalent and $g(0) = f(0)$, then $f(u) \subset g(u)$.

If $f \in A_p$ is given by (1.1) and $g \in A_p$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}.$$

Then Hadamard product (or convolution) is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, b_{p+n} z^{p+n} \quad .$$

The linear operator $J_{\mu,v}^{\lambda,p}(a,c): A_p \to A_p$ defined by

$$J_{\mu,v}^{\lambda,p}(a,c)f(z) = \emptyset_{\mu,v}^{\lambda,p}(a,c;z) * f(z), \quad \left(f \in A_p, z \in U\right), \quad (1.2)$$

where

$$\emptyset_{\mu,v}^{\lambda,p}(a,c;z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n (p+1)_n (p+1-\mu+v)_n}{(c)_n (p+1-\mu)_n} z^{p+n} \quad (1.3)$$

and

$$d_n = \begin{cases} 1 & n = 0 \\ d(d+1)(d+2)\dots(d+n-1) & n \in N \end{cases} .$$

For $a \in R, c \in R \setminus z_\circ^-$, where $z_\circ^- = \{0,-1,-2,\dots\}$, $0 \le \lambda < 1$, $\mu, v \in R$ and $\mu - v - p < 1$ and $f \in A_p$ .Then linear operator $I_{\mu,v}^{\lambda,p,\alpha}(a,c): A_p \longrightarrow A_p$ (see[9]) is defined by

$$I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) \coloneqq \psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z) * f(z), \quad (1.4)$$

where $\psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z)$ is the function defined in terms of the Hadamard product by the following condition:

$$\emptyset_{\mu,v}^{\lambda,p}(a,c;z) * \psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z) = \frac{z^p}{(1-z)^{\alpha+p}} \quad (\alpha > -p). \quad (1.5)$$

We can easily find from (1.3) - (1.5) that

$$I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(c)_n (p+1-\lambda+v)_n (\alpha+p)_n (p+1-\mu)_n}{(a)_n (p+1)_n (p+1-\mu+v)_n n!} a_{p+n} z^{p+n} \quad (1.6)$$

It is easily verified from (1.6) that

$$\boldsymbol{z\left(I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)\right)' = (\alpha+p) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) - \alpha I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z).} \quad \textit{(1.7)}$$

Note that the linear operator $I_{\mu,v}^{\lambda,p,\alpha}(a,c)$ unifies many other operators considered earlier. In particular

1) $I_{0,v}^{0,p,\alpha}(a,c) \equiv J_p^{\alpha}(a,c)$ (see Cho al.[5]).
2) $I_{0,v}^{0,p,\alpha}(a,a) \equiv D^{\alpha+p-1}$ (see Goel and Sohi[6]).
3) $I_{0,v}^{0,p,1}(p+1-\lambda,1) \equiv \Omega_Z^{(\lambda,P)}$ (see Srivastava and Aouf[16]).
4) $I_{0,v}^{0,p,\alpha-1}(a,c) \equiv J_p^{a,\alpha}$ (see Hohlov[8]).
5) $I_{0,v}^{0,1-\alpha,\alpha}(a,c) \equiv L_P(a,c)$ (see Saition[13]).
6) $I_{0,v}^{0,p,1}(p+\alpha,1) \equiv J_{\alpha,P,\alpha} \in z, \alpha > -p$ (see Liu an Noor[10]).

The main object of this idea is to find sufficient conditions for certain normalized analytic functions $\boldsymbol{f}$ to satisfy:

$$\boldsymbol{q_1(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec q_2(z),}$$

and

$$q_1(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q_2(z) ,$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $q_1(0)$and $q_2(0) = 1$.

2- Preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas .

Journal of AL-Qadisiyah for computer science and mathematics Vol.11 No.1 Year 2019
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Definition 2.1. [11]: Denote by Q the set of all functions q that are analytic and injective on $\bar{U}\setminus E(q)$, where

$\bar{U}=U\cup\{z\in\partial U\}$, and

$$E(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\} \quad (2.1)$$

and are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U\,/\,E(q)$.

Further ,let the subclass of Q for which $q(0)=a$ be denoted by $Q(a), Q(0)\equiv Q_0$ *and* $Q(1)\equiv Q_1$.

Lemma 2.1.[1]: Let $q(z)$ be convex univalent function in U, let $\alpha\in\mathbb{C}.\beta\in\mathbb{C}\setminus\{0\}$ and suppose that

$$Re(1+\frac{zq''(z)}{q'(z)})>\max\{0,-Re(\frac{\alpha}{\beta})\} .$$

If $p(z)$ is analytic in U and

$$\alpha p(z)+\beta zp'(z)\prec\alpha q(z)+\beta zq'(z),$$

then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 2.2. [3]: Let $\boldsymbol{q}$ *be univalent in* $\boldsymbol{U}$ *and let* $\emptyset$ *and* $\boldsymbol{\theta}$ *be analytic in the domain* $\boldsymbol{D}$ *containing* $\boldsymbol{q}(\boldsymbol{U})$ *with* $\emptyset(\boldsymbol{w})\neq\boldsymbol{0}$, *when* $\boldsymbol{w}\in\boldsymbol{q}(\boldsymbol{U})$.

Set $Q(z)=zq'(z)\emptyset(q(z))$ and $h(z)=\theta(q(z))+Q(z)$, suppose that

1- Q is starlike univalent in U,

2- $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right)>0,\ z\in U.$

If p is analytic in U with $p(0)=q(0), p(U)\subseteq D$ and

$$\emptyset(p(z))+zp'(z)\emptyset(p(z))\prec\emptyset(q(z))+zq'(z)\emptyset(q(z)),$$

then $p\prec q$, and q is the best dominant .

Lemma 2.3.[12]: Let $q(z)$ be convex univalent in the unit disk U and let θ and $\emptyset$ be analytic in a domain D containing $q(U)$. Suppose that

$1-Re\{\frac{\theta'(q(z))}{\emptyset(q(z))}\}>0$ for $z\in U$,

$2-zq'(z)\emptyset(q(z))$ is starlike univalent in $z\in U$.

If $p\in\mathcal{H}[q(0),1]\cap Q$, with $p(U)\subseteq D$, and $\theta(p(z))+zp'(z)\emptyset(p(z))$ is univalent in U , and

$$\theta(q(z))+zq'(z)\emptyset(q(z))\prec\theta(p(z))+zp'(z)\emptyset(p(z)), \quad (2.2)$$

then $q\prec p$, and q is the best subordinant.

Lemma 2.4.[12]: Let $q(z)$ be convex univalent in U and $q(0)=1$. Let $\beta\in\mathbb{C}$, that $\mathrm{Re}(\beta)>0$. If $p(z)\in\mathcal{H}[q(0),1]\cap Q$ and $p(z)+\beta zp'(z)$ is univalent in U , then

$$q(z)+\beta zq'(z)\prec p(z)+\beta zp'(z),$$

which implies that $q(z)\prec p(z)$ and $q(z)$ is the best subordinant.

3-Subordination Results

Theorem 3.1. Let $q(z)$ be convex univalent in U with $q(0)=1,\eta,\delta\in\mathbb{C}\setminus\{0\}$. Suppose that

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-Re\left(\frac{\delta}{\eta}\right)\right\}. \quad (3.1)$$

If $f\in W$ is satisfies the subordination

$$G(z)\prec q(z)+\frac{\eta}{\delta}zq'(z), \quad (3.2)$$

where

$$G(z)=\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\times\left(1+\eta\left(\frac{(pt_2-t_2\alpha)I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)(z)+(t_2-t_1\alpha+t_2p-pt_1)}{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right.\right.$$
$$\left.\left.\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+(t_1\alpha-t_1p)I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right), \quad (3.3)$$

then

$$\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\prec q(z), \quad (3.4)$$

and $q(z)$ is the best dominant.

Proof: Define a function $k(z)$ by

$$k(z)=\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}, \quad (3.5)$$

then the function $k(z)$ is analytic in $U$and $q(0)=1$, therefore,differentiating (3.5) logarithmically with respect to z and using the identity (1.7) in the resulting equation,

$$G(z)=\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\times\left(1+\eta\left(\frac{(pt_2-t_2\alpha)I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)(z)+(t_2-t_1\alpha+t_2p-pt_1)}{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right.\right.$$
$$\left.\left.\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+(t_1\alpha-t_1p)I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right)$$

Thus the subordination (3.2) is equivalent to

$$k(z)+\frac{\eta}{\delta}zk'(z)\prec q(z)+\frac{\eta}{\delta}zq'(z).$$

An application of Lemma (2.1) with $\beta=\frac{\eta}{\delta}$ and $\alpha=1$, we obtain (3.4).

Taking $q(z)=\frac{1+Az}{1+Bz}$,$(-1\leq B<A\leq 1)$, in Theorem (3.1), we obtain the following Corollary.

Corollary 3.1. Let $\eta,\delta\in\mathbb{C}\setminus\{0\}$*and* $(-1\leq B<A\leq 1)$.Suppose that

$$Re\left(\frac{1-Bz}{1+Bz}\right)>\max\left\{0,-Re\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f\in W$ is satisfy the following subordination condition:

$$G(z)\prec\frac{1+Az}{1+Bz}+\frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2},$$

where $G(z)$ given by (3.3) , then

$$\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant .

Taking $A=1$ *and* $B=-1$ in Corollary (3.1), we get following result.

Corollary 3.2. Let $\eta, \delta \in \mathbb{C} \setminus \{0\}$ and suppose that

$Re\left(\frac{1+z}{1-z}\right) > \max\{0, -Re\left(\frac{\delta}{\eta}\right)\}$.

If $f \in W$ is satisfy the following subordination

$$G(z) \prec \frac{1+z}{1-z} + \frac{\eta}{\delta}\frac{2z}{(1-z)^2},$$

where

$G(z)$ given by (3.3), then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Theorem 3.2. Let $q(z)$ be convex univalent in unit disk U with $q(0) = 1$, let $\eta, \delta \in \mathbb{C}\{0\}, \gamma, t, \psi, \tau \in \mathbb{C}, f \in W$, and suppose that f and q satisfy the following conditions:

$$Re\left\{\frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0, \qquad (3.6)$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0. \qquad (3.7)$$

If $r(z) \prec t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)}$, (3.8)

where

$$r(z) = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\left(\psi + t\gamma\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right) + t + s_{\delta}(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right)\right), \qquad (3.9)$$

then

$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q(z)$, and $q(z)$ is best dominant.

Proof : Define analytic function $k(z)$ by

$$k(z) = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}. \qquad (3.10)$$

Then the function $k(z)$ is analytic in U and $g(0) = 1$,
differentiating (3.10) logarithmically with respect to z, we get

$$\frac{zk'(z)}{k(z)} = \delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right). \qquad (3.11)$$

By setting $\theta(w) = t + \psi w + \tau\gamma w^2$ and $\phi(w) = \frac{s}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}\setminus\{0\}$.
Also, if we let

$$\phi(z) = zq'(z)\phi(q(z)) = s\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)},$$

we find $Q(z)$ is starlike univalent in U, we have

$$h'(z) = \psi q'(z) + 2\tau\gamma q(z)q'(z) + s\frac{q'(z)}{q(z)} + sz\frac{q''(z)}{q(z)} - sz\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)},$$

hence that

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0.$$

By using (3.11), we obtain

$$\psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\left(\psi + \tau\gamma\gamma\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\right) + t + \left(s_{\delta}(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right)\right).$$

By using (3.8), we have

$$\psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} \prec \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)} \qquad (3.8)$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $k(z) \prec q(z)$ and the function $q(z)$ is the best dominant.

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem (3.2), the condition (3.6) becomes.

$$Re\left(\frac{\psi}{s}\frac{1+Az}{1+Bz} + \frac{2\tau\gamma}{s}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0, \qquad (3.12)$$

hence, we have the following Corollary.

Corollary 3.3. Let $(-1 \le B < A \le 1), s, \delta \in \mathbb{C}\setminus\{0\}, \gamma, t, \tau, \psi \in \mathbb{C}$. Assume that (3.12) holds.
If $f \in W$ and

$$r(z) \prec t + \psi\frac{1+Az}{1+Bz} + \tau\gamma\left(\frac{1+Az}{1+Bz}\right)^2 + s\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $r(z)$ is defined in (3.9), then

$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is best dominant.

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(0 < p \le 1)$, in Theorem (3.2), the condition (3.6) becomes

$$Re\left\{\frac{\psi}{s}\left(\frac{1+z}{1-z}\right)^{\rho} + \frac{2\tau\gamma}{s}\left(\frac{1+z}{1-z}\right)^{2\rho} + \frac{2z^2}{1-z^2}\right\}0, (s \in \mathbb{C}\setminus\{0\}), \qquad (3.13)$$

hence, we have the following Corollary.

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ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Corollary3.4. Let $0<\rho\le 1$,$S,\delta\in\mathbb{C}\setminus\{0\}$ $\gamma,t,\tau,\psi\in\mathbb{C}$. Assume that (3.13) holds. If $f\in W$ and

$$r(z)\prec t+\psi\left(\frac{1+z}{1-z}\right)^{\rho}+\tau\gamma\left(\frac{1+z}{1-z}\right)^{2\rho}+s\frac{2\rho z}{1-z^2},$$

where $r(z)$ is defined in (3.9), then $\left(\frac{I_{M,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\rho}$, and $\left(\frac{1+z}{1-z}\right)^{\rho}$ is the best dominant.

4-Superordination Results

Theorem 4.1. Let $q(z)$ be convex univalent $U$with $q(0)=1,\delta\in\mathbb{C}\setminus\{0\},Re\{\eta\}>0$, if $f\in W$,such that

$$\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\neq 0$$

and

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\mathcal{H}[q(0),1]\cap Q. \qquad (4.1)$$

If the function $G(z)$ defined by (3.3) is univalent and the following superordination condition:

$$q(z)+\frac{\eta}{\delta}zq'(z)\prec G(z), \qquad (4.2)$$

holds , then

$$q(z)\prec\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \qquad (4.3)$$

and $q(z)$ is the best subordinant.

Proof: Define a function $k(z)$ by

$$k(z)=\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}. \qquad (4.4)$$

Differentiating (4.4) with respect to ***z*** logarithmically, we get.

$$\frac{zk'(z)}{k(z)}=\delta\left(\frac{t_1\left(z\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)'\right)+t_2\left(z\left(I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)\right)'\right)}{t_1\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)+t_2\left(I_{\mu,V}^{\lambda,P,\alpha}(a,c)f(z)\right)}-\frac{pt_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+pt_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{t_1\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)+t_2\left(I_{\mu,V}^{\lambda,P,\alpha}(a,c)f(z)\right)}\right) \qquad (4.5)$$

A simple computation and using (1.7) from (4.5), we get

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\times\left(1+\eta\left(\frac{(pt_2-\alpha t_2)I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)+(t_2-\alpha t_1+pt_2-pt_1)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+(\alpha t_1+pt_1)I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right)$$

$$=k(z)+\frac{\eta}{\delta}zk'(z),$$

now , by using Lemma(2.4), we get the desired result .

Taking$q(z)=\frac{1+Az}{1+Bz}(-1\le B<A\le 1)$, in Theorem (4.1),we get the following Corollary.

Corollary 4.2. *Let* $Re\{\eta\}>0,\delta\in\mathbb{C}\setminus\{0\}$ and $-1\le B<A\le 1$, such that

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q.$$

.

If the function $G(z)$ given by (3.3) is univalent in U and $f\in W$ satisfies the following superordination condition:

$$\frac{1+Az}{1+Bz}+\frac{\eta}{\delta}\frac{(A-B)Z}{(1+BZ)^2}\prec G(z),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant .

Theorem 4.2. Let $q(z)$ be convex univalent in unit disk U, Let $\delta,s\in\mathbb{C}\setminus\{0\}.\gamma,t,\psi,\tau\in\mathbb{C},q(z)\neq 0$, *and* $f\in W$. Suppose that

$$Re\left\{\frac{q(z)}{s}(2\tau\gamma q(z)+\psi)\right\}q'(z)>0,$$

and satisfies the next conditions

$$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q, \qquad (4.6)$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\neq 0\,.$$

If the function $r(z)$ is given by (3.9) is univalent in U,

$$t+\psi q(z)+\tau\gamma q^2(z)+s\frac{zq'(z)}{q(z)}\prec r(z) \qquad (4.7)$$

implies

$q(z)\prec\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}$, *and* $q(z)$ is the best subordinant.

Proof: Let the function $k(z)$ defined on U by (3.14).

Then a computation show that

$$\frac{zk'(z)}{k(z)}=\delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}-1\right), \qquad (4.8)$$

by setting $\theta(w)=t+\psi\omega+\tau\gamma\omega^2$ *and* $\emptyset(w)=\frac{s}{\omega}$,it can be easily observed that$\theta(w)$is analytic in $\mathbb{C}$,$\phi(w)$is analytic in $\mathbb{C}\setminus\{0\}$ *and that* $\emptyset(w)\neq 0$ $(W\in\mathbb{C}\setminus\{0\})$.

Also , we get $Q(z)=zq'(z)\emptyset(q(z))=s\frac{zq'(z)}{q(z)}$, it observed that $Q(z)$ is starlike univalent in U .

Since $q(z)$ is convex , it follows that

$Re\left(\frac{z\theta'(q(z))}{\phi(q(z))}\right) = Re\{\frac{q(z)}{s}(2\tau\gamma q(z)) + \psi\}\acute{q}(z) > 0.$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$$\theta\left(q(z) + zq'(z)\phi(q(z))\right) = \theta\left(k(z) + zk'(z)\phi(k(z))\right) \quad ,$$

thus , by applying Lemma (2.3), the proof is completed.

5.Sandwich Results

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich Theorem.

Theorem 5.1. Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1). Suppose that $Re\{\eta\} > 0, \eta, \delta \in \mathbb{C} \setminus \{0\}$.
If $f \in W, such\ that$

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q,$$

and the function $G(z)$ defined by (3.3) is univalent and satisfies

$$q_1(z) + \frac{\eta}{\delta} z q_1'(z) \prec G(z) \prec q_2(z) + \frac{\eta}{\delta} z q_2'(z), \quad (5.1)$$

then

$$q_1(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec q_2(z),$$

where q_1 and q_2 are respectively , the subordinant and the best dominant of (5.1).
Combining Theorem (3.2) with Theorem (4.2), we obtain the following sandwich Theorem.

Theorem 5.2. Let q_i be two convex univalent functions in U , such that $q_i(0) = 1$, $q_i(0) \neq 0$ (i=1,2).Suppose that q_1 and $q_2$satisfies (3.8) and (4.8), respectively.
If $f \in W$ and suppose that f satisfies the next conditions:

$$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \in \mathcal{H}[Q(0),1] \cap Q,$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0 ,$$

and $r(z)$ is univalent in U , then

$$t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq_1'(z)}{q_1(z)} \prec t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq_1'(z)}{q_1(z)},$$

implies

$$q_1(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q_2(z) ,$$

and $q_1$and $q_2$are the best subordinant and the best dominant respectively and $r(z)$ is given by (3.9).

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على نظريات الساندويتش التفاضلية من وظائف متعددة التكافؤ المحددة من قبل المشغل الخطي

وقاص غالب عطشان سلوى كلف كاظم
قسم الرياضيات ، كلية علوم الحاسوب وتكنلوجيا المعلومات ، جامعة القادسية ، الديوانية-العراق

المستخلص :

ا لهدف الرئيسي من هذا البحث هو استخلاص بعض النتائج للوظائف التحليلية متعددة التكافؤ التي يحددها المشغل الخطي باستخدام التبعية التفاضلية والإخضاع .