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Some Properties related N-Functions with Young's Inequality

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Abstract

In this paper, we will summarize the necessary facts about a special class of convex functions called N-functions and introduce some theorems related N-functions and complementary N-functionswith Young's inequality.

Keyword:N-function, complementary, comparable, Young's inequality , composition.

1. Introduction and Background

The idea of N-function was introduced by Krasnoselskii and Rutickii in 1961 [3]. The class of convex functions play an important role in many branches of mathematics. One of these classes represented the class of N-functions.

We begin with recalling some basic concepts about N-functions and the complementary of N-functions.

***Definition 1.1*: [1]**

Let $f:[0,\infty) \longrightarrow [0,\infty)$ be a right continuous, monotone increasing function with

1. $f(0)=0$
2. $\lim_{t\to\infty} f(t)=\infty$;
3. $f(t)>0$ whenever $t>0$, then the function defined by

$$F(u)=\int_0^{|u|} f(t)dt$$

is called an N-function.

In that case if $f=F'_+$(f the right derivative of F), then f satisfies $f(0)=0$; $\lim_{t\to\infty} f(t)=\infty$; whenever $t>0$; and $F(u)=\int_0^{|u|} f(t)dt$.

The following proposition gives an alternative view of N-functions.

***Proposition 1.2*:[4]**

The function F is an N-function, if and only if, it is continuous, even and convex with

1. $\lim_{u\to 0} \frac{F(u)}{u}=0$;
2. $\lim_{u\to\infty} \frac{F(u)}{u}=\infty$;
3. $F(u)>0$ if$u>0$.

For example: $F(u)=u^2\sqrt{|u|}$ and $F(u)=|u|^{\alpha}(\ln|u|+1)$ are N-functions.[2]

***Proposition1.3:*[3]**

Any N-function $F:R \longrightarrow R$ is continuous from the right on zero.

***Remark 1.4*:[3]**

We denote by $F^{-1}(v)$ for $0 \leq v < \infty$ the inverse of the N-function $F(u)$ for $-\infty \leq u < 0$, note that $F^{-1}(v)$ is concave down and satisfies inequality

$$F^{-1}(a+b) \leq F^{-1}(a) + F^{-1}(b).$$

***Remark 1.5*:[1]**

Every N-function F is the composition of two another N-functions. That is, there are N-functions F_1, F_2 and so that $F = F_2 \circ F_1$

N-functions come in mutually complementary pairs. In fact, we have the following definition:

***Definition 1.6*: [1]**

For an N-function F define

$$G(v) = \int_0^{|v|} g(t)dt,$$

where g is the right inverse of the right derivative of F (see Figure 1).$G$is an N-function called the complement of F. Furthermore, it is plain that complement of G is F.

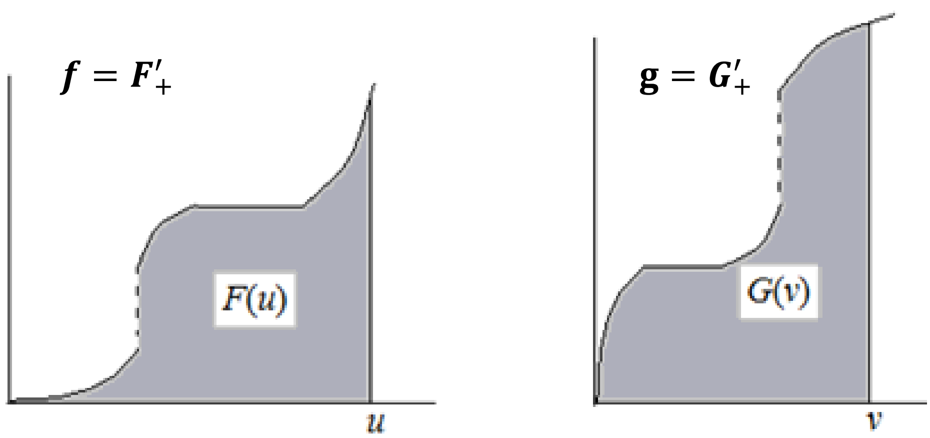
***Theorem (Young's Inequality) 1.7*:[1]**

If F and G are two mutually complementary N-functions, then

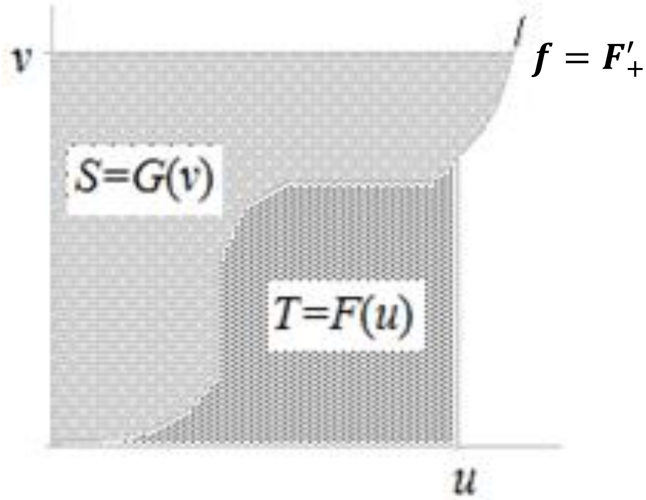
$$uv \leq F(u) + G(v) \quad \forall\, u, v \in R \quad \text{(see Figure 2)}$$

Consequently we have an alternative definition for an N-function F and complementary function G:

$$F(u) = \max_{v \geq 0}\{|u|v - G(v)\}$$
$$G(v) = \max_{u \geq 0}\{u|v| - F(u)\}$$



Figure(1): A pair of complementary N-functions



Figure(2): A geometric interpretation of Young's Inequality

2.*The Main Results*

In this section, we will study and debate some Theorems that related N-function and complementray of N-function with Young's inequality.
Firstly, we begin with the following definition:

Definition 2.1:

Let F_1, F_2 be N-functions, we say that F_1 comparable F_2(we write $F_1 \prec F_2$) if there exist positive constants$c$ and u_0 such that

$$\max_{v\geq 0}\{|u|v - G_1(v)\} \leq \max_{v\geq 0}\{c|u|v - G_2(v)\}.$$

Remark 2.2:

We say that theN-functions $F_1(u)$ and $F_2(u)$ are comparable if one of the relations $F_1 \prec F_2$ or $F_2 \prec F_1$holds.

If $F_1 \prec F_2$ and $F_2 \prec F_1$ then we say that F_1 and F_2 are equivalent and we write $F_1 \sim F_2$.

Theorem 2.3:

Suppose thatF_1, F_2 are N-functions with complements G_1 and G_2 respectively. Suppose that

$$\max_{v\geq 0}\{|u|v - G_1(v)\} \leq \max_{v\geq 0}\{|u|v - G_2(v)\} \text{ for } u \geq u_0.$$

Then

$$\max_{u\geq 0}\{u|v| - F_2(u)\} \leq \max_{u\geq 0}\{u|v| - F_1(u)\} \text{ for } v \geq v_0 = f_2(u_0) = F'_+(u_0).$$

Proof:

Let f_1, f_2, g_1 and g_2 be the right derivaties of $F_1, F_2, \mathrm{G}_1$and G_2 respectively.
Since $v \geq v_0 = f_2(u_0)$, then $\mathrm{g}_2(v) \geq v_0$. Note that (equality case of Young's inequality), we have

$$\mathrm{g}_2(v).v = \max_{v\geq 0}\{|\mathrm{g}_2(v)|.v - G_2(v)\} + \max_{u\geq 0}\{\mathrm{g}_2(v).|v| - F_2(\mathrm{g}_2(v))\} \quad \dots (1)$$

and also

$$g_2(v).v \le \max_{v\ge 0}\{|g_2(v)|.v - G_1(v)\} + \max_{u\ge 0}\{g_2(v).|v| - F_1(g_2(v))\}. \quad ...(2)$$

From (1) and (2), we have

$$\max_{v\ge 0}\{|g_2(v)|.v - G_2(v)\} + \max_{u\ge 0}\{g_2(v).|v| - F_2(g_2(v))\}$$
$$\le \max_{v\ge 0}\{|g_2(v)|.v - G_1(v)\} + \max_{u\ge 0}\{g_2(v).|v| - F_1(g_2(v))\}$$

But $\max_{v\ge 0}\{|u|v - G_1(v)\} \le \max_{v\ge 0}\{|u|v - G_2(v)\}$. Therefore

$$\max_{u\ge 0}\{u|v| - F_2(u)\} \le \max_{u\ge 0}\{u|v| - F_1(u)\}. \qquad \blacksquare$$

<u>Theorem 2.4:</u>

Suppose that $F_1(u)$ and $F_2(u)$ are N-functions, if

$$\max_{v\ge 0}\{|u|v - G_1(v)\} \prec \max_{v\ge 0}\{|u|v - G_2(v)\}$$

Then

$$\max_{u\ge 0}\{u|v| - F_2(u)\} \prec \max_{u\ge 0}\{u|v| - F_1(u)\}.$$

Proof:

By (Definition 2.1), there is $c > 0, u_0 > 0$ such that

$$\max_{v\ge 0}\{|u|v - G_1(v)\} \le \max_{v\ge 0}\{c|u|v - G_2(v)\} \quad ...(3)$$

Let

$$\max_{v\ge 0}\{|u|v - G(v)\} \le \max_{v\ge 0}\{c|u|v - G_2(v)\}.$$

The function

$$\max_{u\ge 0}\{u|v| - F(u)\} \le \max_{u\ge 0}\left\{u\frac{|v|}{c} - F_2(u)\right\}.$$

Inequality (3) can be rewritten in the form

$$\max_{v\ge 0}\{|u|v - G_1(v)\} \le \max_{v\ge 0}\{|u|v - G(v)\}.$$

From (Theorem 2.3), we have

$$\max_{u\ge 0}\{u|v| - F(u)\} \le \max_{u\ge 0}\{u|v| - F_1(u)\}.$$

It follows that $\max_{u\ge 0}\{u|v| - F_2(u)\} \le \max_{u\ge 0}\{cu|v| - F_1(u)\}$. Therefore

$$\max_{u\ge 0}\{u|v| - F_2(u)\} \prec \max_{u\ge 0}\{u|v| - F_1(u)\}. \blacksquare$$

<u>Corollary 2.5:</u>

Suppose that the N-function F_1 and F_2 are equivalent, i.e. $\max_{v\ge 0}\{|u|v - G_1(v)\} \sim \max_{v\ge 0}\{|u|v - G_2(v)\}$, then the complementary N-functions G_1 and G_2 are equivalent, i.e. $\max_{u\ge 0}\{u|v| - F_1(u)\} \sim \max_{u\ge 0}\{u|v| - F_2(u)\}$.

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بعض الخصائص المتعلقة بالدوال-N مع متباينة يونك

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الخلاصة :

في هذا البحث سوف نوجز بعض الحقائق الضرورية حول صف خاص من الدوال المحدبة والتي تسمى بالدوال-N وتقديم بعض المبرهنات المتعلقة بها وبالدوال المكملة-N مع متباينة يونك.

الكلمات المفتاحية: الدالة-N ، مكملة الدالة-N ، المقارنة، متباينة يونك، التركيب.