

A Class of Meromorphic p-valent Functions

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Received : 23\12\2018

Revised : //

Accepted : 13\1\2019

Available online : 28 /1/2019

DOI: 10.29304/jqcm.2019.11.1.479

Abstract:

In this paper, we define a class of meromorphic p-valent functions and study some properties as coefficient inequality, closure theorem , growth and distortion bounds , arithmetic mean, radius of convexity, Convex linear combination and partial sums .

Keywords: Meromorphic p-valent function, Convex function, Integral operator .

Mathematics Subject Classification: 30C45.

1-Introduction: Let A_p^* denote the class of functions f of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, (a_n \geq 0, n \geq p, p \in N), \quad (1.1)$$

which are analytic and p -valent in the punctured unit disk $U^* = \{z \in C: 0 < |z| < 1\}$. Jum-Kim Srivastara [2] defined an integral operator $I_p^\sigma f(z)$ for $f \in A_p^*$ as follows

$$I_p^\sigma f(z) = \frac{1}{z^{p+1}\Gamma(\sigma)} \int_0^z (\log \frac{z}{t})^{\sigma-1} t^p f(z) dt, (n \in N). \quad (1.2)$$

If $f(z)$ is of the form (1.1), then

$$I_p^\sigma f(z) = z^{-p} + \sum_{n=p}^{\infty} \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n (n \geq p, p \in N). \quad (1.3)$$

In particular, when $p=1$ we have:

$$I_p^\sigma f(z) = z^{-1} + \sum_{n=1}^{\infty} \left(\frac{1}{n+2}\right)^\sigma a_n z^n (n \geq p, p \in N).$$

Let f and g be analytic in unit disk U , then g is said to be subordinate of f , written as $g \prec f$ or $g(z) \prec f(z)$, if there exists a schartz function ω which is analytic in U with $\omega(0)=0$ and $|\omega(z)| < 1 (z \in U)$ such that $g(z) = f(\omega(z))$.

In particular, if the function f is univalent in U , we have the following equivalence ([3],[4]).

$$g(z) \prec f(z) (z \in U) \Leftrightarrow g(0) = f(0) \text{ and } g(U) \subseteq f(U).$$

Definition(1.1): A function $f \in A_p^*$ is said to be in the class $A_p^*(\sigma, b, x, y)$ of functions of the form (1.1), which satisfies the condition

$$p - \frac{1}{b} \left\{ 1 + \frac{z^2(I_p^\sigma f(z))''}{z(I_p^\sigma f(z))'} + p \right\} \prec p \frac{1+xz}{1+yz}, \quad (1.4)$$

where

$$-1 \leq y \leq x \leq 1, p \in N, \sigma <$$

$0, b$ non zero complex number.

We can re-write the condition (1.4) as

$$\left| \frac{z(I_p^\sigma f(z))'' + (1+p)(I_p^\sigma f(z))'}{yz(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp](I_p^\sigma f(z))'} \right| < 1. \quad (1.5)$$

2.Coefficient inequality:

In the following theorem, we give a sufficient and necessary condition to be the function in the class $A_p^*(\sigma, b, x, y)$.

Theorem (2.1): Let $f \in A_p^*$ be given by (1.1). Then $f \in A_p^*(\sigma, b, x, y)$ if and only if

$$\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b| \left(\frac{x}{-y}\right)] \left(\frac{1}{n+p+1}\right)^\sigma a_n \leq p^2|b|(x-y). \quad (2.1)$$

The results is sharp for the function f given by

$$f(z) = z^{-p} + \left(\frac{p^2|b|(x-y)}{[n(n+p)(1-y) - np|b|(x-y)]}\right) (n+p+1)^\sigma z^n, (n \geq p, n \in N). \quad (2.2)$$

Proof: Assuming that the inequality (2.1) holds true and $|z| = 1$. Then we have

$$\begin{aligned} & \left| z^2(I_p^\sigma f(z))'' + (1+p)z(I_p^\sigma f(z))' \right| \\ & - |yz^2(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp]z(I_p^\sigma f(z))'| \\ & = \left| \sum_{n=p}^{\infty} n(n+p) \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n \right| - \left| p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n \right| \\ & \leq \sum_{n=p}^{\infty} n(n+p) \left(\frac{1}{n+p+1}\right)^\sigma a_n |z|^n - p^2|b|(x-y) - \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma a_n |z|^n \\ & = \sum_{n=p}^{\infty} n(n+p) \left(\frac{1}{n+p+1}\right)^\sigma a_n - p^2|b|(x-y) - \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma a_n \leq 0, \end{aligned}$$

by hypothesis.

Hence, by the Maximum Modulus Theorem, we have $f(z) \in A_p^*(\sigma, b, x, y)$.

Conversely, suppose that $f(z) \in A_p^*(\sigma, b, x, y)$. Then from (1.5), we have

$$\begin{aligned} & \left| \frac{z^2(I_p^\sigma f(z))'' + (1+p)z(I_p^\sigma f(z))'}{yz^2(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp]z(I_p^\sigma f(z))'} \right| \\ & = \left| \frac{\sum_{n=p}^{\infty} n(n+p) \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n}{p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n} \right| \\ & < 1. \end{aligned}$$

Since $\text{Re}(z) \leq |z|$ for all $z(z \in U)$, we have

$$\text{Re} \left(\frac{\sum_{n=p}^{\infty} n(n+p) \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n}{p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma a_n z^n} \right) \leq 1.$$

We choose the value of z on the real and $z \rightarrow 1^-$, we get

$$\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|] \left(\frac{x}{-y}\right) \left(\frac{1}{n+p+1}\right)^{\sigma} a_n \leq p^2|b|(x-y),$$

which give (2.1). Sharpness of the result follows by setting

$$f(z) = z^{-p} + \left(\frac{p^2|b|(x-y)}{[n(n+p)(1-y) - np|b|(x-y)]}\right) (n+p+1)^{\sigma} z^n, \quad (n \geq p, n \in N).$$

Corollary (2.1) : Let $f(z) \in A_p^*(\sigma, b, x, y)$. Then

$$a_n \leq \frac{p^2|b|(x-y)}{[n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma}}, \quad (n \geq p, n \in N).$$

3. Growth and the Distortion Bounds:

In the following theorems, we obtain the growth and the distortion theorems for the function in the class $A_p^*(\sigma, b, x, y)$.

Theorem (3.1) : If the function $f(z)$ defined by (1.1) is in the class $A_p^*(\sigma, b, x, y)$, then for $0 < |z| = r < 1$, we have:

$$r^{-p} - \left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - |b|(x-y)}\right) r^p \leq |f(z)| \leq r^{-p} + \left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - |b|(x-y)}\right) r^p, \quad (3.1)$$

where equality holds true for the function

$$f(z) = z^{-p} + \left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - |b|(x-y)}\right) z^p. \quad (3.2)$$

Proof: Since $f(z) \in A_p^*(\sigma, b, x, y)$. Then from (2.1)

$$2p^2(1-y) - p^2|b|(x-y) \left(\frac{1}{2p+1}\right)^{\sigma} \sum_{n=p}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma} a_n \leq p^2|b|(x-y),$$

we conclude that

$$\sum_{n=p}^{\infty} |a_n| \leq \frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - |b|(x-y)} \quad (3.3)$$

Thus for $0 < |z| = r < 1$,

$$|f(z)| \leq |z|^{-p} + \sum_{n=p}^{\infty} a_n |z|^n \leq r^{-p} + r^p \sum_{n=p}^{\infty} a_n, \quad (3.4)$$

or

$$|f(z)| \leq r^{-p} - \left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - |b|(x-y)}\right) r^p, \quad (3.5)$$

and

$$|f(z)| \geq |z|^{-p} - \sum_{n=p}^{\infty} a_n |z|^n \geq r^{-p} - r^p \sum_{n=p}^{\infty} a_n,$$

or

$$|f(z)| \geq r^{-p} - \left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - |b|(x-y)}\right) r^p.$$

On using (3.4) and (3.5) inequality (3.1) follows.

Theorem (3.2) : If $f \in A_p^*(\sigma, b, x, y)$ then

$$r^{-(p+1)} - \left(\frac{p|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - p^2|b|(x-y)}\right) r^{p-1} \leq |f(z)'| \leq r^{-(p+1)} + \left(\frac{p|b|(x-y)(2p+1)^{\sigma}}{2(1-y) - p^2|b|(x-y)}\right) r^{p-1}.$$

The result is sharp for the function f is given by (1.3)

Proof: The proof is similar to that of Theorem (3.1).

4. Extreme Points

In the next theorems, we obtain extreme points for the class $A_p^*(\sigma, b, x, y)$.

Theorem (4.1) : Let $f_{p-1}(z) = z^{-p}$ and $f_n(z) = z^{-p} + \left(\frac{p^2|b|(x-y)(n+p+1)^{\sigma}}{[n(n+p)(1-y) - np|b|(x-y)]}\right) z^n$, (4.1)

for $n \geq p$. Then $f(z) \in A_p^*(\sigma, b, x, y)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=p-1}^{\infty} \mu_n f_n(z), \text{ where } \mu_n \geq 0 \text{ and } \sum_{n=p-1}^{\infty} \mu_n = 1. \quad (4.2)$$

Proof: Let

$$f(z) = \sum_{n=p-1}^{\infty} \mu_n f_n(z) = z^{-p} + \sum_{n=p}^{\infty} \left(\frac{p^2|b|(x-y)(n+p+1)^{\sigma} \mu_n}{[n(n+p)(1-y) - np|b|(x-y)]}\right) z^n.$$

Then

$$\frac{[n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma}}{p^2|b|(x-y)} \sum_{n=p}^{\infty} \frac{\left(\frac{1}{n+p+1}\right)^{\sigma}}{p^2|b|(x-y)} \leq \frac{[n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma}}{p^2|b|(x-y)} \sum_{n=p}^{\infty} \mu_n = 1 - \mu_{p-1} \leq 1.$$

Using Theorem (2.1), we easily get $f(z) \in A_p^*(\sigma, b, x, y)$.

Conversely, let $(z) \in A_p^*(\sigma, b, x, y)$.

From the Theorem (2.1), we have

$$a_n \leq \frac{p^2|b|(x-y)(n+p+1)^\sigma}{[n(n+p)(1-y) - np|b|(x-y)]} \text{ for } n \geq p.$$

Setting

$$\mu_n = \frac{n(n+p)(1-y) - np|b|(x-y)}{p^2|b|(x-y)}.$$

$$\left(\frac{1}{n+p+1}\right)^\sigma \text{ for } n \geq p,$$

$$\text{and } \mu_{p-1} = 1 - \sum_{n=p}^{\infty} \mu_n.$$

Then

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n = z^{-p} + \sum_{n=p}^{\infty} \left(\frac{p^2|b|(x-y)(n+p+1)^\sigma \mu_n}{[n(n+p)(1-y) - np|b|(x-y)]} \right) z^n = \mu_{p-1} z^{-p} + \sum_{n=p}^{\infty} \mu_n f_n(z)$$

This completes the proof.

5. Radius of convexity

In the following theorem, we obtain the radius of convexity for the function in the class $A_p^*(\sigma, b, x, y)$.

Theorem (5.1): Let f the function $f(z)$ defined by (1.1) is in the class $A_p^*(\sigma, b, x, y)$. Then f is meromorphically p -valent convex of order λ ($0 \leq \lambda < p$) in the disk $|z| < r_2$, where $r_2 = r_2(p, \sigma, b, x, y) =$

$$\inf_{n \geq p} \left[\frac{(p-\lambda)[(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma}{(n+2p-\lambda)p|b|(x-y)} \right]^{\frac{1}{n+p}} \quad (5.1)$$

The result is sharp for the function f given by (3.4).

Proof: A function f meromorphic p -valent convex of order λ ($0 \leq \lambda < p$) if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \lambda.$$

We must show that

$$\left| \frac{zf''(z)}{f'(z)} + (1+p) \right| < p - \lambda, \quad \text{for } |z| < r_2. \quad (5.2)$$

$$\text{We have } \left| \frac{zf''(z)}{f'(z)} + (1+p) \right| = \left| \frac{zf''(z) + (1+p)f'(z)}{f'(z)} \right| = \left| \frac{\sum_{n=p}^{\infty} n(n+p)a_n z^{n+p}}{-p + \sum_{n=p}^{\infty} n a_n z^{n+1}} \right| \leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n |z|^{n+p}}{p - \sum_{n=p}^{\infty} n a_n |z|^{n+p}}.$$

Thus, (5.2) will be satisfied if

$$\sum_{n=p}^{\infty} \frac{n(n+2p-\lambda)}{p(p-\lambda)} a_n |z|^{n+p} \leq 1. \quad (5.3)$$

Since $f \in A_p^*(\sigma, b, x, y)$, we have

$$\sum_{n=p}^{\infty} \frac{[n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma}{p^2|b|(x-y)} a_n \leq 1.$$

Hence, (5.3) will be true if

$$\frac{n(n+2p-\lambda)}{p(p-\lambda)} |z|^{n+p} \leq \left[\frac{n(n+p)(1-y) - np|b|(x-y) \left(\frac{1}{n+p+1}\right)^\sigma}{p^2|b|(x-y)} \right],$$

or equivalently

$$|z|^{n+p} \leq \frac{(p-\lambda)(n+p)(1-y) - np|b|(x-y) \left(\frac{1}{n+p+1}\right)^\sigma}{(n+2p-\lambda)p^2|b|(x-y)} \geq p, \quad n$$

which follows the result.

6. Convex linear combination:

Theorem (6.1): The class $A_p^*(\sigma, b, x, y)$ is closed under convex linear combinations.

Proof: Let f_1 and f_2 be the chance elements of $A_p^*(\sigma, b, x, y)$. Then for each t ($0 < t < 1$) plus $(a_n, b_n \geq 0)$. we show that $(1-t)f_1 + tf_2 \in A_p^*(\sigma, b, x, y)$. Thus we have

$$(1-t)f_1 + tf_2 = z^{-p} + \sum_{n=p}^{\infty} [(1-t)a_n + tb_n] z^n.$$

Hence

$$\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma [(1-t)a_n + tb_n].$$

$$= (1-t) \sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma a_n$$

$$+ t \sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^\sigma b_n$$

$$\leq (1-t)p^2|b|(x-y) + tp^2|b|(x-y) = p^2|b|(x-y).$$

This completes the proof.

7. The arithmetic mean:

Theorem (7.1): Let the functions f_k sharp by $f_k(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,k} z^n$, ($a_{n,k} \geq 0, n \in N, k = 1, 2, \dots, l$),

be in the class $A_p^*(\sigma, b, x, y)$ for each $k = (1, 2, 3, \dots, l)$, then the function h sharp by

$$h(z) = z^{-p} + \sum_{n=p}^{\infty} e_n z^n, \quad (e_n \geq 0, n \in N)$$

also belong to the class $A_p^*(\sigma, b, x, y)$, where $e_n = \frac{1}{l} \sum_{k=1}^l a_{n,k}$, ($n \geq p, p \in N$).

proof: As $f_k \in A_p^*(\sigma, b, x, y)$, it follows the Theorem (2.1) that

$$\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma} a_{n,k} \leq p^2|b|(x-y),$$

for each $k = 1, 2, 3, \dots, l$ Hence

$$\begin{aligned} & \sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma} e_n \\ &= \sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma} \left(\frac{1}{l} \sum_{k=1}^l a_{n,k}\right) \\ &= \frac{1}{l} \sum_{k=1}^l \left(\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma} a_{n,k} \right) \\ &\leq \frac{1}{l} \sum_{k=1}^l p^2|b|(x-y) \\ &= p^2|b|(x-y). \end{aligned}$$

Then $h \in A_p^*(\sigma, b, x, y)$.

8. Partial sums

Theorem(8.1): Let $f \in A_p^*(\sigma, b, x, y)$ be assumed by (1.1) and $g \in A_p^*(\sigma, b, x, y)$ be assumed by

$$g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n.$$

We define the partial sums $S_1(z)$ and $S_k(z)$ as follows :

$$S_1(z) = z^{-1} \text{ and } S_k(z) = z^{-p} + \sum_{n=p}^{k-1} a_n z^n, \quad (k \in N \setminus \{1\}). \quad (8.1)$$

Also suppose that

$$\sum_{n=p}^{\infty} c_n a_n \leq 1, \quad c_n = \frac{[n(n+p)(1-y) - np|b|(x-y)] \left(\frac{1}{n+p+1}\right)^{\sigma}}{p^2|b|(x-y)}. \quad (8.2)$$

Then, we have $\operatorname{Re} \left\{ \frac{f(z)}{S_k(z)} \right\} > 1 - \frac{1}{c_k}, (z \in U, k \in N),$ (8.3)

and $\operatorname{Re} \left\{ \frac{S_k(z)}{f(z)} \right\} > \frac{c_k}{1+c_k}, (z \in U, k \in N).$ (8.4)

Each of the bounds in (8.3) and (8.4) is the best possible for $k \in N$.

Proof: We can see from (8.2) that $c_{n+1} > c_n > 1, n = p, p+1, p+2, p+3, \dots$

Therefore, we have:

$$\sum_{n=p}^{k-1} a_n + c_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=p}^{\infty} c_n a_n \leq 1. \quad (8.5)$$

By setting

$$g_1(z) = c_k \left[\frac{f(z)}{S_k(z)} - \left(1 - \frac{1}{c_k}\right) \right] = 1 + \frac{c_k \sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=p}^{k-1} a_n z^{n+1}}, \quad (8.6)$$

and applying (8.5) we find that

$$\left| \frac{g_1(z)-1}{g_1(z)+1} \right| \leq \frac{c_k \sum_{n=k}^{\infty} a_n}{2 - 2 \sum_{n=p}^{k-1} a_n - c_k \sum_{n=k}^{\infty} a_n}, \quad (8.7)$$

which readily yields the assertion (8.3) if, we take

$$f(z) = z^{-p} - \frac{z^k}{c_k}. \quad (8.8)$$

Then

$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{c_k} \rightarrow 1 - \frac{1}{c_k} (z \rightarrow 1^-)$, which shows that the bound in (8.3) is the best possible for $k \in N$.

Similarly, if we put

$$g_2(z) = (1 + c_k) \left[\frac{S_k(z)}{f(z)} - \frac{c_k}{1+c_k} \right] = 1 - \frac{(1+c_k) \sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=p}^{k-1} a_n z^{k+1}}, \quad (8.9)$$

and make use of (8.9), we have

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1+c) \sum_{n=k}^{\infty} a_n}{2 - 2 \sum_{n=p}^{k-1} a_n + (1-c_k) \sum_{n=k}^{\infty} a_n}, \quad (8.10)$$

which leads us to the assertion (8.4). The bound (8.5) is sharp for each $k \in N$ with the function given by (6.7). The proof of the theorem is complete.

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