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On Bornivorous Set

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Abstract:

In this paper, we introduce the concept of the bornivorous set and its properties to construct bornological topological space .Also, we introduce and study the properties related to this concepts like bornological base, bornological subbase , bornological closure set, bornological interior set, bornological frontier set and bornological subspace .

Key words: bornivorous set, bornological topological space,b-open set

1.Introduction

The space of entire functions over the complex field C was introduced byPatwardhanwho defined a metric on this space by introducing a real-valued map on it[6]. In(1971), H.Hogbe-Nlend introduced the concepts of bornology on a set [3].Many workers such as Dierolf and Domanski, Jan Haluska and others had studied various bornological properties[2]. In this paper at the second section ,bornivorous set has been introduced with some related concepts. While in the third section a new space "Bornological topological space" has been defined and created in the base of bornivorous set . The bornological topological space also has been explored and its properties .The study also extended to the concepts of the bornological base and bornological subbase of bornological topological space .In the last section a new concepts like bornological closure set, bornological drived set, bornological dense set, bornological interior set, bornological exterior set, bornological frontier set and bornological topological subspace, have been studied with supplementary properties and results which related to them.

Definition1.1[3]

Let A and B be two subsets of a vector space E. We say that:

i. A is circled if $\lambda A \subset A$ whenever $\lambda \in K$ and $|\lambda| \leq 1$.
ii. A is convex if $\lambda A + \mu A \subset A$ whenever λ and μ are positive real numbers such that $\lambda + \mu = 1$.
iii. A is disked, or a disk, if A is both convex and circled.
iv. A absorbs B if there exists $\alpha \in R$, $\alpha > 0$, such that $\lambda A \subset B$ whenever $\alpha \leq |\lambda|$.
v. A is an absorbent in E if A absorbs every subset of E consisting of a single point.

Remark 1.2[3]

(i) If A and B are convex and $\lambda, \mu \in K$, then $\lambda A+\mu B$ is convex.
(ii) Every intersection of circled (respectively convex, disked) sets is circled (respectively convex, disked).
(iii)Let E and F be vector spaces and let u : E →F be a linear map. Then the image, direct or inverse, under u of a circled (respectively convex , disked) subset is circled (respectively convex, disked).

Definition 1.3[7]

Let X be a topological space and let $x \in X$. A subset $N \subseteq X$ is said to be a neighborhood of x iff there exists open set G such that $x \in G \subseteq N$. The collection of all neighborhoods of $x \in X$ is called the neighborhood system at x and shall be denoted by N_x .

Remark 1.4[7]

Let X be topological vector space then
(i) Every topological vector space X has a local base of absorbent and circled sets.
(ii) If A is circled in X , $x \in$cl (A) and $0 \leq |\lambda| \leq 1$, then $\lambda x \in$cl (A).

Remark 1.5[7]

A subset A is bounded iff it is absorbed by any neighborhood of the origin.

Definition 1.6[7]

A metrizable space is a topological space X with the property that there exists at least one metric on the set X whose class of generated open sets is precisely the given topology.

Definition 1.7[1]

A bornology on a set X is a family ß of subsets of X satisfying the following axioms:
(i) ß is a covering of X , i.e. $X = \bigcup_{B \in \beta} B$;
(ii) ß is hereditary under inclusion i.e. if A∈ ß and B is a subset of X contained in A, then B ∈ ß;
(iii)ß is stable under finite union.
A pair (X , ß) consisting of a set X and a bornology ß on X is called a bornological space, and the elements of ß are called the bounded subsets of X .

Example 1.8[3]

The Von–Neumann bornology of a topological vector space, Let E be a topological vector space. The collection ß ={A⊆E: A is a bounded subset of a topological vector space E}forms a vector bornology on E called the Von–Neumann bornology of E. Let us verify that ß is indeed a vector bornology on E, if $ß_0$ is a base of circled neighborhoods of zero in E, it is clear that a subset A of E is bounded if and only if for every B ∈ $ß_0$ there exists $\lambda>0$ such that $A \subseteq \lambda B$. Since every neighborhood of zero is absorbent, ß is a covering of E. ß is obviously hereditary and we shall show that its also stable under vector addition. Let A_1, $A_2 \in$ ß and $B_0 \in ß_0$; there exists B'_0 such that $B'_0 + B'_0 \subseteq B_0$ proposition (1.2.11). Since A_1 and A_2 are bounded in E, there exists positive scalars λ and μ such that $A_1 \subseteq \lambda B'_0$ and $A_2 \subseteq \mu B'_0$. with α= max(λ, μ), we have:

$A_1 + A_2 \subseteq \lambda B'_0 + \mu B'_0 \subset \alpha B'_0 + \alpha B'_0 \subset \alpha(B'_0 + B'_0) \subset \alpha B_0$.

Finally, since $ß_0$ is stable under the formation of circled hulls (resp. under homothetic transformations). Then so is ß, and we conclude that ß is a vector bornology on E. If E is locally convex, then clearly ß is a convex bornology. Moreover, since every topological vector space has a base of closed neighborhoods of 0, the closure of each bounded subset of E is again bounded.

Definition 1.9[6]

Let X and Y be two bornological spaces and u: $X \to$ Y is a map of X into Y. We say that u is a bounded map if the image under u of every bounded subset of X is bounded in Y i.e. $u(A) \in ß_y$, $\forall A \in ß_x$. Obviously the identity map of any bornological space is bounded.

Definition 1.10[3]

Let E be a bornological vector space. A sequence $\{x_n\}$ in E is said to be converge bornologically to 0 if there exists a circled bounded subset B of E and a sequence $\{\lambda_n\}$ of scalars tending to 0, such that $x_n \in B$ and $x_n \in \lambda_n B$, for every integer $n \in N$.

2. Bornivorous Set

In this section, we introduce the definition of bornivorous set and study the properties of it.

Definition 2.1:

A sub set A of a bornonological vector space E is called bornivorous if it absorbs every bounded subset of E

Proposition 2.2:

Let E be a topological vector space with Von Neumann bornology, then every neighbourhood of 0 is bornivorous set.

Proof:

Let A be neighbourhood of 0since E is a topological vector space then every bounded subset of E is absorbed by each neighbourhood of 0, i.e each neighbourghood of 0 absorbed every bound set of E.Then neighbourghood of 0 is bornivorous set.

Proposition 2.3:

Let E be a metrizable topological vector space then a circled set that absorbs every sequence converging to 0 is a neighbourhood of 0 and every bornivorous subset of E is a neighbourhood of 0.

Proof:

Let A be a circled set absorbs every sequence converging topologically to 0.

Since E is a materizable topological vector space then every topologically convergent sequence is bornologically convergent.Then A is a circled set absorbs every sequence converging bornologically to 0. Since every neighbourhood of 0 absorbed every bounded set of E.By definition 1.1 0 thus A is a neighbourhood of 0.

Let B be bornivorous subset of E by definition 2.1 B absorbs every bounded subset of E.E is a metrizable topological space then every topologically convergent sequence is bornologically convergent.Since each neighbourhood of 0 absorbed every bounded set of E. Then B is neighbourhood of 0.Then every bornivorous subset of E is a neighbourhood of 0.

Proposition 2.4:

Let E be a bornological vector space then:

i. Every bornivorous set contains 0.
ii. Every finite intersection of bornivorous set is bornivorous.
iii. If B is bornivorous and $B \subset A$ then A is bornivorous .

Proof:

i. It is clear from proposition 2.2.
ii. Let B_i is bornivorous set for every i=1,…,n the set B_i absorbs every bounded subset of E $\forall i = 1, \dots, n$then $\bigcap_{i=1}^{n} B_i$ absorbs every bounded subset of E then$\bigcap_{i=1}^{n} B_i$ is bornivorous set.
iii. Let B is bornivorous set then B absorbs every bounded subset of E since $B \subset A$ then absorbs every bounded subset of E, thus A is bornivorous set.

Proposition 2.5:

Let E,F be bornological vector space and let $u: E \to F$ be a bounded linear mab then the inverse image of a bornivorous subsets of F is bornivorous in E.

Proof: clear

3.Bornological Topological Space

In this section we define bornological open set depended on bornivorous set and construct bornological topological space and investigate the properties concerning with them.

Definition 3.1:

Let E be a bornonological vector space , subset A of E is called bornological open (for brief b-open) set, if the set $A - \{a\}$ is bornivorous for every $a \in$A, .The complement of bornological open set is called bornological closed (for brief b-closed) set.

Proposition 3.2:

i. The finite intersection of b-open sets is b-open set.
ii. The union of b-open sets is b-open set.

Proof:

(i)Let $A_1, A_2, \dots, A_n$ be b-open sets then by definition 3.1 the set $A_i - \{a_{ij}\} \forall a_j \in A_i$, $i = 1, \dots, n$ (such that j is counter of elements of A_i) is bornivorous set and then $\cap (A_i - \{a_{ij}\}) = A_1 - \{a_{j1}\} \cap A_2 - \{a_{j2}\} \cap \dots A_n - \{a_{jn}\}$.Since $A_i - \{a_{ji}\}$ is bornivorous set then the finite intersection of bornivorous set is bornivorous set and then $\cap (A_i - \{a_{ij}\}$ is bornivorous .Thus $\cap A_i$ is b-open set.

(ii).Let Aλ be b-open set such that $\lambda \in \Lambda$ then $A_\lambda - \{a_{\lambda i}\}$ is bornivorous $\forall a_{\lambda i} \in A_\lambda$ since$A_\lambda - \{a_{\lambda i}\} \subseteq \cup(A_\lambda - \{a_{\lambda i}\})$,$\forall \lambda \in \Lambda$, and λi $\in A_\lambda$ then$\cup(A_\lambda - \{a_{\lambda i}\})$ is bornivorous set. Thus $\bigcup_{\lambda \in \Lambda} A_\lambda$ is bornivorous set.

Remark 3.3:

As a consequence of the proposition 3.2 the family of all bornological open(b-open) sets define a topology on E called the bornological topology denoted by b-topology and the pair (E,T) called bornological topological space(for brief b-topological space).

Proposition 3.4:

(i)The finite union of b-closed sets of b-topological space E is b-closed set.

(ii)The intersection of b-closed sets of b-topological space E is b-closed set.

Proof:

i. Let $F_1, F_2, \dots, F_n$b-closed sets then $F_1^c, F_2^c, \dots, F_n^c$ are b-open set, then $F_1^c \cap F_2^c \cap \dots \cap F_n^c$is b-open set (proposition 3.2 (i))

$\Rightarrow (F_1 \cup F_2 \cup \dots \cup F_n)^c$is b-open set

$\Rightarrow (\bigcup_{i=1}^{n} F_i)^c$is b-open set

$\Rightarrow \bigcup_{i=1}^{n} F_i$ is b-closed set.

ii. Clear

Definition 3.5:

Let (E,T) be a bornological topological space a collection U of subsets of E is said to form a bornological base denoted by(b- base) for b-topology T if ,

1. $B \subset T$.
2. For each point $x \in E$ and each b-open set A of x there exist some $B \in U$ such that $x \in B \subset A$.

Proposition 3.6:

Let (E,T) be a b-topological space a sub collection U of T is b-base for b-topology T if every b-open set can be expressed as the union of members of U.

Proof:clear

Proposition 3.7:

Let(E,T) be a b- topological space and U be a b-base for T , then U has the following properties :

1. For every $x \in E$, there exists a $B \in U$ such that $x \in B$ that is $E = \cup\{B: B \in U\}$.
2. For every $B_1 \in U, B_2 \in U$ and every point $x \in B_1 \cap B_2$there exists a $B \in U$ such that $\in B \subset B_1 \cap B_2$, that is the intersection of any two members of B is a union of members of U.

Proof: Clear

Definition3.8 :

Let (E,T) be a bornological topological space a collection U_* of subsets of E is called bornological sub base denoted by (b-subbase) for the b-topology T if $U_* \subset T$ and finite intersection of member of U_* form a b-base of b-topology T .

Proposition 3.9:

Let (E,T) be a b- topological space and U_* be a collection of subsets of T then U_* is a b-subbase of T if the finite intersection of members of U_* form a b- base for b-topology T .

Proof: Clear

4.Certain Types of Bornological Sets

In this section we define several notions of bornological topological space such as bornological closure set, bornological dense set, bornological interior set, bornological exterior set , bornological frontier set and finally bornological topological subspace. We study several properties of them.

Definition4.1:

Let E be a bornological topological space and $A \subset E$ the intersection of all b-closed subsets of E containing A is called bornological closure set (b-closure set) denoted by $b-\bar{A}$.

Definition4.2:

Let E be a bornological topological space and $A \subset E$. A point $x \in E$ is called bornological limit point of A, if every b-open set contains x contain point of A other than x. The set of all bornological limit points of A is called bornological drived set and denoted by b-D(A)

Proposition 4.3:

Let A be a subset of a b- topological space E , then

i. $b-\bar{A}$ is the smallest b-closed set containing A .
ii. A is b-closed set if and only if $b-\bar{A}=A$.

Proof:

i. This following from definition 4.1.
ii. If A is b-closed , then A itself is the smallest b-closed set containing A and hence $b-\bar{A}=A$ .Conversely if $b-\bar{A}=A$ by (i) $b-\bar{A}$ is b-closed and then A is also b-closed.

Proposition 4.4:

Let E be a b- topological space and A, B be any subsets of E then

i . $A \subset b-\bar{A}$.
ii. If $A \subset B \Rightarrow b-\bar{A} \subset b-\bar{B}$.
iii. $b-\overline{(A \cup B)}= b-\bar{A} \cup b-\bar{B}$.
iv. $b-\overline{(A \cap B)} \subset b-\bar{A} \cap b-\bar{B}$.
v. $b-\bar{\bar{A}}=b-\bar{A}$.

Proof:

i. By proposition 3.2 (i) $b-\bar{A}$ is the smallest b-closed set containing A and so, $A \subset b-\bar{A}$.
ii. By (i) $B \subset b-\bar{B}$. Since $A \subset B$, we have $A \subset b-\bar{B}$. But $b-\bar{B}$ is a b-closed set. Thus $b-\bar{B}$ is a b-closed set containing A, since $b-\bar{A}$ is the smallest b-closed set containing A,we have $b-\bar{A} \subset b-\bar{B}$.
iii. Since $A \subset A \cup B$and $B \subset A \cup B$, then $b-\bar{A} \subset b-\overline{(A \cup B)}$ and $b-\bar{B} \subset b-\overline{(A \cup B)}$ by (ii).Hence $b-\bar{A} \cup b-\bar{B} \subset b-\overline{(A \cup B)}$ …(1).Since $b-\bar{A}$ and $b-\bar{B}$

are b-closed sets ,$b-\bar{A} \cup b-\bar{B}$ is also b-closed set . Also $A \subset b-\bar{A}$ and $B \subset b-\bar{B}$ then $A \cup B \subset b-\bar{A} \cup b-\bar{B}$ thus $b-\bar{A} \cup b-\bar{B}$ is a b-closed set containing $A \cup B$ since $b-\overline{(A \cup B)}$ is the smallest b-closed set containing $A \cup B$, we have $b-\overline{(A \cup B)} \subset b-\bar{A} \cup b-\bar{B}$…(2).From (1) and (2) we have $b-\overline{(A \cup B)} = b-\bar{A} \cup b-\bar{B}$.

iv. $A \cap B \subset A \Rightarrow b-\overline{(A \cap B)} \subset b-\bar{A}andA \cap B \subset B \Rightarrow b-\overline{(A \cap B)} \subset b-\bar{B}$. Hence $b-\overline{(A \cap B)} \subset b-\bar{A} \cap b-\bar{B}$.

v. Since $b-\bar{A}$ is b-closed set we have by proposition4.3(ii)$b-\bar{\bar{A}} = b-\bar{A}$.

vi.

Definition4.5 :

Let E be a bornological topological space and $A \subset E$, then :
A is said to be bornological dense (b-dense) in E if $b-\bar{A} = E$.

Definition4.6 :

Let E be a bornological topological space and $A \subset E$. A point $x \in E$ is said to be bornological interior point of A if there exists b-open set A_1 such that $x \in A_1 \subset A$. The set of all bornological interior points of A is called bornological interior of A and denoted by b-Int(A).

Proposition 4.7:

Let E be b-topologicalspace and $A \subset E$ then $b-Int(A) = \cup \{A_1 : A_1 is\ b-openset, A_1 \subset A\}$.

Proof:

$x \in b-Int(A) \Leftrightarrow$ there exists a b-open set A_1 such that $x \in A_1 \subset A$
$\Leftrightarrow \cup \{A_1 : A_1 isb-openset, A_1 \subset A\}$ hence $b-Int(A) = \cup \{A_1 : A_1 isb-openset, A_1 \subset A\}$.

Proposition 4.8:

Let E be a b- topological space and let $A \subset E$, then;

i. b-Int(A) is a b-open set.
ii. b-Int(A) is largest b-open set contained in A .
iii. A is b-open if and only if $b-Int(A) = A$.
iv.

Proof:

i. Let x be any point of b-Int(A) , then x is bornological interior point of A hence by definition 4.6 then there exists a b-open set A_1 such that $x \in A_1 \subset A$. Since A_1 b-open , then every point of A is a bornological interior point of A so that $A \subset \mathrm{b-Int(A)}$ and consequently then b-Int(A) is b-open set.

ii. Let A_1 be any b-open subset of A and let $x \in A_1$ so that $x \in A_1 \subset A$. Since A_1 is b-open then $x \in \mathrm{b-Int(A)}$ thus we have $x \in A_1$ then $x \in \mathrm{b-Int(A)}$ and so $A_1 \subset \mathrm{b-Int(A)} \subset \mathrm{A}$ hence b-Int(A) contains every b-open subset of A and it is therefore the largest b-open subset of A.

iii. Let A=b-Int(A) by (i) b-Int(A) is b-open set and therefore A also b-open set,conversely let A is b-open then A surely identical with the largest b-open subset A but by (ii) b-Int(A) is the largest b-open subset of A hence A=b-Int(A).

Proposition 4.9:

Let E be a b- topological space and A, B are any subset of E , then

i. $\mathrm{b}-\mathrm{Int}(\mathrm{A}) \subset A$.
ii. If $A \subset B \Rightarrow \mathrm{b}-\mathrm{Int}(\mathrm{A}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{B})$.
iii. $\mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B}) = \mathrm{b}-\mathrm{Int}(\mathrm{A}) \cap \mathrm{b}-\mathrm{Int}(\mathrm{B})$.
iv. $\mathrm{b}-\mathrm{Int}(\mathrm{A}) \cup \mathrm{b}-\mathrm{Int}(\mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A} \cup \mathrm{B})$.
v. $\mathrm{b}-\mathrm{Int}\left(\mathrm{b}-\mathrm{Int}(\mathrm{A})\right) = \mathrm{b}-\mathrm{Int}(\mathrm{A})$.

Proof:

i. Let $x \in \mathrm{b}-\mathrm{Int}(\mathrm{A}) \Rightarrow$ x is a b-interior point of A $\Rightarrow x \in$ A then $\mathrm{b}-\mathrm{Int}(\mathrm{A}) \subset A$.
ii. Let $x \in \mathrm{b}-\mathrm{Int}(\mathrm{A}) \Rightarrow$ x is a b-interior point of A so A is b-open subset containing x since $A \subset B$,then B is b-open set containing x then $x \in \mathrm{b}-\mathrm{Int}(\mathrm{B}) \Rightarrow$ $\mathrm{Int}(\mathrm{A}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{B})$.
iii. since $\mathrm{A} \cap \mathrm{B} \subset A$ and $\mathrm{A} \cap \mathrm{B} \subset B$ we have by (ii) $\mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A})$ and $\mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{B})$ then $\mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A}) \cap \mathrm{b}-$ $\mathrm{Int}(\mathrm{B})$...(1)
Let $x \in (\mathrm{b}-\mathrm{Int}(\mathrm{A}) \cap \mathrm{b}-\mathrm{Int}(\mathrm{B})) \Rightarrow x \in \mathrm{b}-\mathrm{Int}(\mathrm{A})$ and $x \in \mathrm{b}-\mathrm{Int}(\mathrm{B})$ hence x is a b-interior point of each of the sets A and B then exists an b-open set containing x that there intersection $\mathrm{A} \cap \mathrm{B}$ is also b-open set containing x, hence $x \in \mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B})$ then $\mathrm{b}-\mathrm{Int}(\mathrm{A}) \cap \mathrm{b}-\mathrm{Int}(\mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B})$…(2).From (1) and (2) we get $\mathrm{b}-\mathrm{Int}(\mathrm{A} \cap \mathrm{B}) = \mathrm{b}-\mathrm{Int}(\mathrm{A}) \cap \mathrm{b}-\mathrm{Int}(\mathrm{B})$.
iv. By(ii) $\mathrm{A} \subset \mathrm{A} \cup \mathrm{B} \Rightarrow \mathrm{b}-\mathrm{Int}(\mathrm{A}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A} \cup \mathrm{B})$.
$$\mathrm{B} \subset \mathrm{A} \cup \mathrm{B} \Rightarrow \mathrm{b}-\mathrm{Int}(\mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A} \cup \mathrm{B})$$
Hence$\mathrm{b}-\mathrm{Int}(\mathrm{A}) \cup \mathrm{b}-\mathrm{Int}(\mathrm{B}) \subset \mathrm{b}-\mathrm{Int}(\mathrm{A} \cup \mathrm{B})$.
v. By (i) of proposition 4.8 $\mathrm{b}-\mathrm{Int}(\mathrm{A})$ is a b-open set .Hence by (iii) proposition 4.8 $\mathrm{b}-\mathrm{Int}\left(\mathrm{b}-\mathrm{Int}(\mathrm{A})\right) = \mathrm{b}-\mathrm{Int}(\mathrm{A})$.

Definition 4.10:

Let A be a subset of bornological topological space E. A point $x \in X$ is said to be a bornological exterior point of A if it is bornological interior point of the complement of A, that is if there exists an b-open set such that $x \in B \subset A^{C}$. Or $x \in B$and $B \cap A = \phi$. The set of all bornological exterior point of A is called the bornological exterior of A and is denoted by b-ext(A).

Definition 4.11:

A point x of bornological topological space E is said to be a bornological frontier point of a subset A of E if it is neither a bornological interior nor a bornological exterior point of A .The set of all bornological frontier point of A is called the bornological frontier of A and denoted by b-Fr(A).

Proposition 4.12:

Let A be any sub set of b-topological space E then $\mathrm{b-Fr(A)} = (\mathrm{b-Int(A)} \cup b-ext(A))^C$ and b-Fr(A) is a b-closed set .

Proof:

By definition 4.10 b-ext(A) $= \mathrm{b-Int(A^C)}$, also $\mathrm{b-Int(A)} \subset A$ and $\mathrm{b-Int(A^C)} \subset \mathrm{A}^C$.Since $A \cap \mathrm{A}^C = \phi$. It follows $\mathrm{b-Int(A)} \cap \mathrm{b-ext(A)} = \mathrm{b-Int(A)} \cap b-Int(\mathrm{A}^C) = \phi$. Again by definition 4.11, we have:

$x \in \mathrm{b-Fr(A)} \Leftrightarrow x \notin \mathrm{b-Int(A)}andx \notin \mathrm{b-ext(A)}$

$$\Leftrightarrow x \notin (\mathrm{b-Int(A)} \cup \mathrm{b-ext(A)})$$

$$\Leftrightarrow x \in (\mathrm{b-Int(A)} \cup \mathrm{b-ext(A)})^C$$

b-Fr(A)=$(\mathrm{b-Int(A)} \cup \mathrm{b-ext(A)})^C$…(1), since b-Int(A) and b-ext(A) are b-open sets from (1) we have b-Fr(A) is b-closed set.

Proposition 4.13:

Let E be b- topological space and let $A \subset E$ then :

i. $\mathrm{b-Int(A)} = \left(b-(\overline{A^C})\right)^C$.

ii. $b-(\overline{A^C}) = (\mathrm{b-Int(A)})^C$.

iii. $b-\bar{A} = (\mathrm{Int(A^C)})^C$.

Proof : Clear

Proposition 4.14:

Let E be b-topological space and let A, B be subsets of E ,then :

i. $\mathrm{b-Fr(A)} = \mathrm{b}-\bar{A} \cap b-(\overline{A^C}) = (b-\overline{A}) - (\mathrm{b-Int(A)})$.

ii. $\mathrm{b-Int(A)} = A - (\mathrm{b-Fr(A)})$.

iii. $(\mathrm{b-Fr(A)})^C = \mathrm{b-Int(A)} \cup b-\mathrm{Int(A^C)}$.

iv. $\mathrm{b-Fr}(\mathrm{b-Int(A)}) \subset \mathrm{b-Fr(A)}$.

v. $\mathrm{b-Fr}(\mathrm{b}-\bar{A}) \subset \mathrm{b-Fr(A)}$.

vi. $\mathrm{b-Fr(A \cup B)} \subset \mathrm{b-Fr(A)} \cup \mathrm{b-Fr(B)}$.

vii. $\mathrm{b-Fr(A \cap B)} \subset \mathrm{b-Fr(A)} \cup \mathrm{b-Fr(B)}$

Proof:

i. We have $\mathrm{b-Fr(A)} = \left(\mathrm{b-Int(A)} \cup b-ext(A)\right)^C$

$$= \left(b-\mathrm{Int(A)}\right)^C \cap \left(b-ext(A)\right)^C$$

$$= ((b-(\overline{A^C}))^C)^C \cap ((b-\bar{A})^C)^C$$

$$= b-\overline{A^C} \cap b-(\bar{A})$$

Now $b-(\bar{A}) \cap b-(\overline{A^C}) = b-(\bar{A}) - (b-(\overline{A^C}))^C$

$= \mathrm{A} - (\mathrm{b-Int(A)})$by proposition 4.13

Hence $\mathrm{b-Fr(A)} = \mathrm{b}-\bar{A} \cap b-(\overline{A^C}) = b-\bar{A} - (\mathrm{b-Int(A)})$

ii. $\mathrm{A} - (\mathrm{b-Fr(A)}) = \mathrm{A} - (\mathrm{b}-\bar{A} - (\mathrm{b-Int(A)})) = \mathrm{b-Int(A)}$ by (1)

iii. We have $(\mathrm{b-Fr(A)})^C = (\mathrm{b}-\bar{A} \cap b-(\overline{A^C}))$ by (1)

$$(b-\bar{A})^C \cup (b-(\overline{A^C}))^C$$

By proposition 4.13$(b-(\overline{A^C}))^C = \mathrm{b-Int(A)}$ and so

$$b-Int(A^C)=\left(b-\left((\overline{A^C})^C\right)\right)^C=(b-(\overline{A}))^C$$

then $\mathrm{b-Fr(A)}=b-\mathrm{Int}(\mathrm{A}^C)\cup \mathrm{b-Int(A)}=\mathrm{b-Int(A)}\cup b-\mathrm{Int}(\mathrm{A}^C)$

iv. $\mathrm{b-Fr}\big(\mathrm{b-Int(A)}\big)=\mathrm{b-}\overline{(\mathrm{b-Int(A)})}\cap b-\overline{((\mathrm{b-Int(A)})^C)}$ by (1)

$$=\mathrm{b-}\overline{(\mathrm{b-Int(A)})}\cap b-\overline{((b-(\overline{A^C}))^{CC})}$$

$=\mathrm{b-}\overline{(\mathrm{b-Int(A)})}\cap b-\overline{(\overline{A^C})}\subset b-(\bar{A})\cap b-(\overline{A^C})=b-Fr(A)$by (1)

Hence $b-\mathrm{Fr(b-Int(A))}\subset b-\mathrm{Fr(A)}$

v. $\mathrm{b-Fr(b-\overline{A})}=b-\overline{(\mathrm{b-\overline{A}})}\cap b-\overline{((\mathrm{b-\overline{A}})^C)}$by (1)

$$=(\mathrm{b-\overline{A}})\cap b-\overline{(\mathrm{b-\overline{A}})^C}$$

Now $A\subset(\mathrm{b-\overline{A}})\Rightarrow(\mathrm{b-\overline{A}})^C\subset A^C\Rightarrow b-\overline{(\mathrm{b-\overline{A}})^C})\subset\big(\mathrm{b-}\overline{A^C}\big)$

Hence $\mathrm{b-Fr(b-}\bar{A})\subset \mathrm{b-}\bar{A}\cap \mathrm{b-}\overline{A^C}=b-Fr(A)$

vi. $\mathrm{b-Fr(A\cup B)}=b-(\overline{\mathrm{A\cup B}})\cap b-((\overline{\mathrm{A\cup B}})^C)$

$=(\mathrm{b-\overline{A}}\cup b-\bar{B})\cap(b-\big(\overline{A^C\cap B^C}\big))\subset \mathrm{b-\overline{A}}\cup b-\bar{B}\cap(\mathrm{b-}\overline{A^C}\cap \mathrm{b-}\overline{B^C})$

$$=\left(\mathrm{b-\overline{A}}\cap\left(\mathrm{b-}\overline{A^C}\cap\mathrm{b-}\overline{B^C}\right)\right)\cup\left(b-\bar{B}\cap\left(\mathrm{b-}\overline{B^C}\cap\mathrm{b-}\overline{A^C}\right)\right)$$

$$=((\mathrm{b-\overline{A}}\cap\mathrm{b-}\overline{A^C})\cap\mathrm{b-}\overline{B^C})\cup((b-\bar{B}\cap\mathrm{b-}\overline{B^C})\cap\mathrm{b-}\overline{A^C})$$

$=(\mathrm{b-Fr(A)}\cap\mathrm{b-}\overline{B^C})\cup(\mathrm{b-Fr(B)}\cap\mathrm{b-}\overline{A^C})$by (1)

$$\subset\mathrm{b-Fr(A)}\cup\mathrm{b-Fr(B)}$$

vii. $\mathrm{b-Fr(A\cap B)}=b-(\overline{\mathrm{A\cap B}})\cap b-\big((\overline{\mathrm{A\cap B^C}})\ \big)$by (i)

$$\subset(\mathrm{b-\overline{A}}\cap b-\bar{B})\cap(\mathrm{b-}\overline{A^C}\cup\mathrm{b-}\overline{B^C})$$

$$=((\mathrm{b-\overline{A}}\cap b-\bar{B})\cap\mathrm{b-}\overline{A^C})\cup((\mathrm{b-\overline{A}}\cap b-\bar{B})\cap\mathrm{b-}\overline{B^C})$$

$$=((\mathrm{b-\overline{A}}\cap\mathrm{b-}\overline{A^C})\cap b-\bar{B})\cup(\mathrm{b-\overline{A}}\cap\big(b-\bar{B}\cap\mathrm{b-}\overline{B^C}\big))$$

$=(\mathrm{b-Fr(A)}\cap b-\bar{B})\cup\big(\mathrm{b-\overline{A}}\cap\mathrm{b-Fr(B)}\big)$by(1). Then $\mathrm{b-Fr(A\cap B)}\subset\mathrm{b-Fr(A)}\cup\mathrm{b-Fr(B)}$

Proposition 4.15:

Let E be b-topological space and $A\subset E$ then :

i. If A is b-open, then $\mathrm{b-Fr(A)}=(\mathrm{b-\overline{A}})-\mathrm{A}$.
ii. $\mathrm{b-Fr(A)}=\phi$ if and only if A is b-open as well as b-closed.
iii. A is b-open if and only if $A\cap\mathrm{b-Fr(A)}=\phi$ that is, if $\mathrm{b-Fr(A)}\subset \mathrm{A^C}$.
iv. A is b-closed if and only if $\mathrm{b-Fr(A)}\subset\mathrm{A}$.

Proof:

i. By Proposition 4.14 (i) , we have :

$\mathrm{b-Fr(A)}=(\mathrm{b-\overline{A}})-\mathrm{Int(A)}$, since A is b-open, $-\mathrm{Int(A)}=\mathrm{A}$, hence $\mathrm{b-Fr(A)}=(\mathrm{b-\overline{A}})-\mathrm{A}$.

ii. Let$\mathrm{b-Fr(A)}=\phi$, we have

$\mathrm{b-Fr(A)}=\phi\Rightarrow\mathrm{b-\overline{A}}-\big(\mathrm{b-Int(A)}\big)=\phi$ by proposition 4.14 (i)

$$\Rightarrow\mathrm{b-\overline{A}}\subset\mathrm{b-Int(A)}$$

$$\Rightarrow\mathrm{b-\overline{A}}\subset\mathrm{A}$$

$\Rightarrow\mathrm{b-D(A)}\subset\mathrm{A}\Rightarrow$ A is b-closed.

Now $\mathrm{b-Fr(A)}=\phi\Rightarrow\mathrm{b-\overline{A}}-\big(\mathrm{b-Int(A)}\big)=\phi$

$$\Rightarrow\mathrm{b-\overline{A}}\subset\mathrm{b-Int(A)}$$

$$\Rightarrow A \cup b-D(A) \subset b-Int(A)$$
$$\Rightarrow A \subset b-Int(A)$$

But $-Int(A) \subset A$. Hence $-Int(A) = A$, then A is a b-open set, then A is b-open as well as b-closed. Conversely, let A be b-open and b-closed

$$b-Fr(A) = A-(b-Int(A))$$

since A is b-closed, we have $A = b-\overline{A}$
since A is b-open then $A = b-Int(A)$
Hence $b-Fr(A) = A-A = \phi$.

iii. By (i) proposition 4.14
$b-Fr(A) = b-\overline{A} \cap b-\overline{A^C}$…(1)
let A be b-open, then A^C is b-closed , hence $b-\overline{A^C} = A^C$ …(2)
Now $A \cup b-Fr(A) = A \cap (b-\overline{A} \cap b-\overline{A^C})$ by (1)
$= A \cap (b-\overline{A} \cap A^C)$by (2)
$= (A \cap b-\overline{A}) \cap A^C$ [since $A \subset b-\overline{A}$]

$$= A \cap A^C = \phi$$

Conversely, let $A \cap b-Fr(A) = \phi$
Then by (1) $A \cap b-Fr(A) = \phi \Rightarrow A \cap \left(b-\overline{A} \cap b-\overline{A^C}\right) = \phi$

$$\Rightarrow (A \cap b-\overline{A}) \cap b-\overline{A^C} = \phi$$
$$\Rightarrow A \cap b-\overline{A^C} = \phi$$
$$\Rightarrow A \subset (b-\overline{A^C})^C$$
$$\Rightarrow A \subset b-Int(A)$$

But $b-Int(A) \subset A$ then $b-Int(A) = A$ then A is b-open.

iv. Let A is b-closed then $b-\overline{A} = A$
Hence$b-Fr(A) = b-\overline{A} \cap b-\overline{A^C}$

$$= A \cap b-\overline{A^C} \subset A$$

Conversely, let $b-Fr(A) \subset A$, then $A \cup b-Fr(A) = A$
But $A \cup b-Fr(A) = b-\overline{A}$(since$A = A \cup b-Fr(b-\overline{A})$)
It follows that $A = b-\overline{A}$ then A is b-closed.

Proposition4.16:

Let (E,T) be a b-topological space and let $Y \subset E$ *be* bornivorous set then the collection $T_Y = \{A \cap Y : A \text{ is } b-open \text{ in } E\}$ is a b-topology on Y .

Proof:

We have proof the family of all b-open sub set of Y define a bornological topology on Y since (E,T) be a bornological space then (Y, T_Y) sub space of (E, T). We shall prove every $A_1 \in T_Y$ be b-open set in Y. let $A_1 \in T_Y$then $A_1 = A \cap Y$. let $a \in A_1$ then $A_1 - \{a\} = (A \cap Y) - \{a\} = A \cap Y \cap \{a\}^C = A \cap (Y \cap \{a\}^C) = A \cap (\{a\}^C \cap Y) = (A \cap \{a\}^C) \cap Y = (A - \{a\}) \cap Y$ since $A - \{a\}$is bornivorous set (A is b-open set in E) . Since Y bornivorous set then the intersection of two bornivorous sets is bornivorous set proposition 2.4(ii), then if $A_1 \in T_Y$ then A_1 is b-open set in Y. Since T is the family of all b-open set in E and T_Y is the intersection of every b-open set with Y then T_Y is a family of all b-open set in Y .Then T_Y is bornological topology on Y.

Definition4.17 :

Let (E,T) be a bornological topological space and $Y \subset E$ such that Y is bornivorous set. The pair (Y , T_Y) is called bornological subspace of (E,T).

Proposition 4.18:

Let (Y , T_Y) be bornological sub space of b-topological space (E , T) and let (Z , T_Z) be a bornological subspace of (Y , T_Y) such that Y,Z are bornivorous set then (Z , T_Z) is a bornological subspace of (E ,T).

Proof:

Since $Z \subset Y \subset E$then $Z \subset E$. Let $A_2 \in T_Z$ .Since (Z , T_Z) is bornological sub space of (Y ,T_Y), then there exist $A_1 \in T_Y$ such that $A_2 = A_1 \cap Z$ since (Y , T_Y) be bornological sub space of bornological space (E , T), then there exist $A \in T$ such that $A_1 = A \cap Y$ thus $A_2 = (A \cap Y) \cap Z = A \cap (Y \cap Z) = A \cap Z$. Hence by definition 4.16 Then (Z , T_Z) is a bornological subspace of (E , T).

Proposition 4.19:

Let (Y , T_Y) be a bornological subspace of b-topological space (E , T) , then :

i. A subset A_2 of Y is b-closed in Y if and only if there exist a set b-closed set A in E such that $A_2 = A \cap Y$.
ii. For every $A \subset Y$then $(b-\bar{A})_Y = (b-\bar{A})_E \cap Y$.
iii. For every $A \subset Y, b-Int_E(A) \subset b-Int_Y(A)$.
iv. Every$A \subset Y, b-Fr_Y(A) \subset b-Fr_E(A)$.

Proof:

i. Since A_2 is b-closed set in Y $\Leftrightarrow Y - A_2$ is b-open set in Y $\Leftrightarrow Y - A_2 = A_1 \cap Y$ for some b-open sub set A_1 of E (proposition 4.16) $\Leftrightarrow A_2 = Y - (A_1 \cap Y) = (Y - A_1) \cup (Y - Y) \Leftrightarrow A_2 = Y - A_1 \Leftrightarrow A_2 = Y \cap A_1^C$ where $A_1^C = A$ is b-closed set in E(A_1 is b-open set in E) $\Leftrightarrow A_2 = Y \cap A$.
ii. $(b-\bar{A})_Y = \cap\{A_1 : A_1$ is b-closed set in Y and $A \subset A_1\}$
$= \cap\{B \cap Y : B\ is\ b-closed\ set\ in\ E\ and\ A \subset B \cap Y\}$by (1)
$= \cap\{B \cap Y : B\ is\ b-closed\ set\ in\ E\ and\ A \subset B\}$
$= [\cap\{B : B\ is\ b-closed\ set\ in\ E\ and\ A \subset B\}] \cap Y$
$= (b-\bar{A})_E \cap Y$.
iii. Let $x \in b-Int_E(A)$ then x is bornology interior point of A of b-closure T then A is a b-open set in E containing x then $A \cap Y$ is b-open in b-closure T_Y containing x by proposition 4.16 .Since $A \subset Y$ then $A \cap Y = A$ thus $x \in b-Int_Y(A)$, hence $b-Int_E(A) \subset b-Int_Y(A)$.
iv. Clear

Proposition 4.20:

Let Y be bornological subspace of b-topological space E if $A \subset Y$ is b-open (b-closed) in E then A is also b-open (b-closed) in Y .

Proof:

Since $A \subset Y$, we have $A = A \cap Y$ so that A is the intersection of Y with a set b-open (b-closed) A in E . Hence by proposition 4.16 and proposition 4.19 (i) A is b-open (b-closed) in Y .

Proposition 2.21:

Let Y be bornological subspace of b-topological space E , every sub set B of Y which is b-open (b-closed) in Y is b-open (b-closed) in E if and only if Y is b-open (b-closed) in E.

Proof:

Let B any sub set of E which is b-open (b-closed) in Y be also b-open (b-closed) in E. since Y is b-open (b-closed) in Y then Y is b-open (b-closed) in E.Now let B any sub set of Y which is b-open (b-closed) in Y and let Y is b-open (b-closed) in E since B is b-open (b-closed) in Y there exist a sub set A b-open (b-closed) in E such that $B = A \cap Y$. Since Y is b-open (b-closed) in E and B b-open (b-closed) in E , being the intersection of two b-open (b-closed) sub set of E is b-open (b-closed) set in E, then every sub set of Y which is b-open (b-closed) in Y is b-open (b-closed) in E.

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الخلاصة :-

في هذا البحث قدمنا تعريف مجموعة البرونوفوريس لانشاء الفضاء التبولوجي البرنولوجي. ثم تطرقنا الى اهم الخواص المرتبطة بهذا المفهوم مثل القاعده البرنولوجية والقاعده الجزئية البرنولوجية و كذلك تم دراسة الانغلاق والداخل والحدود البرنولوجي للمجموعات وكذلك الفضاء الجزئي البرنولوجي وذلك من خلال وضع التعاريف ودراسة الخواص المرتبطة بهذه المفاهيم.

الكلمات المفتاحية: b،المجموعه بورنوفوريس، الفضاء التبولوجي البرنولوجي المجموعةالمفتوحة.