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On The Sub-implicative Ideal of a BH-algebra

حول المثالية شبه الاستنتـاجية في جبر-BH

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Abstract

In this paper, we study the notions of sub-implicative ideal of a BH-algebra and we state and prove some theorems which determine the relationships among this ideal with the intersection, union, image of function, inverse function for sub-implicative ideals of BH-algebra and also we give some properties of this ideal and relate it with other types of concepts of a BH-algebra.

المستخلص :

في هذا البحث، درسنا مفاهيم حول المثالية شبه الإستنتاجية من الجبر- BH و اعطينا و برهنا بعض النظريات التي تحدد العلاقات بين هذه المثالية مع التقاطع، الاتحاد، صورة الدالة، معكوس الدالة لمثاليات شبه الاستنتاجية في الجبر-BH و كذلك أعطينا بعض خصائص هذه المثالية وصلتها مع أنواع أخرى من المفاهيم جبر-BH.

1. Introduction:

In 1998, Jun et al, the notion of BH-algebras and more Characteristics on BH-algebras was formulated Y. B. Jun, E. H. Roh, H. S.Kim [6]. H. H. Abbass and H. D. Dahham submit the concept of a completely closed ideal and on the other hand, defined a new concept, namely b-completely closed ideal of a BH-algebra [2]. In 2014, H. H. Abbass and S. A. Neamah introduced the notions of an implicative ideal of a BH-algebra and On the other hand, a new concept, namely a b-implicative ideal of a BH-algebra [3].

2. Preliminaries :

This section is devoted to some basic ordinary concepts of BH-algebra, ideal, sub-implicative ideal and homomorphism in BH-algebra, we give some basic concepts about the image of function, the inverse image, positive implicative and translation ideal of a BH-algebra with some propositions and theorems.

Definition(2.1): [6]

A **BH-algebra** is a nonempty set X with a binary operation * satisfying the following conditions:

i. $a * a = 0$, for all $a \in X$.
ii. $a * b = 0$ and $b * a = 0$ imply $a = b$, for all $a, b \in X$.
iii. $a * 0 = a$, for all $a \in X$.

Definition (2.2): [6]

Let I be a nonempty subset of a BH-algebra X. Then I is named an **ideal** of X if it satisfies:

i. $0 \in I$.
ii. $a*b \in I$ and $b \in I$ imply $a \in I$.

Definition (2.3): [3]

A nonempty subset I of a BH-algebra X is named **sub-implicative** ideal of X if:

i. $0 \in I$.
i. $((a*(a*b))*(b*a))*c \in I$ and $c \in I$ imply $b*(b*a) \in I$, $\forall a, b, c \in X$.

Proposition (2.4): [3]
If X be a BH-algebra and I be a sub-implicative ideal of X. Then I is an ideal of X.

Remark (2.5): [7]
Let X and Y be BH-algebras. A mapping h: X→Y is claims a **homomorphism** if h(a*b) = h(a)*h(b), ∀ a, b ∈X. A homomorphism h is called a **monomorphism** (resp., **epimorphism**) if it is injective. For any homomorphism h : X →Y, the set {a∈X: h(a)=0'} is called the **kernel** of h, denoted by ker(h), and the set {h(a):a∈X} is called the **image** of h, Symbolized by Im(h). Notice that h(0)=0', ∀ homomorphism h. '

Remark (2.6): [2]
Let (X,*, 0) be a BH-algebra and let N be a normal subalgebra of X. Define a relation $\sim_N$ on X by a $\sim_N$ b if and only if a*b ∈N and b*a ∈N, where a, b ∈ X .Then $\sim_N$ is an equivalence relation on X. Denote this by $[a]_N$, i.e., $[a]_N$={b∈ X| a $\sim_N$ b} and X/N={$[a]_N$ | a ∈ X}. And define $[a]_N \oplus [b]_N$ =$[a*b]_N$, then ((X/N),⊕,$[0]_N$) is a BH-algebra.

Theorem (2.7): [3]
Let N be a normal subalgebra of BH-algebra X. If I is an ideal of X, then I/N is an ideal of X/N.

Definition (2.8): [5]
An ideal A of a BH-algebra X is said to be a translation ideal of X if x*y∈A and y*x∈A, then (x*z)*(y*z) ∈A and (z*x)*(z*y)∈A,∀ x, y, z ∈X.

Remark (2.9): [6]
Let A be a translation ideal of X and let (X,*, 0) be a BH-algebra. Define a relation $\sim_A$ on X by a $\sim_A$ b if and only if a*b ∈A and b*a ∈A, where a, b ∈ X .Then ∼A is an equivalence relation on X. $[a]_A$={b∈X |a $\sim_A$ b} and X/A={ $[a]_A$ | a ∈X}, Define $[a]_A \oplus [b]_A = [a*b]_A$, then ((X/A),⊕,$[0]_A$) is BH-algebra.

Theorem (2.10): [6]
Let A be a translation ideal of a BH-algebra (X, *, 0). If we define $[a]_A \oplus [b]_A = [a*b]_A$ for all a and b ∈ X, then (X/A, ⊕, $[0]_A$) is a BH-algebra.

Definition (2.11): [4]
Let X be a BH-algebra. For a fixed b∈X, we define a map R_b: X→X such that **$R_b(x)$ =x*b,** ∀ x∈X and call R_b a **right map** on X. Symbolize the set of all right maps on X by R(X). A left map L_b is defined by a similar way, we define a map L_b: X→X such that **$L_b(x)$ = b*x,** ∀ x∈X and call L_b a **left map** on X.

Definition (2.12): [1,4]
A BH-algebra (X,*, 0) is said to be **a positive implicative** if it satisfies ∀ a, b and c∈X, (a*c)*(b*c) = (a*b)*c.

Theorem (2.13): [4]
If X is a positive implicative BH-algebra, then (L(X), ⊕, L_0) is a positive implicative BH-algebra.

Remark (2.14):
Let X be a BH-algebra and let I be a subset of X. we will denote to the set{ L_a ∈ L(X) ; a ∈ I } by L(I).

Remark (2.15): [4]

Suppose that X be a positive implicative BH-algebra, defined ⊕ an operation in L(X) is $(L_a \oplus L_b)(x) = L_a(x) * L_b(x)$ and $(L_a \oplus L_b)(x) = L_{a*b}(x)$, $\forall L_a, L_b \in L(X)$ and $\forall x \in X$.

Definition (2.16): [2]

A BH-algebra X is called an **associative BH-algebra** if:
$(x * y) * z = x * (y * z)$, $\forall x, y, z \in X$.

Theorem (2.17): [2]

Let X be an associative BH-algebra. Then the following properties are hold:

i. $0*a=x$; $\forall a \in X$
ii. $a*b=b*a$; $\forall a, b \in X$
iii. $a*(a*b)=b$; $\forall a, b \in X$
iv. $(c*a)*(a*b)=a*b$; $\forall a, b, c \in X$
v. $a*b=0 \Rightarrow a=b$; $\forall a, b \in X$
vi. $(a*(a*b))*b=0$; $\forall a, b \in X$
vii. $(a*b)*c=(a*c)*b$; $\forall a, b, c \in X$
viii. $(a*c)*(b*d)=(a*b)*(c*d)$; $\forall a, b, c, d \in X$

3. The Relationship the Sub-implicative Ideal with Other Notions:

We should mentioned that BH-algebra is not necessary associative.

Theorem (3.1):

Let I be an ideal of a BH-algebra X. If $a*(a*b) = b$, then I is a sub-implicative ideal of X.

<u>Proof</u>:

Suppose that I be an ideal of X. Then

i. $0 \in I$. [By definition (2.2)(i)]
ii. Let a, b, c $\in X$ such that $((a*(a*b))*(b*a))*c \in I$ and $c \in I$
$\Rightarrow (a*(a*b))*(b*a) \in I$. [Since I is an ideal of X. By definition (2.2)(ii)]
$\Rightarrow b*(b*a) \in I$. [By the condition $a*(a*b)=b$]

Then I is a sub-implicative ideal of X.■

Theorem (3.2):

Let $\{ I_i , i \in \Gamma\}$ be a family of sub-implicative ideals of a BH-algebra X. Then $\bigcap_{i\in\Gamma} I_i$ is a sub-implicative ideal of X.

<u>Proof:</u>

To prove $\bigcap_{i\in\Gamma} I_i$ is a sub-implicative ideal of X.

i. $0 \in I_i$, $\forall i \in \Gamma$. [Since each I_i are sub-implicative ideal of X,$\forall i \in \Gamma$. By definition (2.3)(i))]
$\Rightarrow 0 \in \bigcap_{i\in\Gamma} I_i$.

ii. Let x, y, z $\in X$ such that $((x*(x*y))*(y*x))*z \in \bigcap_{i\in\Gamma} I_i$ and $z \in \bigcap_{i\in\Gamma} I_i$
$\Rightarrow ((x*(x*y))*(y*x))*z \in I_i$ and $z \in I_i$, $\forall i \in \Gamma$.
$\Rightarrow y*(y*x) \in I_i$,$\forall i \in \Gamma$.
[Since each I_i is sub-implicative ideal of X, $\forall i \in \Gamma$. By definition (2.3)(ii))]
$\Rightarrow y*(y*x) \in \bigcap_{i\in\Gamma} I_i$.

Therefore, the intersection of a family of sub-implicative ideals is a sub-implicative ideal of X. ■

Theorem (3.3):

Let $\{I_i,\ i\in\Gamma\}$ be a chain sub-implicative ideals of a BH-algebra X. Then $\bigcup_{i\in\Gamma} I_i$ is a sub-implicative ideal of X.

Proof :

We must show that $\bigcup_{i\in\Gamma} I_i$ is a sub-implicative ideal of X.

i. $0\in I_i, \forall i\in\Gamma$. [Since each I_i are sub-implicative ideal of X, $\forall i\in\Gamma$.By definition (2.3)(i)]

$\Rightarrow 0\in \bigcup_{i\in\Gamma} I_i$.

ii. Let x, y, z $\in$X such that $((x*(x*y))*(y*x))*z \in \bigcup_{i\in\Gamma} I_i$ and $z \in \bigcup_{i\in\Gamma} I_i$

There exist I_j , $I_k \in \{ I_i \}_{i\in\Gamma}$, such that $((x*(x*y))*(y*x))*z \in I_j$ and $z \in I_k$

$\Rightarrow$ either $I_j \subseteq I_k$ or $I_k \subseteq I_i$ [Since $\{I_i\}_{i\in\Gamma}$ is a chain]

$\Rightarrow((x*(x*y))*(y*x))*z \in I_j$ and $z\in I_j$ or $(x*(y*x))*z \in I_k$ and $z\in I_k$

$\Rightarrow$ either $y*(y*x) \in I_j$ or $y*(y*x) \in I_k$.

[Since I_j and I_k are sub-implicative ideals of X. By definition (2.3)(ii)]

$\Rightarrow y*(y*x)\in \bigcup_{i\in\Gamma} I_i$.

Therefore, $\bigcup_{i\in\Gamma} I_i$ is a sub-implicative ideal of X. ■

Proposition(3.4):

Let g : (X,*,0)→(Y,*', 0') be a BH-epimorphism. If I is a sub-implicative ideal of X, then g(I) is a sub-implicative ideal of Y.

Proof :

Let I be a sub-implicative ideal of X. Then

i. g(0) = 0'. [Since g is an epimorphism. By Remark (2.5)]

$\Rightarrow 0'\in g(I)$

ii. Let x, y, z$\in$Y such that $((x*'(x*'y))*'(y*'x))*'z \in g(I)$ and $z \in g(I)$

$\Rightarrow \exists$ a ,b, c $\in$ I such that g(a)=x , g(b)=y and g(c)=z

$\Rightarrow((x*'(x*'y))*'(y*'x))*'z = ((g(a)*'(g(a)*'g(b)))*'(g(b)*'g(a)))*'g(c)$

$= g(((a*(a*b))*(b*a))*c) \in g(I)$,

[Since g is an epimorphism. By Remark (2.5)]]

$\Rightarrow$ $((a*(a*b))*(b*a))*c \in I$ and $c\in I$, [Since $g(I)=\{g(x) ; x\in I\}$]

$\Rightarrow$ $b*(b*a) \in I$, [Since I is a sub-implicative ideal of X.]

$\Rightarrow g(b*(b*a)) \in g(I)$. [Since $g(I)=\{g(x) ; x\in I\}$]

$\Rightarrow g(b*(b*a)) = g(b)*(g(b)*g(a)) =y*'(y*'x)\in g(I)$.

Thus, g(I) is a sub-implicative ideal of Y.■

Proposition(3.5):

Let g: (X,*, 0) → (Y,*', 0') be a BH-homomorphism. If I is a sub-implicative ideal of Y, then $g^{-1}(I)$ is a sub-implicative ideal of X.

Proof:

Let I be a sub-implicative ideal of Y. Then

i. g(0) = 0' [Since g is a homomorphism. By Remark (2.5)]

$\Rightarrow 0 \in g^{-1}(I)$.

ii. Let x, y, z $\in$X such that $(x*(x*y))*(y*x))*z \in g^{-1}(I)$ and $z \in g^{-1}(I)$

$\Rightarrow g(((x*(x*y))*(y*x))*z) \in I$ and $g(z) \in I$

$\Rightarrow g(((x*(x*y))*(y*x))*z)=((g(x)*'(g(x)*'g(y)))*'(g(y)*'g(x)))*'g(z)\in I$ and $g(z)\in I$, [Since g is a homomorphism. By Remark (2.5)]
$\Rightarrow g(y)*'(g(y)*'g(x))\in I$,[Since I is a sub-implicative ideal of Y. By Definition (2.3)(ii)]
$\Rightarrow g(y)*'(g(y)*'g(x))= g(y*(y*x))\in I$
$\Rightarrow y*(y*x)\in g^{-1}(I)$. [Since g is a homomorphism, by Remark (2.5)]
Then $g^{-1}(I)$ is a sub-implicative ideal of X. ■

Theorem (3.6):

Let N be a normal subalgebra of BH-algebra X. If I is a sub-implicative ideal of X, then I/N is a sub-implicative of X/N.

Proof:

Suppose that I be a sub-implicative ideal of X.
$\Rightarrow$ I is an ideal of X. [By proposition (2.4)]
$\Rightarrow$ I/N is an ideal of X/N. [By theorem (2.7)]
i. $[0]_N\in I/N$. [Since $0\in I$. By definition (2.2)(i)]
ii. Let $[x]_N, [y]_N, [z]_N\in X/N$ such that
$(([x]_N*([x]_N*[y]_N))*([y]_N*[x]_N))*[z]_N\in I/N$ and $[z]_N\in I/N$,
$\Rightarrow (([x]_N*[x*y]_N)*[y*x]_N)*[z]_N\in I/N$ and $[z]_N\in I/N$, [Since $[x]_N*[y]_N=[x*y]_N$]
$\Rightarrow ([x*(x*y)]_N*[y*x]_N)*[z]_N\in I/N$ and $[z]_N\in I/N$,
$\Rightarrow [(x*(x*y))*(y*x))*z]_N\in I/N$ and $[z]_N\in I/N$,
$\Rightarrow ((x*(x*y))*(y*x))*z\in I$ and $z\in I$, [Since $I/N = \{[x]_N | x\in I\}$.By Remark (2.6)]
$\Rightarrow y*(y*x)\in I$, [Since I is a sub-implicative ideal. By definition (2.3)(ii)]
$\Rightarrow [y*(y*x)]_N\in I/N$.
Therefore, I/N is a sub-implicative ideal of X/N.■

Proposition (3.7):

Let A be a translation ideal of a BH-algebra X. If I is a sub-implicative ideal of X, then I/A is a sub-implicative of X/A.

Proof:

Assume that I be a sub-implicative ideal of X. Then
i. $[0]\in I/A$. [By definition(2.3)(i)]
ii. Let $[x]_A,[y]_A, [z]_A\in X/A$ such that
$(([x]_A\oplus([x]_A\oplus[y]_A))\oplus([y]_A\oplus[x]_A))\oplus[z]_A\in I/A$ and $[z]_A\in I/A$
$\Rightarrow (([x]_A\oplus[x*y]_A)\oplus[y*x]_A)\oplus[z]_A\in I/A$ and $[z]_A\in I/A$, [Since $[x]_A\oplus[y]_A=[x*y]_A$]
$\Rightarrow ([x*(x*y)]_A\oplus[y*x]_A)\oplus[z]_A\in I/A$ and $[z]_A\in I/A$,
[Since$[x]_A\oplus[y]_A=[x*y]_A$. By Remark (2.9)]
$\Rightarrow [(x*(x*y))*(y*x))*z]_A\in I/A$ and $[z]_A\in I/A$,
$\Rightarrow ((x*(x*y))*(y*x))*z\in I$ and $z\in I$, [Since $I/A = \{[x]_A | x\in I\}$.By Remark (2.9)]
$\Rightarrow y*(y*x)\in I$, [Since I is a sub-implicative ideal. By Definition (2.3)(ii)]
$\Rightarrow [y*(y*x)]_A\in I/A$.
Then, I/A is a sub-implicative ideal of X/N.■

Proposition (3.8):

Let X be a positive implicative BH-algebra. If I is a sub-implicative ideal of X, then L(I) is a sub-implicative ideal of $(L(X),\oplus,L_0)$.

Proof:

Let I be a sub-implicative ideal of X.
i. $0\in I$ [By definition (2.3)(i)]
$\Rightarrow L_0\in L(I)$
ii. Let $L_a, L_b, L_c\in L(I)$ such that $((L_a\oplus(L_a\oplus L_b))\oplus(L_b\oplus L_a))\oplus L_c\in L(I)$ and $L_c\in L(I)$.
$\Rightarrow ((a*(a*b))*(b*a))*c\in I$ and $c\in I$,

[Since $((L_a \oplus (L_a \oplus L_b)) \oplus (L_b \oplus L_a)) \oplus L_c = L_{((a*(a*b))*(b*a))*c} \in L(I)$]
⇒ b*(b*a) ∈I, [By definition (2.3)(ii)]
⇒ $L_{b*(b*a)} \in L(I)$.
Therefore, L(I) is a sub-implicative ideal of $(L(X),\oplus,L_0)$.■

Theorem (3.9):
Let X be a positive implicative BH-algebra and let

$$H_t = \{a \in X \mid a * t = 0 , t \in X\}$$

be a subset of X. If (a*t)*(b*t) = b*t with a*t ≠ b*t, ∀ a, b, t ∈ X, then $H_t \cup \{0\}$ is a sub-implicative ideal of X.

Proof:
We must show that $H_t \cup\{0\}$ is a sub-implicative ideal of X.
i. we have $0 \in H_t \cup\{0\}$.
ii. Let $a * b \in H_t$ and $b \in H_t$.
⇒ (a*b)*t=0 and b*t =0
⇒ (a*t)*(b*t) = 0 [(a*b)*t=(a*t)*(b*t), since X is positive implicative BH-algebra. By definition (2.12)]
⇒ (a*t)*0=0 [Since b*t =0]
⇒ a*t=0 [Since X is BH- algebra ; x*0=x]
⇒ $a \in H_t$.
⇒ $H_t \cup \{0\}$ is an ideal of X.
iii. Let a, b, c ∈ X such that $((a*(a*b))*(b*a))*c \in H_t$ and $c \in H_t$.
⇒ $((a*(a*b))*(b*a)) \in H_t$ [Since $H_t \cup \{0\}$ is an ideal]
⇒ ((a*(a*b))*(b*a))*t =0
⇒ ((a*t)* ((a*t)*(b*t)))*((b*t)*(a*t))=0
[(a*b)*t=(a*t)*(b*t), since X be positive implicative BH-algebra.]
Case 1: if a*t= b*t, then
⇒ ((a*t)*((a*t)* (a*t)))* ((a*t)* (a*t))=0
⇒ ((a*t)*((a*t)* (a*t))* 0= 0 [Since X is BH-algebra, x*x=0]
⇒ ((a*t)*((a*t)* (a*t))= 0. [Since X is BH-algebra, a*0=a]
⇒ (b*t)*((b*t)* (a*t))= 0 [Since a*t= b*t]
⇒ (b*(b*a))*t= 0
⇒ $b*(b*a)) \in H_t$.
Therefore, $H_t \cup\{0\}$ is a sub-implicative ideal of X.
Case 2: if a*t ≠ b*t, then
Suppose that a, b, t ∈ X such that ((a*t)*((a*t)*(b*t)))* ((b*t)*(a*t)) = 0.
⇒ ((a*t)*(b*t))*((b*t)*(a*t)) = 0.
[By the condition (a*t)*(b*t)=b*t with a*t ≠ b*t, ∀ a , b, t ∈ X]
⇒ (b*t)*((b*t)*(a*t)) = 0.
⇒ (b*(b*a))*t = 0.
⇒ $(b*(b*a)) \in H_t$.
Therefore, $H_t \cup\{0\}$ is a sub-implicative ideal of X.■

Theorem (3.10):

If g : (X, *, 0) → (Y,*', 0') be a homomorphism from an associative BH-algebra X into BH-algebra Y, then ker(g) is a sub-implicative ideal of X.

Proof:

We must show that ker(g) is a sub-implicative ideal of X .

i. g(0)=0'. [Since g be a homomorphism]

⇒ 0∈ker(g).

ii. Let (x*(x*y))*(y*x))*z ∈ ker(g) and z∈ ker(g)

⇒ g((x*(x*y))*(y*x))*z)=0' and g(z)=0' [By Remark (2.5)]

⇒ g((x*(x*y))*(y*x))*z)*'g(z)=0'. [Since g is a homomorphism.]

⇒ g((x*(x*y))*(y*x))*0= 0' [Since g(z)=0']

⇒ g((x*(x*y))*(y*x))= 0' [Since Y is a BH-algebra; g(x)*'0'=g(x)]

⇒ g((x*x)*y)*(y*x)=0' [Since X is an associative. By definition (2.17)]

⇒g((0*y)*(y*x))=0' [Since X is a BH-algebra ; x*x=0]

⇒g(y*(y*x))=0' [Since X is an associative; 0*y=y. By theorem (2.17)(i)]]

⇒ y*(y*x)∈ ker(g).

"Therefore, ker(g) is a sub-implicative ideal of X".

or In other way ,by Proposition 3.5, since $\ker g = g^{-1}(0)$ so $\{0'\}$ is sub-implicative ideal of Y. ■

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