New Version Of Korovkin Approximation Theorem In $L_{p,\alpha}$-Space , $1 \le p < \infty$, $\alpha > 0$

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Abstract:

In this paper, we use the positive linear operators and the notion of statistical summability $(C,1)_{p,\alpha}$ to obtain new version of Korovkin approximation theorem by using the test functions $1, e^{-x}, e^{-2x}$, in $L_{p,\alpha}$-space.

Introduction:

In 1970 [1] Boyanov and Veslinov have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions $1, e^{-x}, e^{-2x}$.

Recently, Mörciz in 2002 [3] has defined the concept of statistical summability (C,1) as follows:

For a sequence $x = (x_k)$, let us write: $t_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k$. A sequence $x = (x_k)$ is statistically

summable (C,1) of st- $\lim_{n\to\infty} t_n = L$ in this case, write: $L = C_1(st) - \text{Lim } x$

In 2011, Mohiuddine [2] has obtained application of all almost convergence for single sequence Korovkin-type approximation theorem and proved some related results.

Also, Abdu Mohiuddin in 2012 [4] proved the Korovkin-type approximation theorem by using the notion of summability and the test functions $1, e^{-x}, e^{-2x}$.

In this paper, we use the results above to obtain a new version of Korovkin theorem and statistical summability for $L_{p,\alpha}$-space, $1 \le p < \infty$, $\alpha > 0$.

The Main Result:

In our paper, we use the norm [4]:

$$\|f\|_{p,\alpha} = \left(\int_a^b | f(x)e^{-\alpha x} |^p \, dx\right)^{1/p}, \ (1 \le p < \infty),\ \alpha > 0$$

and suppose that:

$$L_{p,\alpha} = \left\{ f : f \text{ is unbounded and } |f(x)| \le M w_\alpha(x), \text{ such that } w_\alpha(x) = e^{\alpha x} \text{ and } w_{-\alpha} = e^{-\alpha x}, \left(\int_a^b | f(x) w_{-\alpha}(x) |^p \, dx \right)^{1/p} < \infty \right\}$$

Let $L_n : C_{p,\alpha}(I) \longrightarrow C_{p,\alpha}(I)$, where $I = [0,\infty)$ and $C_{p,\alpha}(I)$ be a Banach space with norm $\|\cdot\|_{p,\alpha}$.

We say that L_n is a positive operator if $L_n(f,x) \ge 0$, $\forall f(x) \ge 0$.

From above, we find the relationship between the moduls $w_\alpha(f,\delta)$ and $w(f,\delta)$ and prove some theorems, where:

$$w_\alpha(f,\delta) = \sup_{|h|<\delta} \|f(x+h)e^{-\alpha x} - f(x)e^{-\alpha x}\|$$

$$w(f,\delta) = \sup_{|h|<\delta} |f(x+h) - f(x)|$$

Theorem (1):

If $T_k : C_{p,\alpha}(I) \longrightarrow C_{p,\alpha}(I)$, where T_k be a sequence of positive linear operators, then $\forall f \in C_{p,\alpha}(I)$, we have:

$$\text{st-}\lim_{k\to\infty} \|T_k(f,x) - f(x)\|_{p,\alpha} = 0 \qquad \dots(1)$$

if and only if:

i- $\text{st-}\lim_{k\to\infty} \|T_k(1;x) - 1)\|_{p,\alpha} = 0$ …(2)

ii- $\text{st-}\lim_{k\to\infty} \|T_k(e^{-s};x) - e^{-x})\|_{p,\alpha} = 0$ …(3)

iii- $\text{st-}\lim_{k\to\infty} \|T_k(e^{-2s};x) - e^{-2x}\|_{p,\alpha} = 0$ …(4)

Proof:

Suppose that (2, 3 and 4) are satisfied, then we have to prove 1,

Each 1, e^{-x}, e^{-2x} belong to $C_{p,\alpha}(I)$ in fact:

$$\|T_k(1;x) - f(1)\|_{p,\alpha} \ge \|f(1)\|_{p,\alpha} - \|T_k\|_{p,\alpha}$$

$$\|T_k(1;x) - f(1)\|_{p,\alpha} + \|T_k\|_{p,\alpha} \ge \|f(1)\|_{p,\alpha}$$

Since $\|f(1)\|_{p,\alpha} < \infty$, let $f \in C_{p,\alpha}(I)$, then there exist $M > 0$, such that $|f(x)| \le Me^{\alpha x}$, $\forall x \in I$ and therefore:

$$|f(s) - f(x)| \le |f(s)| + |f(x)| \le Me^{\alpha x} + Me^{\alpha x} < 2Me^{\alpha x}, \qquad -\infty < s, x < \infty \qquad \dots(5)$$

Now, $\exists\, \varepsilon > 0$ and $\delta > 0$, such that:

$|f(s) - f(x)| \le \varepsilon$, whenever $|e^{-s} - e^{-x}| < \delta$, $\forall\ x \in I$ …(6)

We putting $(\psi_1) = \psi_1(s;x) = (e^{-s} - e^{-x})^2$, using (5) and (6)

$$|f(s) - f(x)| < \varepsilon + \frac{2Me^{\alpha x}}{\delta^2}(\psi_1),\ \forall\ |s - x| < \delta$$

$$-\varepsilon - \frac{2Me^{\alpha x}}{\delta^2}(\psi_1) < f(s) - f(x) < \varepsilon + \frac{2Me^{\alpha x}}{\delta^2}(\psi_1)$$

$$T_k(1;x)\left(-\varepsilon - 2\frac{Me^{\alpha x}}{\delta^2}(\psi_1) < f(s) - f(x) < \varepsilon + 2\frac{Me^{\alpha x}}{\delta^2}(\psi_1)\right)$$

Where $T_k(f;x)$ is monotone and linear. Now, we obtain:

$$T_k(1;x)\left(-\varepsilon - 2\frac{Me^{\alpha x}}{\delta^2}(\psi_1)\right) < T_k(1;x)(f(s) - f(x)) < T_k(1;x)\left(\varepsilon + 2\frac{Me^{\alpha x}}{\delta^2}(\psi_1)\right)$$

f(x) is constant and x a fixed point, therefore:

$$-\varepsilon T_k(1;x) - 2\frac{Me^{\alpha x}}{\delta^2}T_{j,k}(\psi_1;x) < T_k(f;x) - f(x)T_k(1;x) < \varepsilon T_k(1;x) + 2\frac{Me^{\alpha x}}{\delta^2}T_k(\psi_1;x)$$

…(7)

Now, $T_k(f;x) - f(x) = T_k(f;x) - f(x)T_k(1;x) + f(x)T_k(1;x) - f(x)$

$$T_k(f;x) - f(x) = T_k(f;x) - f(x)T_k(1;x) + f(x)[T_k(1;x) - 1] \quad \text{…(8)}$$

From (7), we get:

$$T_k(f;x) - f(x)T_k(1;x) < \varepsilon T_k(1;x) + 2\frac{Me^{\alpha x}}{\delta^2}T_k(\psi_1;x)$$

$$T_k(f;x) - f(x)T_k(1;x) + f(x)[T_k(1;x) - 1] < \varepsilon T_k(1;x) + 2\frac{Me^{\alpha x}}{\delta^2}T_k(\psi_1;x) + f(x)[T_k(1;x) - 1]$$

Then:

$$T_k(f;x) - f(x) < \varepsilon T_k(1;x) + 2\frac{Me^{\alpha x}}{\delta^2}T_k(\psi_1;x) + f(x)[T_k(1;x) - 1] \quad \text{…(9)}$$

Now, $T_k(\psi_1;x) = T_k((e^{-s} - e^{-x})^2,x)$

$$T_k(\psi_1;x) = T_k((e^{-2s} - 2e^{-s}e^{-x} + e^{-2x});x)$$
$$= T_k(e^{-2s};x) - 2e^{-x}T_k(e^{-s};x) + e^{-2x}T_k(1;x)$$
$$= T_k(e^{-2s};x) - 2e^{-2x} - 2e^{-x}T_k(e^{-s};x) + 2e^{-2x} + 2e^{-2x}T_k(1;x)$$
$$= T_k(e^{-2s};x) - e^{-2x} - 2e^{-x}T_k(e^{-s};x) + 2e^{-2x} + e^{-2x}T_k(1;x) - 2e^{-2x}$$
$$= [T_k(e^{-2s};x) - e^{-2x}] - 2e^{-x}[T_k(e^{-s};x) - e^{-x}] + e^{-2x}[T_k(1;x) - 1]$$

By using (9), we have:

$$T_k(f;x) - f(x) < \varepsilon T_k(1;x) + \frac{2Me^{\alpha x}}{\delta^2}\Big\{\Big[T_k(e^{-2s};x) - e^{-2x}\Big] - 2e^{-x}\Big[T_k(e^{-s};x) - e^{-x}\Big] + e^{-2x}\Big[T_k(1;x) - 1\Big]\Big\} + f(x)[T_k(1;x) - 1]$$

$$< \varepsilon T_k(1;x) - \varepsilon + \varepsilon + \frac{2Me^{\alpha x}}{\delta^2}\Big\{\Big[T_k(e^{-2s};x) - e^{-2x}\Big] - 2e^{-x}\Big[T_k(e^{-s};x) - e^{-x}\Big] + e^{-2x}\Big[T_k(1;x) - 1\Big]\Big\} + f(x)[T_k(1;x) - 1]$$

$$< \varepsilon + [\varepsilon + f(x)][T_k(1;x) - 1] - \frac{2Me^{\alpha x}}{\delta^2}\Big\{\Big[T_k(e^{-2s};x) - e^{-2x}\Big] - 2e^{-x}\Big[T_k(e^{-s};x) - e^{-x}\Big] + e^{-2x}\Big[T_k(1;x) - 1\Big]\Big\} + f(x)[T_k(1;x) - 1]$$

Since $|f(x)| < Me^{\alpha x}$

$$|T_k(f;x) - f(x)| \le \varepsilon + (\varepsilon + Me^{\alpha x})|T_k(1;x) - 1| + \frac{2Me^{\alpha x}}{\delta^2}|T_k(e^{-2s} - e^{-2x})| - \frac{4Me^{\alpha x}}{\delta^2}|e^{-x}|\,|T_k(e^{-s};x) - e^{-x}| + \frac{2Me^{\alpha x}}{\delta^2}|e^{-2x}|\,|T_k(1;x) - 1|$$

Since:

$$\left(\int_X \left[T_k(f;x) - f(x)\right]^p e^{-\alpha x p}\,dx\right)^{1/p} \le \left(\int_X \operatorname{Sup}\left[T_k(f;x) - f(x)\right]^p e^{-\alpha x p}\,dx\right)^{1/p}$$

$$= \left\|T_k(f;x) - f(x)\right\|_{p,\alpha}$$

$$= \operatorname{Sup}[T_k(f;x) - f(x)]e^{-\alpha x}$$

and $|e^{-x}| < 1$, $\forall$ x $\in$ I and $e^{-\alpha x} \longrightarrow 0$. Suppose that:

$$K = \max\left\{\varepsilon + Me^{\alpha x} + \frac{2Me^{\alpha x}}{\delta^2}, \frac{2Me^{\alpha x}}{\delta^2}, \frac{4Me^{\alpha x}}{\delta^2}\right\}$$

Hence, we get:

$$\|T_k(f;x) - f(x)\|_{p,\alpha} \le \varepsilon + K(\|T_k(1;x) - 1\|_{p,\alpha} + \|T_k(e^{-s};x) - e^{-x}\|_{p,\alpha} + \|T_k(e^{-2s};x) - e^{-2x}\|_{p,\alpha} \quad \ldots(10)$$

Now, replacing $T_k(.;x)$ by $\frac{1}{m+1}\sum_{k=0}^{m} T_k(.,x)$ and then by $B_m(.,x)$ in (10) on both sides.

For given $\varepsilon > 0$, choose $\varepsilon' > 0$, such that $\varepsilon' < r$, and define the following sets:

$$A = \{m \le n : \|B_m(f;x) - f(x)\|_{p,\alpha} \ge r\}$$

$$A_1 = \{m \le n : \|B_m(1;x) - 1\|_{p,\alpha} \ge \frac{r-\varepsilon'}{3k}\}$$

$$A_2 = \{m \le n : \|B_m(t;x) - e^{-x}\|_{p,\alpha} > \frac{r-\varepsilon'}{3k}\}$$

$$A_3 = \{m \le n : \|B_m(t^2;x) - e^{-2x}\|_{p,\alpha} \ge \frac{r-\varepsilon'}{3k}\}$$

and $A \subset A_1 \cup A_2 \cup A_3$, so $\delta(A) \le \delta(A_1) + \delta(A_2) + \delta(A_3)$. Then we get:

$$C_1(st) - \lim_n \|Tn(f;x)\|_{p,\alpha} = 0. \quad \blacksquare$$

We study the rate of weighted convergence of sequence of positive linear operator defined from $C_{p,\alpha}(I)$ into $C_{p,\alpha}(I)$ by using the modulus $w_\alpha(f;\delta)$ and find the relationship between $w(f;\delta)$ and $w_\alpha(f;\delta)$; where:

$$w_\alpha(f;\delta) = \sup_{|h|<\delta} \|f(x+h) - f(x)\|_{p,\alpha}$$

Now:

$$w(f;\delta) = \sup_{|h|<\delta} |f(x+h) - f(x)|$$

$$< \sup_{|h|<\delta} |f(x+h) - f(x)|e^{-\alpha x}$$

$$w(f;\delta) \le \sup_{|h|<\delta} |f(x+h)e^{-\alpha x} - f(x)e^{-\alpha x}|$$

$$\le \sup_{|h|<\delta} \|f(x+h) - f(x)\|_{p,\alpha}$$

$= w_\alpha(f;\delta)$

Now, we use the definition from [4].

Definition (4):

Let (a_n) be a positive nonincreasing sequence, we say that the sequence $x = (x_k)$ is statistically summability (C,1) to the number L with the rate $O(a_n)$ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n}\left|\{m \le n : |t_m - L| > \varepsilon\}\right| = 0$$

In this case, we write $x_k - L = C_1(st) - O(a_n)$.

Now, we have to prove theorem.

Theorem (2):

For any sequence of positive linear operators T_k from $C_{p,\alpha}(I)$ to $C_{p,\alpha}(I)$ satisfying:

i- $\|T_k(1;x) - 1\|_{p,\alpha} = C_1(st) - O(a_n)$.

ii- $w_\alpha(f;\lambda_k) = C_1(st) - O(b_n)$, where:

$\lambda_k = \sqrt{T_k(\psi_k;x)}$ and $\psi_k(x + h) = (e^{-\alpha(x + h)} - e^{-\alpha x})^2$

We have:

$\|T_k(f;x) - f(x)\|_{p,\alpha} = C_1(st) - O(c_n)$, $\forall f \in C_{p,\alpha}(I)$ and $c_n = \max\{a_n, b_n\}$

Proof:

Let C(p) be a constant, such that:

$|f(x + h) - f(x)| \le C(p)\|f(x + h) - f(x)\|_{p,\alpha}$

Now, T_k be a bounded linear operator:

$$\|T_k(f;x) - f(x)\|_{p,\alpha} \le \|T_k(f(x + h) - f(x))\|_{p,\alpha} + \|f(x)(T_k(1;x) - 1)\|_{p,\alpha}$$

$$\le \|T_k(C(p)\|f(x + h) - f(x)\|_{p,\alpha})\|_{p,\alpha} + \|f(x)(T_k(1;x) - 1)\|_{p,\alpha}$$

From [4]:

$$|f(y) - f(x)| < w(f;\delta)\left(\frac{e^{-y} - e^{-x}}{\delta} + 1\right)$$

We get:

$$|f(x + h) - f(x)|e^{-\alpha x} \le w_\alpha(f;\delta)\left(\frac{e^{-\alpha(x+h)} - e^{-\alpha x}}{\delta} + 1\right)$$

Then, we have:

$\|T_k(f;x) - f(x)\|_{p,\alpha} \le$

$$C(p)\left\| T_k\left(\frac{|e^{-\alpha(x+h)} - e^{-\alpha x}|}{\delta} + 1\right) w_\alpha(f;\delta)\right\|_{p,\alpha} + \|f(x)(T(1;x) - 1)\|_{p,\alpha}$$

$$\le C(p)w_\alpha(f;\delta)\left\| T_k\left(\frac{|e^{-\alpha(x+h)} - e^{-\alpha x}|}{\delta} + 1\right)\right\|_{p,\alpha} + \|f(x)(T(1;x) - 1)\|_{p,\alpha}$$

$$\le C(p)w_\alpha(f;\delta)\left\| T_k(1;x) + \frac{1}{\delta^2}T_k(\psi_x;x)\right\|_{p,\alpha} + \left(\int_X \left|f(x)(T_k(1;x)-1)e^{-\alpha x}\right|^p dx\right)^{1/p}$$

$$\le C(p)w_\alpha(f;\delta)\left\| T_k(1;x) + \frac{1}{\delta^2}T_k(\psi_x;x)\right\|_{p,\alpha} - w_\alpha(f;\delta) + w_\alpha(f;\delta) + \|f\|_{p,\alpha}\|T_k(1;x) - 1\|_{p,\alpha}$$

$$\le C(p)w_\alpha(f;\delta)\|T_k(1;x) - 1\|_{p,\alpha} + w_\alpha(f;\delta) + \|f\|_{p,\alpha}\|T_k(1;x) - 1\|_{p,\alpha} + \frac{1}{\delta^2}\, w_\alpha(f;\delta)T_k(\psi_x;x)$$

Now, put $\delta = \lambda_k = \sqrt{T_k(\psi_k;x)}$

$$\le C(p)w_\alpha(f;\lambda_k)\|T_k(1;x) - 1\|_{p,\alpha} + \|f\|_{p,\alpha}\|T_k(1;x) - 1\|_{p,\alpha} + \frac{1}{\delta^2} w_\alpha(f;\lambda_k)\delta^2 + w_\alpha(f;\lambda_k)$$

$$\le \|f\|_{p,\alpha}\|T_k(1;x) - 1\|_{p,\alpha} + C(p)2w_\alpha(f;\lambda_k)\|T_k(1;x) - 1\|_{p,\alpha}$$

Let $K = \max\{\|f\|_{p,\alpha}, 2\}$.

Now, replacing $T_k(.,x)$ by $\frac{1}{n+1}\sum_{k=0}^{n} T_k(.,x) = L_n(.,x)$

$$\|L_n(f;x) - f(x)\|_{p,\alpha} \le K\{\|L_n(1;x) - 1\|_{p,\alpha} + w_\alpha(f;\lambda_k) + w_\alpha(f;\lambda_k)\|T_k(1;x) - 1\|_{p,\alpha}\}$$

Hence, by using the above definition and (i), (ii), we get what is required. ■

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نوع جديد لنظرية التقريب من نوع كوروفكين في الفضاء- $L_{p,\alpha}$ $(1 \leq p < \infty)$، $\alpha > 0$

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الخلاصة :-

في هذا البحث إستخدمنا المؤثرات الخطية الموجبة ورمز المجموع الإحصائي $(C,1)_{p,\alpha}$ لكي نجد نظرية التقريب من نوع كوروفكين بواسطة الدوال $1, e^{-x}, e^{-2x}$ في فضاء $-L_{p,\alpha}$.