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New Characterizations of C-compact Spaces

Atallah Th. Al-Ani

Irbid National University- College of Science and Information Technology--Irbid-Jordan

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Abstract: Some new properties and characterizations of C-compact spaces are given.

Kewords: C-compact spaces, compact space, H-closed space, multifunctions , θ-closed graph.

Introduction and preliminaries.

C-compact spaces are defined in [4]. They form a class lies between compact spaces and quasi H-closed spaces. These spaces had been studied by several authors, as in [1] , [2] , [3] and [5]. A space X is C-compact iff every closed set is H-set. A subset A of X is H set in X iff every cover of A by open sets in X has a finite subcollection such that the closures of its members in X cover A. A space X is quasi H-closed (H-closed) cover if it is H-set in X (and Hsusdorff). (X,τ) is maximal C-compact (minimal Hausdorff) iff every topology on X strictly finer (coarser) than τ is not C-compact (not Hausdorff) . A Hausdorff space X is called functionally compact [2] iff every continuous function on X into a Hausdorff space is closed. A multifunction $\alpha:X\to Y$ is a subset of $X \times Y$, such that $\alpha(x) \neq \phi$ for every $x \in X$. α is called closed graph iff its graph is closed in $X \times Y$, α is called C-closed graph iff for every $(x,y)\in X\times Y\setminus G(\alpha)$, where $G(\alpha)$ is the graph of α , there is an open set V in X such that $x \in clV$ and an open set W in Y such that $y \in W$ and $(clV \times W) \cap G(\alpha) = \phi$. A multifunction $\alpha:X\to Y$ is called θ-closed graph iff its graph is θ-closed in $X \times Y$. A subset A of a space X is θ-closed iff A is equals to its θ–closure $cl_\theta A = \{x\in X: \exists$ V open in X, $x\in V$ and $\overline{V} \cap A \neq f\}$. If $\alpha : X \to Y$ is a multifunction and $K \subset Y$ then $\alpha^{-1}(K) = \{x\in X : \alpha(x) \cap K \neq\phi\}$. α is called upper semi continuous (u. s. c) iff $\alpha^{-1}(V)$ is open in X for every open set V in Y. By $\gamma(x)$ we mean {cl V : V open in X , $x\in V$} and by $\Gamma(x)$ we mean {clV : V open in X , $x \in$ cl V } , $\Gamma(K)$ will denote {clV:V open and $K \subset clV$} . If Ω is a collection of subsets of X then ad $\Omega= \cap\{cl K : K \in\Omega\}$ the adherence of Ω, $ad_\theta \Omega=\cap\{cl_\theta K : K\in \Omega\}$ the θ-adherence of Ω. The adherence of $\alpha(\Gamma(x))$ is denoted by $S(\alpha,x)$. And for $K\subset X$, $S(\alpha,K)= \cup\{S(\alpha,x): x \in K \}$. A subset A of X is called regular closed in X iff A equals to the interior of its closure in X. The collection of all regular closed sets containing a subset K is denoted by $\nabla(K)$. The small inductive dimension ind X, $X\neq\phi$, is the smallest integer n > -1 such that for every $x\in X$ and every open set O containing x there is and open set V such that $x\in V\subset O$ and ind $b(V) \leq n - 1$. b(V) denotes the boundary of V. Taking a closed set in the above definition instead of x we get the definition of large inductive dimension Ind X of X. If $X = \phi$, ind X = -1 = Ind X.

Some properties of C-compact spaces

We start with elementary properties, some of them are known. As their proofs are straightforward we omit them.

1. A C-compact space is maximal compact and minimal Hausdorff.
2. Every H-set in a Hausdorff space is closed.
3. A C-compact space in which every H-set is closed is maximal C-compact.
4. If every closed subspace of C-compact space is C-compact then the space is compact .
5. C-compact space is functionally compact.
6. Completely regular functionally compact space is compact.

3- Characterization of C-compact spaces

The following theorem is easy to prove and needed further.

Theorem

The following are equivalent

(a) *X is C-compact*

(b) *For every closed subset K of X and every filterbase W on X such that FÇC¹ f is satisfied for every FÎW and regular closed set C containing K we have K Ç ad_q W ¹ f.*

(c) *For every closed set K of X and every open filterbase W on X and VÇC ¹ f is satisfied for every V*

Î W and regular closed set C containing K we have KÇ ad W ¹ f.

Theorem

The following are equivalent about spaces X and Y an u. s. c multifunction a : X®Y.

(a) X is C-compact.

(b) S(a, K) = ad a (G (K)) for each K closed in X

(c) S(a, K) is closed in Y for each K closed in X.

Proof (a) $\rightarrow$ (b) for each $x \in K$ we have $\alpha\Gamma(K) \subset \alpha(\Gamma(x))$.

Consequently $S(\alpha, K) = \cup\{ \text{ad}\ \alpha(\Gamma(x)) : x \in K\} \subset \text{ad}\ \alpha(\Gamma(K))$. Now let X be C-compact and K be closed in X. Then K is an H-set in X. Let $z \in \text{ad}\alpha(\Gamma(K))$. Let Δ be a local base at z. Then $z \in cl_Y\alpha cl_X(V)$ for each V open in X with $K \subset clV$. For $W \in \Delta$ we have $W \cap cl_Y\alpha cl_X(V) \neq f$ and since W is open in Y, we have $W \cap \alpha(clV) \neq f$.

Then $\alpha^{-1}(W) \cap cl\ V \neq f$. Thus by the above theorem we have $K \cap ad\ \alpha^{-1}(K) \neq f$ for each $x \in K \cap ad\alpha^{-1}(\Delta)$ we have $x \in K$ and $x \in ad\alpha^{-1}(\Delta)$. So, $x \in clV$ for each V open in X such that $K \subset clV$ and $x \in \alpha^{-1}(W)$ so that $clV \cap \alpha^{-1}(W) \neq f$. Consequently $\alpha(clV) \cap W \neq f$ for each $clV \in \Gamma(x)$, and $W \in \Delta$. Thus $z \in S(\alpha, x)$. So that $S(\alpha, K) \subset S(\alpha, x)$ for some $x \in K$. Thus $S(\alpha, K) = ad\alpha(\Gamma(K))$

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) let Ω be an open filter base on X such that $W \cap clV \neq f$ is satisfied for every V open in X and $K \subset clV$ with $W \in \Omega$. Let $y_o \notin X$ .Let $Y = X \cup \{y_o\}$. Define a topology on Y be taking A open in Y iff A is open in X or $y_o \in A$ and there exists $W \in \Omega$ such that $W \subset A$. Let $\alpha : X \rightarrow Y$ be identity function. Then α is continuous, and by hypothesis $S(\alpha, K)$ is closed in Y. So that $y_o \in S(\alpha, K)$. Thus $y_o \in S(\alpha, x)$ for some $x \in K$. For such an x we have $clV \cap (W \cup \{y_o\}) \neq f$, for every V open in X with $x \in clV$ and $W \in \Omega$. Thus $clV \cap W \neq \phi$ for every V open in X with $K \subset clV$ and $W \in \Omega$. So $K \cap ad\ \Omega \neq f$. Thus K is C-compact.

Corollary

X is C-compact iff a(K) is closed for every closed subset KÌX and a C-closed multifunction a:X®Y.

Proof If α is C-closed graph then $S(\alpha,K) = \alpha(K)$ and the result follows from the above theorem.

Theorem

X is C-compact iff every q-closed graph multifunction on X maps closed sets onto q-closed sets.

Proof If $\alpha: X \rightarrow Y$ and X is C-compact then it is not difficult to prove that $\alpha(K) = ad_\theta\alpha(\nabla(K))$ for every closed set K in X. Conversely suppose that X satisfies the condition in the statement of the theorem . Let Ω be a filterbase on X . Let $y_o \notin X$, and $Y = X \cup \{y_o\}$. Topologize Y by taking every subset of X open and a set containing y_o is open iff it contains a member of Ω. Let $\alpha: X \rightarrow Y$ be the θ-closure in Y of the identity function of X. Then $cl_\theta(X) = ad_\theta\alpha(\nabla(X))$ in Y. Thus $y_o \in \alpha(X)$. So there is $x \in X$ such that $\alpha(x) = \{x, y_0\}$. Then if $V \in \gamma(x)$ in X and $F \in \Omega$, we get $V \cap F \neq \phi$. So that by the above theorem we have $X \cap ad_\theta\Omega \neq \phi$. Consequently, X is C-compact.

Corollary

A Hausdorf space X is C-compact iff every q-closed graph multifunction on X maps q-closed sets onto q-closed sets.

Proof In a Hausdorff C-compact space every closed set is θ-closed.

The following is a characterization of quasi H-closed spaces which are C-compact.

Theorem

A quasi H-closed space X is C-compact iff the boundary of every closed (open) set is C-compact.

Proof If X is C-compact then the boundary of every closed is closed and consequently it is C-compact, conversely let X be quasi H-closed and K be closed in X . Let $\{V_\lambda\}$ be a cover of K by open sets in X. Then $\{V_\lambda\} \cup \{K^c\}$ is an open cover of X and so there is a finite subcollection $\left\{ V_{l_1}, V_{l_2}, \ldots, V_{l_n}, K^c \right\}$ the closures of its members cover X. On the other hand b(K) is C-compact and so there is a subcollection $\left\{ V_{l_{1+1}}, V_{l_{2+2}}, \ldots, V_{lm} \right\}$ of $\{V_\lambda\}$ the closures of its members cover b(K). Then $\left\{ V_{l_1}, V_{l_2}, \ldots, V_{l_m} \right\}$ is a subcollection of $\{V_\lambda\}$ the closures of its members cover K. Thus X is C-compact.

Application

We conclude this paper by the following application of C-compact spaces in dimension theory.

Theorem

A quasi H-closed space X such that the boundary of every closed set is quasi H-closed with ind X £ n (Ind X £ n) is C-compact .

Proof The proof is by induction on n. If n= -1 then X=ϕ and so it is C-compact. Suppose that the result is true for n-1. For x∈X (K closed in X) and G open in X with x∈G(K⊂G) there is an open set V such that x ∈V ⊂G (K⊂V⊂G) with
ind b(V)≤n1.(Ind b(v) ≤ n so ind b(V) is C-compact. Thus X is C-compact.

The above result cannot be weakened by dropping the condition that the boundary of every closed set is H-closed as the following example shows

Example Let

$$Y=\{(\frac{1}{n},\frac{1}{m}):n,m\in N\}\cup\{(\frac{1}{n},0):n\in N\}\subset R^2$$

, where R is the set of real numbers. Let $x_o\in R^2\setminus Y$ and $X=Y\cup\{x_o\}$. Topologize X by taking a subset U of X open if U ∩Y is open in Y (as a subspace of R^2) and if $x_o\in U$ then there exists r ∈N such that

$$U_r=\{(\frac{1}{n},\frac{1}{m}):n\geq r,m\in N\}\subset U.$$

Now

$$b(U_r)=\{(\frac{1}{n},0):n>r\},$$

is an infinite discrete space. So it is not H-closed. Also, ind X= Ind X = 1.

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عطاالله ثامر العاني

E-mail : atairaqi@yahoo.com

الخلاصة

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