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On The Solution of Existence of Nonlinear Integral-and Integrodifferential Equations

Noora L. Husein

Department of Mathematics \ College of Education
University of Mosul

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الخلاصة

في هذا البحث درسنا الوجود والوحدانية لمعادلة فولتيرا– فريدهولم التكاملية ومعادلة فولتيرا– فريدهولم التكاملية التفاضلية مستخدمين نظرية النقطة الثابتة والفضاء المتري .

Aabstract

In this paper we study the existence and uniqueness for mixed Volterra – Fredholm integral and integrodifferential equations By using the extensions of Banach's contraction principle in complete cone metric space.

Keywords : Fixed point theorem ; Cone metric space ; Comparison function ; Banach space.

Introduction

The purpose of this paper is study the existence of solution for Volterra – Fredholm integral equation of the form :

$$x(t) = f(t, x(t)) + \int_0^t k(t,s)k_1(s, x(s))ds + \int_0^b h(t,s)h_1(s, x(s))ds, \; t \in J = [0,b] \ldots\ldots (1)$$

and is study the existence of solution for the integro diffirential Volterra-Fredholm integral equation of the first order of the form :

$$x'(t) = f(t, x(t)) + \int_0^t k(t,s)k_1(s, x(s))ds + \int_0^b h(t,s)h_1(s, x(s))ds \;\; \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$x(0) = x_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $f : J \times Z \to Z. k, h : J \times J \to J. k_1, h_1 : J \times Z \to Z$, are continuous and the

given x_0 is element of Z, Z is a Banach space with norm $\|\cdot\|$

The integrodiffirential equations and integral equations have been

studied by many authors [2,4,5,6,7].Tidke [7] studied the existence of solution for Volterra – Fredholm integro differential equations of second order of the form:

$$x''(t) = Ax(t) + \int_0^t k(t, s, x(s))kds + \int_0^b h(t, s, x(s))ds$$

And B.G.Pachpatte [6] studied the existence of solution for Volterra integral equation in two variables of the form

$$u(x, y) = f(x, y, u(x, y), K(x, y))$$

$$(Ku)(x, y) = \int_a^x \int_b^y K(x, y, m, n, u(m, n))\, dn\, dm$$

Finally in [4], Balachandran and Kim established sufficient conditions for the existence and uniqueness of random solutions of nonlinear Volterra-Fredholm stochastic integral equations of mixed type by using admissibility theory and fixed point theorem .

Preliminaries

Let us recall the concepts of the cone metric space and we refer the reader for the more details . Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if ,

1. P is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in R,\ a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$, where intP denotes the interior of P . The cone P is called normal if there is a number $K > 0$ such thatimplies $0 \leq x \leq y$
The least Positive number satisfying $x, y \in E$, for every $\|x\| \leq K\|y\|$
above is called the normal constant of P

In the following way , we always suppose E is areal Banach space , P is acone

in E with intP $\neq \phi$,and $\leq$ is a partial ordering with respect to P.

Definition 1:[7]

Let X be nonempty set . Suppose that the mapping $d : X \times X \rightarrow E$ satisfies :

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$;

Then d is called a cone metric on X and (X,d) is called cone metric space .The concept of cone metric space is more general than that of metric space .

Definition 2:[7]

Let X be an ordered space .A function $\phi: X \to X$ is said to a comparison function if for every $x, y \in X, x \leq y$ implies that $\phi(x) \leq \phi(y), \phi(x) \leq x$ and $\lim_{n\to\infty} \|\phi^n(x)\| = o$, for every $x \in X$

Definition 3:[8]

The function $x \in B$ given by

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \int_o^t [\int_0^s k(t,\tau)k_1(\tau, x(\tau))d\tau + \int_0^b h(t,\tau)h_1(\tau, x(\tau))d\tau]ds \quad ......(4)$$

Is called the solution of the initial value problem (2)-(3) .

We need the following Lemma for further discussion.

Lemma :[8]

Let (X,d) be a complete cone metric space ,where P is a normal cone with normal constant K .Letbe a function such that there exist a $f: X \to X$ comparison function $\phi: P \to P$

such that $d(f(x), f(y)) \leq \phi(d(x, y))$ for every $x, y \in X$. Then f has a unique fixed point.

-We list the following hypotheses for our convenience :

(H1) Ther exist continuous $P_1, P_2 : J \times J \to R^+$ and a comparison function $\phi: R^2 \to R^2$ such that

$$k(t,s)[\|k_1(s,u) - k_1(s,v)\|, \alpha\|k_1(s,u) - k_1(s,v)\|] \leq M P_1(s)\phi(d(u,v)) \text{ ,}$$

$$h(t,s)[\|h_1(s,u) - h_1(s,v)\|, \alpha\|h_1(s,u) - h_1(s,v)\|] \leq N P_2(s)\phi(d(u,v)) \text{ ,}$$

and

$$[\|f(s,u) - f(s,v)\|, \alpha\|f(s,u) - f(s,v)\|] \leq r\,\phi(d(u,v)), \quad -1 < r \leq 0$$

For every $t,s \in J$ and $u,v \in Z$

(H2) $\sup\limits_{s\in J}\int_0^b [M\,P_1(s) + N\,P_2(s)]ds \le 1$

(H3) $\int_0^b \int_0^b [M\,P_1(s) + N\,P_2(s)]dsdt \le 1$

(H4) $\int_0^b ds \le 1$

Main Results

In this section we will prove the main result of this paper :

<u>Theorem 1:</u> Assume that hypotheses (H1)-(H2) hold then the integral equation (1) has unique solution x on J .

<u>Proof</u> :

The operator $F : B \to B$ Is defined by :

$$Fx(t) = f(t, x(t))dt + \int_0^t k(t,s)k_1(s, x(s))ds + \int_0^b h(t,s)h_1(s, x(s))ds, \quad t \in J \quad ......(5)$$

By using the hypotheses H1 –H2 , we have

$$
\begin{aligned}
&(\|Fx(t)-Fy(t)\|,\alpha\|Fx(t)-Fy(t)\|) \\
&\le (\| f(t,x(t)) + \int_0^t k(t,s)k_1(s,x(s))ds + \int_0^b h(t,s)h_1(s,x(s))ds - f(t,y(t)) \\
&-\int_0^t k(t,s)k_1(s,y(s))ds - \int_0^b h(t,s)h_1(s,y(s))ds \ \|,\alpha\| \ f(t,x(t)) + \\
&\int_0^t k(t,s)k_1(s,x(s))ds + \int_0^b h(t,s)h_1(s,x(s))ds - f(t,y(t)) \_ \\
&-\int_0^t k(t,s)k_1(s,y(s))ds - \int_0^b h(t,s)h_1(s,y(s))ds \ \|) \\
&\le (\|f(t,x(t)) - f(t,y(t))\| + \int_0^t \|k(s,t)[k_1(s,x(s)) - k_1(s,y(s))]\| ds \\
&+\int_0^b \|h(s,t)[h_1(s,x(s)) - h_1(s,y(s))]\| ds, \alpha\|f(t,x(t)) - f(t,y(t))\| \\
&+\alpha\int_0^t \|k(s,t)[k_1(s,x(s)) - k_1(s,y(s))]\| ds + \alpha\int_0^b \|h(s,t)[h_1(s,x(s)) - h_1(s,y(s))]\| ds)
\end{aligned}
$$

$$
\begin{aligned}
&\le (\|f(t,x(t)) - f(t,y(t))\|,\alpha\|f(t,x(t)) - f(t,y(t))\|) + (\int_0^t \|k(s,t)[k_1(s,x(s)) - k_1(s,y(s))]\| ds \\
&,\alpha\int_0^t \|k(s,t)[k_1(s,x(s)) - k_1(s,y(s))]\| ds) + (\int_0^b \|h(s,t)[h_1(s,x(s)) - h_1(s,y(s))]\| ds \\
&,\alpha\int_0^b \|h(s,t)[h_1(s,x(s)) - h_1(s,y(s))]\| ds) \\
&\le r\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) + \int_o^t MP_1(s)\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds + \int_o^b NP_2(s)\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds \\
&\le r\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) + \int_o^b MP_1(s)\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds + \int_o^b NP_2(s)\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds \\
&\le r\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) + \int_o^b [MP_1(s) + NP_2(s)]\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds \\
&\le r\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) + \phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) \\
&= (r+1)\ \phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) \\
&\le \phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) \qquad , \quad |r+1| \le 1
\end{aligned}
$$

This implies that $d(Fx,Fy) \le \phi(d(x,y))$ For every $x, y \in B$ .Now an application of lemma 1, the operator has a unique point in B .This means that the equation (1) has unique solution .This completes the proof of theorem (1).

Theorem 2: Assume that hypotheses (H1)-(H3) and (H4) hold then the initial value problem (2)-(3) has unique solution x on J.

Proof :

The operator $G: B \to B$ Is defined by

$$Gx(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t [\int_0^s k(t,\tau)k_1(\tau, x(\tau))d\tau + \int_0^b h(t,\tau)h_1(\tau, x(\tau))d\tau]ds, \quad t \in J \text{ ....}(6)$$

By using the hypotheses H1,H3,H4 , we have.

$$(\|Gx(t) - Gy(t)\|, \alpha\|Gx(t) - Gy(t)\|)$$

$$\leq (\| \int_0^t f(s, x(s))ds + \int_0^t [\int_0^s k(t,\tau)k_1(\tau, x(\tau))d\tau + \int_0^b h(t,\tau)h_1(\tau, x(\tau))d\tau]ds$$

$$- \int_0^t f(s, y(s))ds - \int_0^t [\int_0^s k(t,\tau)k_1(\tau, y(\tau))d\tau + \int_0^b h(t,\tau)h_1(\tau, y(\tau))d\tau]ds\|, \alpha$$

$$\| \int_0^t f(s, x(s))ds + \int_0^t [\int_0^s k(t,\tau)k_1(\tau, x(\tau))d\tau + \int_0^b h(t,\tau)h_1(\tau, x(\tau))d\tau]ds$$

$$- \int_0^t f(s, y(s))ds - \int_0^t [\int_0^s k(t,\tau)k_1(\tau, y(\tau))d\tau + \int_0^b h(t,\tau)h_1(\tau, y(\tau))d\tau]ds\|)$$

$$\leq (\int_0^t \|f(s,x(s))-f(s,y(s))\|ds+\int_0^t[\int_0^s k(t,\tau)\|k_1(\tau,x(\tau))-k_1(\tau,y(\tau))\|d\tau]ds$$

$$+\int_0^t[\int_0^b h(t,\tau)\|h_1(\tau,x(\tau))-h_1(\tau,y(\tau))\|d\tau]ds,\alpha\int_0^t \|f(s,x(s))-f(s,y(s))\|ds$$

$$+\alpha\int_0^t[\int_0^s k(t,\tau)\|k_1(\tau,x(\tau))-k_1(\tau,y(\tau))\|d\tau]ds+\alpha\int_0^t[\int_0^b h(t,\tau)\|h_1(\tau,x(\tau))-h_1(\tau,y(\tau))\|d\tau]ds)$$

$$\leq (\int_0^t \|f(s,x(s))-f(s,y(s))\|ds,\alpha\int_0^t \|f(s,x(s))-f(s,y(s))\|ds)+(\int_0^t[\int_0^s k(t,\tau)\|k_1(\tau,x(\tau))-k_1(\tau,y(\tau))\|d\tau]ds$$

$$,\alpha\int_0^t[\int_0^s k(t,\tau)\,\|k_1(\tau,x(\tau))-k_1(\tau,y(\tau))\|\,d\tau]\,ds)+(\int_0^t[\int_0^b h(t,\tau)\,\|h_1(\tau,x(\tau))-h_1(\tau,y(\tau))\|\,d\tau]\,ds,\alpha$$

$$\int_0^t[\int_0^b h(t,\tau)\|h_1(\tau,x(\tau))-h_1(\tau,y(\tau))\|d\tau]ds)$$

$$\leq \int_0^t r\,\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds+\int_0^t\int_0^s M\,P_1(s)\,\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)d\tau ds$$

$$+\int_0^t\int_0^b N\,P_2(s)\,\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)d\tau ds$$

$$\leq \int_0^b r\,\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)ds+\int_0^b\int_0^b M\,P_1(s)\,\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)d\tau ds$$

$$+\int_0^b\int_0^b N\,P_2(s)\,\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)d\tau ds$$

$$\leq r\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)+\int_0^b\int_0^b[M\,P_1(s)+N\,P_2(s)]\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)d\tau ds$$

$$\leq r\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)+\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$$

$$=(r+1)\phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$$

$$\leq \phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) \qquad |r+1|\leq 1$$

This implies that $d(Fx,Fy)\leq\phi(d(x,y))$ For every $x,y\in B$

Now an application of lemma 1 the operator has a unique point in B .This means that the equation (1) –(2) has unique solution .This couplets the proof of theorem (2).

Application

In this section we give example to illustrate the usefulness of our result.
In equation (1) , we define:

$$k(t,s)=s\ ,\quad k_1(s,x(s))=\frac{x}{2},\ h(t,s)=s,\ h_1(s,x(s))=\frac{x^2}{2},\ f(s,x(s))=\lambda x+s$$
$$s,t\in[0,1],\quad x\in C([0,1],R),\ \lambda\ \text{constant}$$

And metric $d(x,y)=(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$ on $C([0,1],R)$ and $\alpha\geq 0$. Then clearly $C([0,1],R)$ is complete cone metric space.

Now we have

$$(|k(t,s)[k_1(s,x(s))-k_1(s,y(s))]|,\alpha|k(t,s)[k_1(s,x(s))-k_1(s,y(s))]|)$$
$$=\left(\left|s[\frac{x}{2}-\frac{y}{2}]\right|,\alpha\left|s[\frac{x}{2}-\frac{y}{2}]\right|\right)$$
$$=(\frac{s}{2}|x-y|,\alpha\frac{s}{2}|x-y|)$$
$$=\frac{s}{2}(|x-y|,\alpha|x-y|)$$
$$\leq\frac{s}{2}(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$$
$$=M\,P_1^*\Phi^*\,(\|x-y\|_\infty,\alpha\|x-y\|_\infty),$$

Where $P_1^*(s)=s$, which is continuous function of $[0,1]\times[0,1]$ into R^+ and a comparison function $\Phi^*:R^2\to R^2$ such that $\Phi^*(x,y)=(x,y)$. Similarly, we can show that

$(|h(t,s)[h_1(s,x(s)) - h_1(s,y(s))]|, \alpha|h(t,s)[h_1(s,x(s)) - h_1(s,y(s))]|) \le P_2^* \phi^* (\|x-y\|_\infty, \alpha\|x-y\|_\infty),$

wher $P_2^*(s) = s$, which is contiuous function of $[0,1]\times[0,1]$ into R^+. Similary, we can show that

$(|f(s,x(s)) - f(s,y(s))|, \alpha|f(s,x(s)) - f(s,y(s))|) \le \lambda \phi^* (\|x-y\|_\infty, \alpha\|x-y\|_\infty),$

$-1 < \lambda \le 0$

Morever,

$$\int_0^1 [M P_1^*(s) + N P_2^*(s)]ds = \int_0^1 [\frac{1}{2}s + \frac{1}{2}s]ds = \frac{1}{2} < 1$$

Also

$$\int_0^1\int_0^1 [M P_1^*(s) + M P_2^*(s)]dsdt = \int_0^1\int_0^1 [\frac{1}{2}s + \frac{1}{2}s]dsdt = \int_0^1 \frac{1}{2}dt = \frac{1}{2} < 1$$

Also

$$\int_0^1 ds = 1$$

With these choices of function , all requirements of Theorem 1. And Theorem 2 are satisfied . Hence the existence and uniqueness are verified.

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