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T-essentially Quasi-Dedekind modules

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Abstract:
In this paper, we introduce and study type of modules namely (t-essentially quasi-Dedekind modules) which is generalization of quasi-Dedekind modules and essentially quasi-Dedekind module. Also, we introduce the class of t-essentially prime modules which contains the class of t-essentially quasi-Dedekind modules.

Keywords: quasi-Dedekind modules, essentially quasi-Dedekind modules, t-essentially quasi-Dedekind modules, essentially prime modules, t-essentially prime modules.

Mathematics Subject Classification: 2010:16 D10, 16D20, 16 D50.

1. Introduction

Let R be a commutative ring with unity and M be a right R-module. A submodule N of M is called quasi-invertible if $Hom\left(\frac{M}{N}, M\right) = 0$ [10]. M is called quasi-Dedekind if every nonzero submodule N of M is quasi-invertible, that is $Hom\left(\frac{M}{N}, M\right) = 0$ for each nonzero submodule N of M. Equivalently M is quasi-Dedekind if for each $f \in End(M), f \neq 0$, then $Ker(f) = 0$ [10]. As a generalization of quasi-Dedekind modules. Tha'ar in [14] introduced the concept essentially quasi-Dedekind (briefly, ess.q-Ded.) by restricting the definition of quasi-Dedekind on essential submodules, where a submodule N of M is called essential in M (denoted by $N \leq_{ess} M$) if $N \cap W \neq 0$ for each nonzero submodule W of M[7]. However, the concept essentially quasi-Dedekind is equivalently to k-nonsingular which is introduced by Roman C.S[12], that M is ess-q-Ded. Module if for each $f \in End(M), Ker(f) \leq_{ess} M$ implies $f = 0$.

In [3] "introduced the concept t-essential submodule, a submodule N of M is called t-essential submodule (denoted by $N \leq_{tes} M$) if $N \cap W \leq Z_2(M)$, then $W \leq Z_2(M)$, where $Z_2(M)$ is the second singular submodule of M and defined by $Z(\frac{M}{Z(M)}) = \frac{Z_2(M)}{Z(M)}$, $Z(M) = \{m \in M{:}mI{=}0$ for some $I \leq_{ess} R\}$[7]. It is clear that $Z(M) = \{m \in M: ann(m) \leq_{ess} R\}$.Also,$Z_2(M) = \{m \in M: mI = 0$ for some $I \leq_{tes} R\} = \{m \in M: ann(m) \leq_{tes} R\}$". It is obvious; every essential submodule is t-essential, but not conversely.

In section two, we define t-essentially quasi-Dedekind module, where an R-module M is called t-essentially quasi-Dedekind if every nonzero t-essential submodule is quasi-invertible, that is $Hom\left(\frac{M}{N}, M\right) = 0$ for each $(0) \neq N \leq_{tes} M$. Analogus characterization of ess.q-Ded. module we have . An R-module M is t-ess.q-Ded. if for each $f \in End(M), Ker(f) \leq_{tes} M$ *implies* $f = 0$. We study t-essentially quasi-Dedekind module. It is clear that every t-essentially quasi-Dedekind module is essentially qusi-Dedekind but not conversely (Remarks and Examples 2.2(2) and every quasi-Dedekind module is t-essentially quasi-Dedekind, but the converse may be not true (Remarks and Examples 2.2(4)). Also we see that every nonsingular module is t-essentially quasi-Dedekind(Remarks and Examples 2.2(3)).

The property of t-essentially quasi-Dedekind is inherited by direct summand (Proposition 2.3); however it is not inherited by direct sum. So we provide necessary and sufficient conditions for a direct sum of t-essentially quasi-Dedekind to be t-essentially quasi-Dedekind.

Beside these some connections between t-essentially quasi-Dedekind modules and other types of modules are investigated.

It is known that every quasi-Dedekind module M is a prime module (that is $annM = annN$ for each $(0) \neq N \leq M$) but the converse may be not true [11]. However implies that every prime modules is ess.q.Ded.. Also, every essentially quasi-Dedekind module M is essentially prime module (that is $annM = annN$ for each $N \leq_{ess} M$) and the converse is not true in general [14, Proposition 2.1.8]. We notice that every t-ess.q.Ded. module M implies $annM = annN$ for each $(0) \neq N \leq_{tes} M$, so this note lead us in section three to introduce and study the concept of t-essentially prime module (that is $annM = annN$ for each, $(0) \neq N \leq_{tes} M$). Thus for a module M, we have the following implications.

t-ess.q-Ded. $\Longrightarrow$ t-ess.prime $\Longrightarrow$ ess.prime.

But none of these implications is reversible(Remarks and Examples 3.3(2),(3)). The concepts essentially prime module and t-essentially prime module are equivalent, under certain conditions(Propositions 3.4,3.7). Also we have that for an R-module , with $annM = ann\overline{M}$($\overline{M}$ is the quasi-injective hull of M) then M is t-essentially prime if and only if $\overline{M}$ is t-essentially prime (Proposition 3.9). Beside these many other properties of t-essentially prime modules, also several connections between this type of modules and other modules are presented.

We list some known results, which will be needed for future use.

Proposition 1.1:[3, Proposition 2.2]. The following statements are equivalent for a submodule A of an R-module M:

(1) A is t-essential in M;

(2) $\frac{(A+Z_2(M))}{Z_2(M)}$ is essential in $\frac{M}{Z_2(M)}$;

(3) $(A + Z_2(M))$ is essential in M;

(4) $\frac{M}{A}$ is Z_2-torsion.

Remark 1.2: [2, Corollary 1.3] Let A_λ be a submodule of M_λ for each $\lambda \in \wedge$

(1) If $\wedge$ is a finite set and $A_\lambda \leq_{tes} M_\lambda$ then $\cap_{\lambda\in\wedge} A_\lambda \leq_{tes} \cap_{\lambda\in\wedge} M_\lambda$;

(2) $\oplus_{\lambda\in\wedge} A_\lambda \leq_{tes} \oplus_{\lambda\in\wedge} M_\lambda$ if and only if $A_\lambda \leq_{tes} M_\lambda$ for each $\lambda \in\wedge$.

Proposition 1.3: [2, Corollary1.2] Let $A \leq B \leq M$. Then $A \leq_{tes} M$ if and only if $A \leq_{tes} B$ and $B \leq_{tes} M$.

2. T-essentially Quasi-Dedekind modules

Definition 2.1: An R-module M is called t-essentially quasi-Dedekind (brifly t-ess.q.Ded.) if every nonzero t-essential submodule N of M is quasi-invertible, that is M is t-ess.q-Ded. if $Hom\left(\frac{M}{N}, M\right) = 0$ for all nonzero t-essential submodule N of M. A ring R is t-ess.q-Ded. if it is t-ess.q-Ded R-module.

Remarks and Examples 2.2:

(1) It is clear that every simple is t-ess.q-Ded. module.

(2) Every t-ess.q-Ded. module is ess.q-Ded. module, since every essential submodule is t-essential. However the converse may be not true, for example: Let $M = Q\oplus Z_2$ as Z-module. M is ess.q-Ded. let $N = Q\oplus(0)$. Then $N + Z_2(M) = (Q\oplus(0)) + ((0)\oplus Z_2) = Q\oplus Z_2 = M \leq_{ess} M$ and so by Proposition 1.1, $N \leq_{tes} M$. It follows that $Hom(\frac{M}{N}, M) \simeq Hom(Z_2, Q\oplus Z_2) \neq 0$ and hence M is not t-ess.q-Ded.

(3) Every nonsingular module is t-ess.q-Ded.

Proof: Let M be a nonsingular module. Then by [11, Proposition 3.13], every essential submodule is quasi-invertible. Hence every t-essential submodule is quasi-invertible by Remark 1.2, and so M is t-ess.q-Ded.. □

(4) It is obvious that every quasi-Dedekind is t-ess.q-Ded, but the converse is not true in general, for example: The Z-module $Z\oplus Z$ is nonsingular, so it is t-ess.q-Ded. (see part (3)), but M is not quasi-Dedekind since $Hom(\frac{M}{Z\oplus(0)}, M) \simeq Hom(Z, Z\oplus Z) \neq 0$.

Similarly each of the Z-module $Q\oplus Z, Q\oplus Q$ is t-ess.q-Ded., but not quasi-Ded.

(5) Let R be a ring. Then the following are equivalent:

(1) R is t-ess.q.-Ded.;

(2) R is ess. Q-Ded.

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(3) R is a nonsingular(R is a semiprime)ring.

Proof: (1)⇒(2) It follows by Remarks and Examples 2.2(2).

(2)⇒(3) It follows by [14, Proposition 2.2.6]

(3)⇒(1) It follows by Remarks and Example 2.2(3). □

(6) For R-module M, $\frac{M}{C}$ is t-ess.q-Ded. for each t-closed submodule C of M, where a submodule C of M is called t-closed if C has no proper t-essential extension in M [3].

Proof: If C is a t-closed submodule, then by [3, Proposition 2.6] $\frac{M}{C}$ is nonsingular. Hence by Remarks and Examples 2.2(4), $\frac{M}{C}$ is t-ess.q-Ded. □

In particular, $\frac{M}{Z_2(M)}$ is t-ess.q-Ded. for any R-module M.

(7) Let M be a t-uniform module (that is, for submodule of M is t-essential[8] . Then M is t-ess.q-Ded. if and only if M is ess.q-Ded.

(8) A homomorphic image of t-ess.q-Ded. need not be a t-ess.q-Ded. for example : Z as a Z-module is t-ess.q-Ded. let $\pi: Z \mapsto \frac{Z}{<4>} \simeq Z_4$ be the natural projection, hence $\pi(Z) = Z_4$ is not t-ess.q-Ded. since $Hom(\frac{Z_4}{(\bar{2})}, Z_4) \neq 0$ and $(\bar{2}) \leq_{tes} Z_4$.

(9) Let M and M'be two isomorphic R-module. Then M is t-ess.q.Ded. if and only if M' is t-ess.q-Ded.

(10) If M is t-ess.q-Ded., then $annM = annN$ for each $N \leq_{tes} M$ and $N \neq 0$

Proof: Since M is t-ess.q-Ded., every $N \leq_{tes} M$, $N \neq 0$ is quasi-invertible submodule. Hence $annM = annN$ for each $0 \neq N \leq_{tes} M$ by [11] □

(11) Let M be an R-module such that $Z_2(M) \leq N$ for all $N \leq M$. Then M is t-ess.q.Ded. if nd only if M is ess.q-Ded.

Proof:⇒ It is clear.

⇐ Let $N \leq_{tes} M$. Then by Remark 1.2, $N + Z_2(M) \leq_{ess} M$, hence $N \leq_{ess} M$ (since $Z_2(M) \leq N$). As M is ess.q-Ded., thus $Hom\left(\frac{M}{N}, M\right) = 0$. □

The property of t-ess.q-Ded. is inherited by direct summand.

Proposition 2.3: A direct summand of t-ess.q-Ded. module M is t-ess.q-Ded.

Proof: Let N be a direct summand of $M (N \leq^{\oplus} M)$. To prove N is a t-ess.q.Ded. Let $(0) \neq K \leq_{tes} N$. As $N \leq^{\oplus} M, M = N \oplus W$, for some $W \leq M$. Since $K \leq_{tes} N$ and $W \leq_{tes} W$, then $K \oplus W \leq_{tes} N \oplus W = M$. By t-essentially quasi-Dedekind of M, $Hom\left(\frac{M}{K \oplus W}, M\right) = 0$; thus , $Hom\left(\frac{N}{K}, M\right) = 0$. Suppose , $Hom\left(\frac{N}{K}, N\right) \neq 0$ that is there exist $f: \frac{N}{K} \mapsto N, f \neq 0$. Hence $i \circ f: \frac{N}{K} \mapsto M, i \circ f \neq 0$, where i is the inclusion mapping. Thus $Hom\left(\frac{M}{K}, M\right) \neq 0$, which is a contradiction. It follows that $Hom\left(\frac{N}{K}, N\right) = 0$ and N is t-ess.q-Ded.

□

Thaa'r in [14, Theorem1.2.3] an R-module is ess.q.Ded. if and only if M is K-nonsingular that is for each $f \in End(M)$ implies $f = 0$.

By similar proof of this result, we get the following.

Theorem 2.4: Let M be an R-module. Then M is t-ess. Q-Ded., if and only if for each $f \in End(M)$, $0 \neq Kerf \leq_{tes} M$ implies $f = 0$.

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Note 2.5: Every semisimple module is ess.q-Ded. [14, Proposition 1.2.4]. However semisimple module may not t-ess. Q-Ded., since $Hom(\frac{Z_6}{<3>}, Z_6) \simeq Hom(Z_3, Z_6) \neq 0$ and $(\bar{3}) \leq_{tes} Z_6$(because $(\bar{3}) + Z_2(Z_6) = (\bar{3}) + Z_6 = Z_6 \leq_{ess} Z_6$).

"Asgari in [4] introduced t-semisimple module, where an R-module M is called t-semisimple if for each $N \leq M$, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. It is clear that every semisimple is t-semisimple but the converse may be not true " [4].

Proposition 2.6: Let M be t-semisimple module and t-ess.q-Ded. module. Then t-closed submodule of M is t-ess.q-Ded.

Proof: Let N be t-closed submodule of M. Then by [3, Lemma 2.5(1)]$N \geq Z_2(M)$, and so[4, Theorem 2.3], N is direct summand . Thus by Proposition 2.3, N is a t-ess. Q-Ded. □

Corollary 2.7: Let R be a t-semisimple ring and t-ess.q-Ded.. Then R is semisimple.

Proof: Since R is t-ess. Q-Ded, R is nonsingular by Remarks and Examples 2.2(5). But R is nonsingular and t-semisimple ring implies R is semisimple. □

"Recall that a module M over a commutative ring R is called scalar module if for each $f \in End(M)$, there exists $0 \neq r \in R$ such that $f(x) = xr$ for each $x \in M$" [13].

" An R-module M is called quasi-prime if $ann(m)$ is a prime ideal of R, for each $m \neq 0$ and $m \in M$" [1].

Theorem 2.8: Let M be a scalar quasi-prime module. Then M is t-ess.q-Ded.

Proof: Let $f \in End(M)$ and suppose that $\neq 0$. Since m is a scalar module, there exists $0 \neq r \in R$ and $f(x) = xr$ for each $x \in M$. Assume $Ker(f) \leq_{tes} M$, hence $Ker(f) + Z_2(M) \leq_{ess} M$ by **Proposition 1.1.** So that for any $m \in M$, there exist $a \in R$ such that $0 \neq ma \in Ker(f) + Z_2(M)$. It follows that $ma = m_1 + m_2$ for some $m_1 \in Kerf, m_2 \in Z_2(M)$. Thus $f(ma) = mar = f(m_1) + f(m_2) = f(m_2) \in Z_2(M)$. If $mar = 0$, then $ar \in ann(m)$. But $ann(m)$ is a prime ideal of R since M is quasi-prime, so either $a \in ann(m)$ or $r \in ann(m)$. If $a \in ann(m)$, then $ma = 0$, which is a contradiction. If $r \in ann(m)$ then $mr = 0$ for each $m \in M$ and $Mr = f(M) = 0$ (that is $f = 0$) which is a contradiction. Thus $0 \neq mar \in Z_2(M)$ which implies that $Z_2(M) \leq_{ess} M$ and so $Z_2(M) \leq_{tes} M$ which a contradiction is since $Z_2(M)$ is t-closed by [3, Corollary 2.7(1)]. Therefor $Ker(f) \not\leq_{tes} M$. Thus M is t-ess.q-Ded. □

Remark 2.9: If M is a t-ess.q-Ded. module, then either $\bar{M}$ or $E(M)$ (quasi-injective hull or injective hull of M) is t-ess.q-Ded. The following example explain this: Let $M = Z_3$ as Z-module. M is t-ess.q-Ded, but $\bar{M} = E(M) = Z_3{}^{\infty}$ is not t-ess q-Ded.

The converse of Remark 2.8 follows directly by the following result, which is an analogous to [14, Proposition 1.2.15].

Proposition 2.10: Let M be a t-ess. q-Ded R-module and it is quasi-injective. If $N \leq_{tes} M$, then N is a t-ess. Q-Ded R-module.

Proof: It is similar to the proof of [14, Proposition 1.2.15] and so is omitted. □

Corollary 2.11: Let M be an R-module. If $\bar{M}$(or $E(M)$ is a t-ess.q-Ded R-module. Then M is tes.q-Ded.

Proof: Since $M \leq_{ess} \bar{M}(M \leq_{ess} E(M))$, so $M \leq_{tes} \bar{M}(M \leq_{tes} E(M))$, the result follows by Proposition 2.10. □

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Now we turn our attention to the direct sum of t-ess.q-Ded modules. First we notice that the direct sum of two t-ess.q-Ded modules need not be t-ess.q-Ded, as the following example: The Z-module Z_2 and Z_3 are t-ess.q-Ded. module, but $Z_2 \oplus Z_3 \simeq Z_6$ is not t-ess.q-Ded.

Definition 2.12: Let M and W be R-module. M is said to be t-ess.q-Ded relative to W for all $f \in Hom(M, W), f \neq 0$ implies $Kerf \not\leq_{tes} M$.

Remarks and Examples 2.13:

(1) Let M be an R-module. M is a t-ess.q-Ded module if and only if M is a t-ess. Q-Ded relative to M.

(2) Let M be a t-ess.q-Ded . Then M is a t-ess. q-Ded. relative to N, for each $N \leq M$.

(3) Z_6 is not t-ess. q-Ded relative to Z_2, since there exists $f: Z_6 \mapsto Z_2$ defined by $f(\bar{0}) = f(\bar{2}) = f(\bar{4}) = \bar{0}_{Z_2}$, $f(\bar{1}) = f(\bar{5}) = f(\bar{3}) = \bar{1}_{Z_2}$

Thus $Ker(f) = \{\bar{0}, \bar{2}, \bar{4}\} \leq_{tes} Z_6$ *and* $f \neq 0$.

The following Theorem is analogous to [14, Theorem 1.3.5].

Theorem 2.14: Let $\{M_i\}_{i \in \wedge}$ be a family of R-modules. Then $M = \{M_i\}_{i \in \wedge}$ is t-ess. q-Ded if and only if M_i t-ess. q-Ded relative to M_j for $i, j \in \wedge$.

Proof: It is similar to Theorem 1.3.5 in [14] and so is omitted. □

3. t-essentially prime Modules

Ali Saba in [11] prove that: If M is a prime module, then for each $f \in End(M)$ and $Ker(f) \leq_{ess} M$ then $= 0$; that is every prime module is ess. q-Ded module. However prime module does not imply t-ess. q-Ded. for example : Let M be the Z-module $Z_2 \oplus Z_2$. M is a prime module but M is not t-ess. q-Ded since M is singular and so every submodule N of M, $N \leq_{tes} M$. Take $N = Z_2 \oplus (0)$. Then $Hom(\frac{M}{N}, M) \neq 0$.

We have the following:

Proposition 3.1: Every faithful prime module is t-ess. q-Ded.

Proof: First we shall show that M is nonsingular. Let $x \in Z(M)$ and suppose that $x \neq 0$. Then $ann(x) \leq_{ess} R$. Hence there exists $x \in R, r \neq 0$ and $r \in ann(x)$ and so $xr = 0$. As M is a prime module and $x \neq 0, r \in annM = 0$ which is a contradiction. Thus $Z(M) = 0$ (M is nonsingular) and so by Remarks and Examples 2.2(3), M is t-ess. q-Ded. □

Notice that the condition M is faithful is necessary in Proposition 3.1 as we have seen $M = Z_2 \oplus Z_2$ as Z-module is prime, not faithful and M is not t-ess. q-Ded.

Now it is known by [14, Proposition 2.1.8], every ess. q-Ded module is an essentially prime module (that is $ann_R M = ann_R N$ for each $N \leq_{ess} M$). Also, by Remarks and Examples 2.2(9), if M is a t-ess. q-ded module, then $ann_R M = ann_R N$ for each $(0) \neq N \leq_{tes} M$. This leads us to introduce the following.

Definition 3.2: An R-module is called t-essentially prime (briefly t-ess.prime) if $ann_R M = ann_R N$ for each $(0) \neq N \leq_{tes} M$.

Remarks and Examples 3.3:

(1) It is clear that every prime module is t-ess. prime is, but the converse is not true in general (see part(3), II).

(2) Every t-ess. prime module is ess. prime, since every essential submodule is t-essential. But the converse may not be true in general, for example. The Z-module Z_6 is ess. prime module, but it is not t-ess. prime since $ann_Z Z_6 \neq ann_Z(\bar{2})$ and $(\bar{2}) \leq_{tes} Z_6$.

(3) A t-ess. prime module need not be t-ess. q-Ded module, as the following examples show :

(I) Let M be the Z-module $Z_2 \oplus Z_2$. M is t-ess. prime, but M is not t-ess. q-Ded as we have seen in the beginning of section three.

(II) Let $M = Z_2 \oplus Z_2$ as Z-module . M is not t-ess. q-Ded , since if $N = Z \oplus (0)$, then $N + Z_2(M) = M \leq_{ess} M$ and so by Proposition 1.1, $N \leq_{tes} M$. But $Hom(\frac{M}{N}, M) \simeq Hom(Z_2, Z \oplus Z_2) \neq 0$. On the other hand, we can show that M is t-ess. prime as follows: Let $W \leq_{tes} M$ then $W + Z_2(\mathrm{M}) \leq_{ess} M$(by Proposition 1.1). As M is an ess. prime module by [14, Example 2.1.12], hence $ann_Z\big(W + Z_2(M)\big) = ann_Z M = (0)$. It follows that $ann_Z W \cap ann_Z Z_2(M) = 0$ and so $ann_Z W \cap 2Z = 0$. (since $Z_2(M) = (0) \oplus Z_2$ and $ann_Z Z_2(M) = 2Z$). Since $2Z \leq_{ess} Z$ then $ann_Z W = 0$. This implies $ann_Z W = ann_Z M$ and M is t-ess. prime. Also, note that M is not prime module.

(4) Let M be a nonsingular module. Then M is an ess. prime if and only if M is a t-ess. prime module.

Proposition 3.4: Let M be a faithful R-module such that $ann_R(Z_2(M)) \leq_{ess} R$. Then M is an ess. prime module if and only if M is t-ess. prime.

Proof: $\Leftarrow$ It is clear.

$\Rightarrow$ Let $0 \neq N \leq_{tes} M$. Then $N + Z_2(M) \leq_{ess} M$. As M is ess. prime, $ann(N + Z_2(M)) = annM = (0)$. Hence $annN \cap ann\big(Z_2(M)\big) = 0$. By hypothesis, $ann(Z_2(M)) \leq_{ess} R$, so that $annN = 0 = annM$. It follows that M is t-ess. prime. □

"Recall that an R-module M is bounded if there exists $x \in M$ such that $ann_R M = ann_R(x)$ " [6].

Proposition 3.5: Let M be a bounded module with $ann_R M$ is a prime ideal of R and $ann_R M < ann(Z_2(M))$. Then M is t-ess. prime.

Proof: Let $(0) \neq N \leq_{tes} M$. Then $N + Z_2(M) \leq_{ess} M$ by proposition 1.1. Since M is bounded with $annM$ is a prime ideal, then by [14, Lemma 2.1.11], M is ess. prime. Hence $ann_R\big(N + Z_2(M)\big) = ann_R M$. It follows that $ann_R \cap ann_R\big(Z_2(M)\big) = ann_R M$. As $ann_R M$ is a prime ideal, either $ann_R N \leq ann_R M$ or $ann_R Z_2(M) = ann_R M$. Thus either $ann_R N \leq ann_R M$ or $ann_R(Z_2(M)) = ann_R M$. But by hypothesis $ann_R M \neq ann_R(Z_2(M))$, so that $ann_R N = ann_R M$ and so M is t-ess. prime. □

Corollary 3.6: Let M be a bounded quasi-prime R-module with $ann_R M \subsetneq ann_R(Z_2(M))$. Then M is t-ess. prime.

Proof : As M is a quasi-prime module, then $ann_R M$ is a prime ideal of R and so by [14, Lemma 2.1.11] M is an ess. prime module. Then by the same procedure of Proposition 3.5, M is a t-ess. prime module. □

As application of Corollary 3.6, $M = Q \oplus Z_2$ as Z-module is t-ess. prime module since M is bounded (where $ann_Z M = ann_Z(1, \bar{1})$, also it is easy to check that M is quasi-prime, and 0=$ann_Z M \subsetneq ann_Z\big(Z_2(M)\big) = ann_Z Z_2 = 2Z$.

" Recall that an R-module is called multiplication if for each $N \leq M$, $N = MI$ for some ideal I of R" [5].

Proposition 3.7: Let M be a faithful multiplication R-module. Consider the following statements:

(1) M is a t-ess. prime .
(2) M is t-ess.q-Ded.
(3) M is ess.prime;
(4) R is t-ess. q-Ded;
(5) R is ess. q-Ded;
(6) $End_R(M)$ is t-ess.q-Ded.

Then (1) ⇔ (2) ⇔(3)⇔ (5) ⇔ (6) and (4) ⇔ (6) if M is a finitely generated module.

Proof: (1) ⇒(2) Since M is t-ess. prime, M is ess. prime. Hence by [14, Proposition 2.1.16], R is ess. q-Ded and so R is nonsingular by [14, Proposition 1.2.6]. On the other hand, M is faithful mulitiplication implies $Z(M) = MZ(R)$ by [5, Corollary 2.1.4]. It follows that $Z(M) = M(0) = 0$; that is M is nonsingular and hence by Remarks and Examples 2.3(3), M is t-ess. q-Ded.

(2)⇒(1) It follows by Remarks and Examples 3.3(3).

(2)⇒(3) M is t-ess.q-Ded implies M is t-ess. prime and hence M ess. Prime (see Remarks and Examples 3.3(2),(3)).

(3)⇒(5) Since M is an ess. prime faithful module then by [14,Lemma 2.1.16], R is ess. q-Ded.

(5)⇒(2) Since R is ess. q-Ded, R is nonsingular which implies M is nonsingular because $Z(M) = MZ(R) = 0$. Thus M is t-ess q-Ded by Remarks and Examples 2.2(3).

(4)⇔(5) It follows by Remarks and Examples 2.2(5).

(4)⇒(6) Since M is a finitely generated multiplication module, then M is scalar R-module [13]. Hence by [10], $E(M) \simeq \frac{R}{annM} \simeq \frac{R}{(0)} \simeq R$. Thus $End(M)$ is t-ess. q-Ded if and only if R is t-ess. q-Ded. □

Remark 3.8: The condition M is a multiplication module cannot be dropped from Theorem 3.7. The following example explains this:

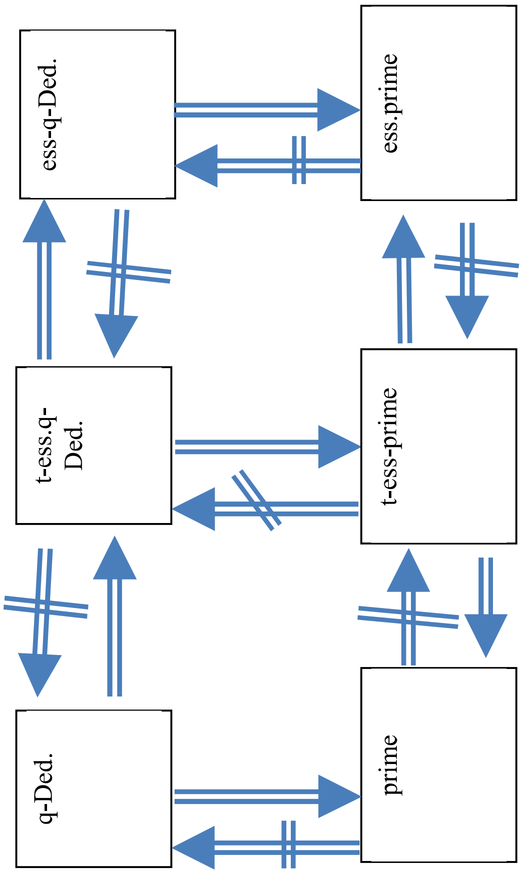
Let $M = Z \oplus Z_2$ as Z-module but not multiplication module. However, M is t-ess. prime Z-module and it is not t-ess. q-Ded (see Remarks and Examples 3.3(3(II)). Also note that R is t-ess. q-Ded.

Proposition 3.9: Let M be an R-module. Then M is t-ess prime and $annM = ann\bar{M}$ if and only if $\bar{M}$ is tes- prime. Where $\bar{M}$ is the quasi-injective hull of M.

Proof: ⇒ Let $(0) \neq N \leq_{tes} \bar{M}$. To prove $ann_R N = ann_R \bar{M}$. Since $M \leq_{ess} \bar{M}$, then $M \leq_{tes} \bar{M}$ and so $N \cap M \leq_{tes} \bar{M}$ by Proposition 1.3. Let $B \leq M$ and $(N \cap M) \cap B \subseteq Z_2(M) - - - - - - - - (I)$. Then $N \cap B \subseteq Z_2(M) \subseteq Z_2(\bar{M})$. It follows that $B \subseteq Z_2(\bar{M})$, since $N \leq_{tes} \bar{M}$ and $B \leq M \leq \bar{M}$. Thus $B \subseteq Z_2(\bar{M}) \cap M = Z_2(M)$; and so by (I) implies $N \cap B \leq_{tes} M$. On the other hand M is t-ess. prime, which implies that $ann_R(N \cap M) = ann_R(M) = ann_R(\bar{M})$. Since $ann_R(N \cap M) \supseteq ann_R(N)$(because $(N \cap M) \leq N$, hence $ann_R(\bar{M}) \supseteq ann_R N$. But $ann_R(\bar{M}) \subseteq ann_R N$. Thus $ann_R(\bar{M}) = ann_R(N)$ and so $\bar{M}$ is t-ess. prime.

⇐ Since $M \leq_{ess} \bar{M}$, then $M \leq_{tes} \bar{M}$. So that by t-essentially prime of M, $ann_R(M) = ann_R(\bar{M})$. Now, let $(0) \neq N \leq_{tes} M$, hence $N \leq_{tes} M \leq_{tes} \bar{M}$ which implies $N \leq_{tes} \bar{M}$. It follows that $ann_R(N) = ann_R(\bar{M})$ (since $\bar{M}$ is t-ess. prime), but by the proof $ann_R(\bar{M}) = ann_R(N)$. Thus $ann_R N = ann_R M$ and M is t-ess. prime. □

Remark 3.10: The condition $ann_R M = ann_R \overline{M}$ can't be dropped from Proposition 3.9 and the following example explains this: Let M be the Z-module Z_P (where P is a prime number). M is a prime module, so it is t-ess. prime, but $\overline{M} = Z_{P^\infty}$ is not t-ess. prime (since $(0) = ann_Z \overline{M} \neq ann_Z \left(\frac{1}{P} + Z\right) = PZ$. Also notice that $PZ = ann_Z M \neq ann_Z \overline{M} = 0$.



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المقاسات شبه الديدكاندية الواسعة من النمط t

فرحان داخل شياع
قسم الرياضيات
كلية التربية
جامعة القادسية

شكرنعمه العياشي
قسم التخطيط الحضري
كليةالتخطيط العمراني
جامعة الكوفة

انعام محمد علي هادي
قسم الرياضيات
كلية التربية-ابن الهيثم
جامعة بغداد

المستخلص :

في هذا البحث قدمنا و درسنا صنف من القاسات اطلقنا عليه المقاسات شبه الديدكاندية الواسعة من النمط t وهي تعميم للمقاسات شبه الديد كاندية الواسعة والمقاسات شبه الديكاندية.كذلك قدمنا صنف المقاسات الاولية الواسعة من النمط t والذي يحتوي على صنف المقاسات شبه الديدكدية الواسعة من النمط t.