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SOME RESULTS ON p –GROUPS

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ABSTRACT

In this paper , we define a certain subgroup ,denoted by $Z^*(G)$,as follows : $Z^*(G)=\{x \in Z(G): x^p = e\}$ of a finite group G , and we give some properties of $Z^*(G)$. Main result for $Z^*(G)$ is given in theorem 3.5 , which state that G is an elementary abelian $p-$ group if and only if $G = Z^*(G)$.

INTRODUCTION

It is interesting to use some information on the subgroups of a finite group G to dete-rmine the structure of the group G. The con-cept of the center of a group plays an impor-
rtant role in the theory of groups especially finite $p-$groups .

Definition 1.1 [2] :

The center , $Z(G)$, of a group G is the subset of elements in G that commute with every element of G. In symb- ols,

$Z(G)=\{x \in G : xy = yx$ for all y in $G\}$.

One of the first standard results, is that cent-er of a non-trivial finite $p-$group cannot be the trivial subgroup[1]. This forms the basis for many inductive methods in $p-$groups.

It is well known that a group G is abelian if and only if G is identical with its center[3] .

Definition 1.2 [4] :

Let G be a group and let $a,b \in G$ .Then $aba^{-1}b^{-1}$ is called a com- mutator of a and b. Let S denote the set of all commutators of G and let G' denote the subgroup of G generated by S then G' is called commutator subgroup of G.

The commutator subgroup G' is the smallest normal subgroup of G such that G/G' is abelian

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$o(G)$ means order of G is defined to be the number of its elements [2].

2. BASIC DEFINITIONS

Definition 2.1. Let G be a group, then a subgroup H of G is said to be a characte-
ristic subgroup [4] of G if $\alpha(H) \subseteq H$ for all automorphism α of G .

Definition 2.2. Let G be a finite $p-$group. Define $Z^*(G)=\{x \in Z(G): x^p = e\}$, where e is the identity of G.

Remark. The subgroup $Z^*(G)$ of a $p-$group G may or may not be identical with $Z(G)$ as the following two examples show that .

Examples 2.3. **(1)** A $p-$group G such that $Z^*(G) \neq Z(G)$.

Let $G = \langle x \rangle$ with $o(G)=8$. Since G is cyclic group , then $G = Z(G)$. Also $x^4 \in Z(G)$ and $(x^4)^2 = e$. We have $x^i \in Z(G)$ for $i = 1,2,3,4,5,6,7$.

But $(x^i)^2 \neq e$, for $i = 1,2,3,5,6,7$. Hence $x^i \notin Z^*(G)$ for $i = 1,2,3,5,6,7$. Therefore $Z^*(G)=\{e, x^4\} \neq Z(G)$.

(2) A $p-$group G such that $Z^*(G)=Z(G)$. Let $G = \{\langle x, y\rangle : x^4 = e, y^2 = e, (xy)^2 = e\}$ Then $Z(G)=\{e, x^2\}$, and $(x^2)^2 = e$. Hence $x^2 \in Z^*(G)$. Therefore $Z^*(G)=Z(G)$.

3. THEORMS

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Theorem 3.1. Let G be a finite $p-$group , then $Z^*(G) \neq \{e\}$.

Pr*oof* . It is obvious that $o(G) = p^n, n \geq 1$. We know that [5], $o(Z(G)) = p^r, 1 \leq r \leq n$. So $p \mid o(Z(G))$, and by Cauchy theorem[6], it fallows that $Z(G)$ contains an element $x \neq e$ of order p , i.e. $x^p = e$. Thus $e \neq x \in Z^*(G)$, which means that $Z^*(G) \neq \{e\}$.

Remark. Finiteness of G in the above theorem is necessary because there are infinite $p-$groups G with $Z^*(G) = \{e\}$.

Lemma . Let G be a finite group and let $\alpha \in Aut(G)$, then

$$o(x) = o(\alpha(x)), \forall x \in G$$

Proof. Since G is finite , then $\forall x \in G$, there is an integer n (depend on x) such that $x^n = e$.

But $(\alpha(x))^n = \alpha(x^n)$
$= \alpha(e)$
$= e$

Now , suppose that there is an integer $m < n$ such that

$(\alpha(x))^m = \alpha(x^m) = e$.

Then $\alpha(x^m) = \alpha(x^n)$. Since α is one-to-one , then $x^m = x^n = e$, so $o(x) = m$,which is a contradiction.Hence $o(\alpha(x)) = n$.

Theorem 3.2. Let G be a finite $p-$group . Then $Z^*(G)$ is a characteristic subgroup[4],

of G. Then

Proof. It is easy to show that $Z^*(G)$ is a normal subgroup of G.. Now let $\alpha \in Aut(G)$, then for every $z \in Z^*(G)$ we have

$$zx = xz, \forall x \in G.$$

So that ,
$\alpha(z)\,\alpha(x) = \alpha(x)\,\alpha(z), \forall \alpha \in Aut(G)$.

Since $z^p = e$, then (by lemma) we have $(\alpha(z))^p = e$.

Thus $\alpha(z) \in Z^*(G)$ which means that $Z^*(G)$ is a characteristic subgroup of G.

Corollary. For every finite $p-$group G **,** there is a natural homomorphism from

$$Aut(G) \text{ into } Aut(G/Z^*(G)) .$$

*Proof***.** Since $Z^*(G)$ is a characteristic subgroup of G , we can define

$\Theta : Aut(G) \to Aut(G/Z^*(G))$ by

$\Theta(\alpha(x\ Z^*(G))) = (\alpha(x))Z^*(G)$

It is easy to show that Θ is a homomor- rphism.

Remark. $Z^*(G)$ is not necessarily fully invariant [5] as shown in the following example. Let $G = \langle x, y, z, z^4 = y^2 = x^2, yx = x^{-1}y,$
$xz = zx, zy = yz \rangle$

It is clearly that $o(G) = 16$, $o(Z^*(G)) = 4$, by the fundamental theorem of finite abelian group [5] , it follows that

$Z^*(G)$ □□ ×□

Define $\alpha : G \to G$ by

$\alpha(x) = \alpha(y) = \alpha(x^3) = \alpha(xyz) = \alpha(yxz) = y$ and $\alpha(x^2) = \alpha(xy) = \alpha(xz) = \alpha(yx) = \alpha(z) = e$.

Then α is an endomorphism of G mapping $Z^*(G)$ into y which is not in $Z^*(G)$.

Theorem 3.3. Let $G_1, G_2, ..., G_n$ be finite $p-$groups . Then

$$Z^*(G_1 \times G_2 \times ... \times G_n) = Z^*(G_1) \times Z^*(G_2) \times ... \times Z^*(G_n)$$

$G = G_1 \times G_2 \times ... \times G_n$ *Proof* . Consider

$$Z(G) = Z(G_1) \times Z(G_2) \times ... \times Z(G_n)$$

(see[3, chapter 5, proposition 2])

Let $z \in Z^*(G)$, so $z = (z_1, z_2, ..., z_n)$, where $z_i \in G_i \quad \forall i, 1 \leq i \leq n$.

Therefore $z \in Z(G_1) \times Z(G_2) \times ... \times Z(G_n)$. By definition of $Z^*(G)$, we have $z^p == e$, consequently $z^p = (z_1^p, z_2^p, ..., z_n^p) = e$, which means that

$$z_i^p = e, \forall i\ 1 \leq i \leq n.$$

Therefore $z_i \in Z^*(G_i), \forall i\ 1 \le i \le n$. Thus

$Z^*(G) \subseteq Z^*(G_1) \times Z^*(G_2) \times ... \times Z^*(G_n) \ldots\ldots(1)$

Conversely suppose that $z_i \in Z^*(G_i), \forall i\ \ 1 \le i \le n$, then $z_i \in Z(G_i)$, and $z_i^p = e, \forall i\ 1 \le i \le n$.

Let $z = (z_1, z_2, ..., z_n) \in Z^*(G_1) \times Z^*(G_2) \times ... \times Z^*(G_n)$. Then $z \in Z(G_1) \times Z(G_2) \times ... \times Z(G_n)$, which means that $z \in Z(G)$.So

$z^p = (z_1, z_2, ..., z_n)^p = (z_1^p, z_2^p, ..., z_n^p) = e$. Thus $z \in Z^*(G)$ and

$$Z^*(G_1) \times Z^*(G_2) \times ... \times Z^*(G_n) \subseteq Z^*(G) ...(2)$$

From (1) and (2) we conclude

$$Z^*(G_1 \times G_2 \times ... \times G_n) = Z^*(G_1) \times Z^*(G_2) \times ... Z^*(G_n)$$, and this completes the proof.

It is clear that the commutator subgroup $(Z^*(G))'$ of $Z^*(G)$ is $\{e\}$ for every finite $p-$group.

Now we get the following theorem as criteria for G to be abelian.

Theorem 3.4 : Let G be a finite $p-$group , then G is abelian $p-$group if and only if $Z^*(G') = \{e\}$, where G' is the commutator subgroup of G.

Proof. The only if part is obvious . To prove the if part , suppose that $Z^*(G') = \{e\}$ and G is non abelian , then $G' \ne \{e\}$. But G is a finite $p-$group , so by theorem 3.1. it follows that $Z^*(G) \ne \{e\}$, which is contradiction . Then G is abelian. $Z^*(G)$ gives an indication about G to be an elementary abelian $p-$group.

Theorem 3.5. Let G be a finite $p-$group . Then $G = Z^*(G)$ if and only if G is an elementary abelian $p-$group.

Proof. If $G = Z^*(G)$, then $G = Z(G)$ which means that G is abelian. Also for each $x \in G$, we have $x \in Z^*(G)$ and so $x^p = e$. Thus G is abelian $p-$group.

Then $G = Z(G)$. Moreover , for each $x \in G$, we have $x \in Z(G)$ and so $x^p = e$. Thus $x \in Z^*(G)$. Hence $G \subseteq Z^*(G)$. Therefore $G = Z^*(G)$.This completes the proof.

Theorem 3.6: Let G be a finite $p-$group and $Z(G)$ is cyclic . Then $o(Z^*(G)) = p$.

Proof . Since G is a finite $p-$group , then $o(G) = p^n\ (n > 1)$ and $o(Z(G)) = p^r$, where $1 \le r \le n$. Then there are two cases :

Case(i) : $r = 1$, in this case $o(Z(G)) = p$, so $o(Z^*(G)) = p$.

Case (ii) : $r > 1$, since $Z(G)$ is cyclic , then $Z^*(G)$ is cyclic (see [8], lemma 2.1]). $Z^*(G_1) \times Z^*(G_2) \times ... \times Z^*(G_n)$ Suppose that $o(Z^*(G)) = p^i$, $1 < i \le r$, then there is $a \in Z^*(G)$ such that $a^p = e$ and $a^{p^i} = e$, where $p^i > p$, which is contradiction.

Therefore $i = 1$, and $o(Z^*(G)) = p$. This Completes the proof.

Corollary : Let G be a finite $p-$group , then $o(Z^*(G)) = p$.

Theorem 3.7 : Let $G = \langle a \rangle$ be a finite cyclic group of order p^n . Then

$$Z^*(G) = \{e, a^{p^{n-1}}, a^{2p^{n-1}}, ..., a^{(p-1)p^{n-1}}\}$$

Proof: We have $G = Z(G)$ and $a^{p^{n-1}} \in Z(G)$.

Then $(a^{p^{n-1}})^p = a^{p^n} = e$ which means that $a^{p^{n-1}} \in Z^*(G)$.

Similarly $a^{i\,p^{n-1}} \in Z^*(G), i = 0,1,..., p-1$. Now suppose that

$$a^{p^{n-r}} \in Z^*(G),\ 2 < r < n.$$

$$Z^*(G) = \{e, a^{p^{n-1}}, a^{2\,p^{n-1}}, . . . , a^{(p-1)\,p^{n-1}}\}$$

Then $(a^{p^{n-r}})^p = e$. i.e. $(a^{p^{n-r+1}}) = e$, which is a contradiction . Therefore and this completes the proof.

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بعض النتائج في الزمر– P

ياسين عبد الواحد الهيتي

الخلاصة

في بحثنا هذا تناولنا دراسة الزمر من نمط (P-group) المنتهية . حيث عرفنا زمرة جزئية جديدة للزمرة G، رمزنا لها بالرمز $Z^*(G)$، و أعطينا عددا من المبرهنات ألتي تحدد بعض خواص $Z^*(G)$. أهم ألنتائج هي (المبرهنة 3.6) حيث أثبتنا أن الزمرة G تكون ابيلية أوليا اذا و فقط اذا كان $G = Z^*(G)$.