



A new Definition of Fractional Derivative and Fractional Integral

Ahmed M. Kareem

Department of Mathematics, College of Science, University of Diyala, Diyala, Iraq.

a.murshed@yahoo.com

ABSTRACT

In this paper, we introduce three different definitions of fractional derivatives, namely Riemann-Liouville derivative, Caputo derivative and the new formula Caputo expansion formula, and some basics properties of these derivatives are discussed. The difference between Caputo and Riemann – Liouville formulas for the fractional derivatives also mentioned. The paper focuses on find approximate values for function derivatives, when the function order is a negative integer, illustrated by some theorems and examples.

Keywords: Gamma function, Beta function, Mittag -Leffler Function, Complementary Error Function, Fractional integration, Riemann-Liouville and Caputo fractional derivatives.

التعريف الجديد في مشتقات وتكاملات الدوال الكسرية

أحمد مرشد كريم

قسم الرياضيات، كلية العلوم، جامعة ديالى، ديالى، العراق.

a.murshed@yahoo.com

الملخص

في هذا البحث نقدم ثلاثة تعريفات مختلفة للمشتقات الكسرية، وهي تعريف الاشتقاق الكسري حسب ريمان - لوفيل، وتعريف الاشتقاق الكسري حسب كابوتو والتعريف الجديد للمشتقات الكسرية صيغة كابوتو الموسعة وناقش بعض الخصائص الأساسية لهذه المشتقات. وكذلك قدمنا الفرق بين خصائص المشتقات الكسرية حسب تعريف ريمان - لوفيل وحسب تعريف كابوتو. البحث يركز على إيجاد قيم تقريبية لمشتقات الدوال عندما يكون اس الدالة عدد صحيح سالب. وهذا يتضح من بعض النظريات والأمثلة.

الكلمات المفتاحية: دالة كاما، دالة بيتا، معادلة Mittag-Leffler، معادلة Complementary Error، التكامل الكسري، مشتقات Riemann-Liouville و Caputo الكسرية.

1.Introduction

The fractional calculus is a field of mathematics that studies the integration and differentiation of functions of any order [1, 2, 3]. The history of fractional calculus started at the same time when classical calculus was established. It was first mention in Leibniz's letter to l' Hospital in 1695, where the idea of semi derivative was suggested [4, 5]. During time fractional calculus was built on formal foundations by many famous mathematicians, who have provided important contributions up to the middle of 20-th century, e.g. Liouville, Riemann, Euler, Lagrange, Caputo, Heaviside, Fourier, Abel, Holmgren, Laplace etc. The fractional calculus finds applications in different fields of science, including theory of fractional, engineering, physics, numerical analysis, biology and economics [6, 7, 8, 9]. In this paper we introduce three different definitions of fractional differentiation. The first one is, Riemann-Liouville fractional derivatives from this definition, we can calculate the derivative of order α when $(0 < \alpha < 1)$ and function $f(t) = t^\mu$, the order of the function (μ must be positive [10,11,12]. The problem with this definition is that, fractional derivative of a constant does not equal zero [13,14,15]. The second one is, Caputo fractional derivatives, Caputo found this definition after reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions. To solve his fractional order differential equations, from this definition, we can calculate the derivative of order α when $0 < \alpha < 1$ and function $f(t) = t^\mu$, the order of the function (μ) also must be positive [16]. In this new definition of Caputo the fractional derivative of a constant equal zero. According to Riemann-Liouville and Caputo fractional derivatives, derivative is assumed to be zero when the order of the function negative value i.e. $f(t) = t^\mu$ and $(\mu < 0$ when $0 < \alpha < 1, t > 0)$, Since the gamma function is undefined for arguments whose real part is negative integer. For this reason we reformulated the Caputo fractional derivative of the power function, in order to find the new formula Caputo Expansion formula. The main aim is, find approximate values for function derivatives when $f(t) = t^\mu$ and μ is negative integer by Caputo Expansion formula. The paper organized as follows: Section 2 is devoted to describe some necessary basic definitions on fractional calculus which will be used thought the paper, in section 3 we present fractional derivative of the power function. Section 4 explains the application of the Riemann-Liouville and Caputo fractional derivative of the

power function. In section 5 we present a new Caputo Expansion formula and some applications. In section 6 some conclusions are given.

2. Basic Definitions

Through this section we explain some mathematical definitions of the fractional calculus which are used in our work.

2.1 Definition: The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$ and defined as.

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \text{Re } z > 0 \tag{1}$$

Below we show some basic properties of Γ function, namely:

$$\Gamma(1) = \Gamma(2) = 1,$$

$$\Gamma(z + 1) = z\Gamma(z)$$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad \text{for negative value of } z .$$

$$\Gamma(n) = (n - 1)!, \quad n \in N_0,$$

$$\Gamma(n + 1) = n!, \quad \text{for } n \in N_0,$$

Whereas N_0 is the set of the non-negative integers. From the above we can get:

$$\text{a) } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\text{b) } \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\text{c) } \Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{\frac{-3}{2}} = \frac{\Gamma\left(\frac{-1}{2}\right)}{\frac{-3}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-3}{2} \cdot \frac{-1}{2}} = \frac{4}{3}\sqrt{\pi}$$

2.2 Definition: The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined by the two – parameter integral

$$\beta(z, w) = \int_0^1 t^{z-1} (1 - t)^{w-1} dt, \quad \text{Re } z > 0, \text{Re } w > 0 \tag{2}$$

Equation (2) is provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the Gamma function. It should also be mention that the Beta function is symmetric, namely:

$$\beta(z, w) = \beta(w, z) \tag{3}$$

2.3 Definition: Mittag -Leffler Function

While the Gamma function is a generalization of the factorial function, The Mittag-Leffler Function is a generalization of the exponential function, and it is one of the most important functions that is related to fractional differential equations. Firstly, we introduce one-parameter function by using series [2], namely

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \alpha \in R, z \in \mathbb{C}, \quad (4)$$

Then, we define the Mittag-Leffler function with two parameters, as:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \alpha, \beta \in R, z \in \mathbb{C}. \quad (5)$$

There are some properties of Mittag-Leffler Function, namely:

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!}, = e^z,$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z - 1 - z}{z^2}$$

In general,

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left[e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right] \quad (6)$$

Easily, we can obtain the following:

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z)$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{z(2k+1)!} = \frac{\sinh(z)}{z}$$

$$E_{2,1}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos(z)$$

$$E_{2,2}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{z(2k+1)!} = \frac{\sin(z)}{z}$$

2.4 Definition: The complementary error function is an entire function defined as:

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt. \quad (7)$$

Special values of the corresponding error function are given below:

$$erfc(-\infty) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = 2,$$

$$erfc(0) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = 1,$$

$$erfc(+\infty) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = 0.$$

The following relations are interesting to be mentioned, namely

$$erfc(-x) = 2 - erfc(x),$$

$$\int_0^{\infty} erfc(x) dx = \frac{1}{\sqrt{\pi}},$$

$$\int_0^{\infty} erfc^2(x) dx = \frac{2 - \sqrt{2}}{\sqrt{\pi}}.$$

2.5 Definition: Fractional integration

In this section, we define the Cauchy's formula

$$J^n f(t) = \int_a^t \int_a^{\tau_1} \dots \int_a^{\tau_{n-1}} f(\tau) d\tau \dots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_a^t f(\tau) (t - \tau)^{n-1} d\tau. \quad (8)$$

2.6 Definition: Suppose that $\alpha > 0, t > a, a, t \in R$. Then we have

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau) (t - \tau)^{\alpha-1} d\tau$$

This is named the Riemann-Liouville fractional integral of order α . And have some Properties:

1- $J^0 f(t) = f(t)$, i.e. $J^0 = I$ is the identity operator.

2- $J^\alpha(\lambda f(t) + g(t)) = \lambda J^\alpha f(t) + J^\alpha g(t)$, $\alpha \in R_+, \lambda \in \mathbb{C}$, (the Linear property).

3- (i)- $\lim_{\alpha \rightarrow 0} J^\alpha f(t) = f(t)$, (ii)- $J^\alpha (J^\beta f(t)) = J^\beta (J^\alpha f(t)) = J^{\alpha+\beta} f(t)$, $\alpha, \beta \in R_+, \lambda \in \mathbb{C}$, (if

$f(t)$ is continuous for $t \geq 0$).

2.7 Definition: Suppose that $\alpha > 0, t > a, \alpha, a, t \in R$. Then we have

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in N, \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in N. \end{cases} \quad (9)$$

This is named the Riemann-Liouville fractional derivative of order α .

2.8 Definition: Suppose that $\alpha > 0, t > a, \alpha, c, t \in R$. The fractional Caputo operator has the form

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in N, \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in N. \end{cases} \quad (10)$$

Remark: The difference between Caputo and Riemann – Liouville formulas for the fractional derivatives leads to the following differences:

- a) Caputo fractional derivative of a constant equals zero, while Riemann – Liouville fractional derivative of a constant does not equal zero.
- b) The non-commutation, in Caputo fractional derivative we have:

$$D_*^\alpha D^m f(t) = D_*^{\alpha+m} f(t) \neq D_*^\alpha f(t) \quad (11)$$

Where $\alpha \in (n-1, n), n \in N, m = 1, 2, 3, \dots$

In general, the Riemann-Liouville derivative is also non-commutation as:

$$D^m D^\alpha f(t) = D^{\alpha+m} f(t) \neq D^\alpha D^m f(t) \quad (12)$$

Whereas $\alpha \in (n-1, n), n \in N, m = 1, 2, 3, \dots$

Note that the formulas in (4) and (5) become equalities under the following additional conditions:

$$f^{(s)}(0) = 0, \quad s = n, n+1, \dots, m. \quad \text{for } D_*^\alpha \text{ and}$$

$$f^{(s)}(0) = 0, \quad s = 0, 1, 2, \dots, m, \text{ for } D^\alpha$$

3. Fractional Derivative of the Power Function

In order to comprehension the fractional derivative of the power function, we review some important theorems related to our work.

3.1 Theorem [2]:

Suppose $t > 0, \alpha \in R, \text{ and } n - 1 < \alpha < n, n \in N$, then the following relation between the Riemann-Liouville and the Caputo operator holds:

$$D_*^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0)$$

Proof: The well-known Taylor series expansion about the point 0 reads as:

$$\begin{aligned} f(t) &= f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \dots + \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)}f^{(k)}(0) + R_{n-1}. \end{aligned}$$

Considering (8) we conclude that

$$R_{n-1} = \int_0^t \frac{f^{(n)}(\tau)(t-\tau)^{n-1}}{(n-1)!}d\tau = \frac{1}{\Gamma(n)} \int_0^t f^{(n)}(\tau)(t-\tau)^{n-1}d\tau = J^n f^{(n)}(t).$$

Now, by using the linearity characteristic of the Riemann-Liouville fractional derivative, we obtain:

$$\begin{aligned} D^\alpha f(t) &= D^\alpha \left(\sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)}f^{(k)}(0) + R_{n-1} \right) \\ &= \sum_{k=0}^{n-1} \frac{D^\alpha t^k}{\Gamma(k+1)}f^{(k)}(0) + D^\alpha R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \frac{t^{k-\alpha}}{\Gamma(k+1)}f^{(k)}(0) + D^\alpha J^n f^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0) + J^{n-\alpha}f^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0) + D_*^\alpha f(t). \end{aligned}$$

This means that:

$$D_*^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0).$$

So, the proof is completed.

3.2 Theorem [2]: The Riemann-Liouville fractional derivative of the power function satisfies:

$$D^\alpha t^r = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \quad n-1 < \alpha < n, r > -1, r \in R. \tag{13}$$

Proof:

Since $n-1 < \alpha < n, n \in \mathbb{N}$, then

$$\begin{aligned} D^\alpha t^r &= D^n [D^{-(n-\alpha)} t^r] = D^n \left[\frac{\Gamma(r+1)}{\Gamma(r+n-\alpha+1)} t^{r+n-\alpha} \right] \\ &= \frac{\Gamma(r+1)}{\Gamma(r+n-\alpha+1)} \cdot \frac{\Gamma(r+n-\alpha+1)}{\Gamma(r+n-\alpha-n+1)} t^{r+n-\alpha-n} \\ &= \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \quad \blacksquare \end{aligned}$$

3.3 Theorem [3]: The Caputo fractional derivative of the power function is:

$$D_*^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, & n-1 < \alpha < n, p > n-1, p \in R, \\ 0, & n-1 < \alpha < n, p \leq n-1, p \in \{-1, -2, -3, \dots\} \end{cases} \tag{14}$$

Proof :

The first case since $n-1 < \alpha < n, p > n-1, p \in R$, then

$$\begin{aligned} D_*^\alpha t^p &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{(\tau^p)^n}{(t-\tau)^{\alpha+1-n}} d\tau \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\Gamma(p+1)}{\Gamma(p-n+1)} (\tau^{p-n})(t-\tau)^{n-\alpha-1} d\tau \end{aligned}$$

And using substitution $\tau = \lambda t, 0 \leq \lambda \leq 1$

$$D_*^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} \int_0^t (\lambda t)^{p-n} ((1-\lambda)t)^{n-\alpha-1} t d\lambda$$

$$\begin{aligned}
 &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} t^{p-\alpha} \int_0^t \lambda^{p-n} (1-\lambda)^{n-\alpha-1} d\lambda \\
 &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} t^{p-\alpha} \beta(p-n+1, n-\alpha) \\
 &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} t^{p-\alpha} \frac{\Gamma(p-n+1)\Gamma(n-\alpha)}{\Gamma(p-\alpha+1)} \\
 &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}
 \end{aligned}$$

The second case $D_*^\alpha t^p = 0$,

when $(n-1 < \alpha < n, p \leq n-1, p \in \{-1, -2, -3, \dots\})$ follows the design of the proof of the differentiation of the constant function, since $(t^p)^n = 0$

for $p \leq n-1, p, n \in \mathbb{N}$. So, the proof of the theorem is complete.

3.4 Theorem [2]: Let $f(t) = t^\mu$ where $\mu > -1, t > 0$ and $\text{Re } v > 0$, then the Riemann-Liouville fractional integral of the power function satisfies:

$$D^{-v} f(t) = \frac{\Gamma(\mu+1)}{\Gamma(v+\mu+1)} t^{\mu+v}$$

Proof:

$$\begin{aligned}
 D^{-v} f(t) &= \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} f(\xi) d\xi \\
 &= \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} \xi^\mu d\xi \\
 &= \frac{t^{v-1}}{\Gamma(v)} \int_0^t \left(1 - \frac{\xi}{t}\right)^{v-1} \xi^\mu d\xi \\
 &= \frac{t^v}{\Gamma(v)} \int_0^t \left(1 - \frac{\xi}{t}\right)^{v-1} \xi^\mu \frac{1}{t} d\xi
 \end{aligned}$$

By substituting $y = \frac{\xi}{t}$

$$= \frac{t^{v+\mu}}{\Gamma(v)} \int_0^1 (1-y)^{v-1} \left(\frac{\xi}{t}\right)^\mu dy$$

$$\begin{aligned}
 &= \frac{t^{v+\mu}}{\Gamma(v)} \int_0^t (1-y)^{v-1} y^\mu dy \\
 &= \frac{\Gamma(v)\Gamma(\mu+1)t^{v+\mu}}{\Gamma(v+\mu+1)\Gamma(v)} \int_0^t \frac{\Gamma(v+\mu+1)}{\Gamma(v)\Gamma(\mu+1)} (1-y)^{v-1} y^\mu dy \\
 &= \frac{\Gamma(\mu+1)}{\Gamma(v+\mu+1)} t^{\mu+v}
 \end{aligned}$$

So, the proof of the theorem is complete.

4. Applications

In this section we presented, four important examples of the fractional derivatives and fractional integral, in order to comprehend the Riemann-Liouville fractional derivative of the power function and Caputo fractional derivative of the power function.

4.1 Example:

Suppose that $f(t) = t^\mu = k$, and k is constant when we apply the form:

$$D_*^{-\alpha} t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \alpha, t > 0, \mu > -1 \text{ we will obtain:}$$

$$D_*^{-\alpha} k = \frac{k}{\Gamma(\alpha+1)} t^\alpha$$

4.2 Example:

Suppose that $f(t)=t^p$ when $n-1 < \alpha < n$, $p > n-1$, $p \in R$, $t > 0$, (α is the order of differentiation). Four cases are considered, namely fractional derivatives of the functions t^4, t^5, t^6 and t^7 , i.e., $p = 4$, $p = 5$, $p = 6$ and $p = 7$ respectively. by applying the form

$$D_*^\alpha f(t) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \text{ we will obtain:}$$

The first case: if $p = 4$, and $\alpha = \frac{3}{2}$

$$D_*^{3/2} t^4 = \frac{\Gamma(4+1)}{\Gamma(4-\frac{3}{2}+1)} t^{4-\frac{3}{2}} = \frac{4!}{\Gamma(\frac{7}{2})} t^{5/2} = \frac{192}{15\sqrt{\pi}} t^{5/2}$$

The second case: when $p = 5$ and $\alpha = \frac{3}{2}$

$$D_*^{3/2} t^5 = \frac{\Gamma(5+1)}{\Gamma(5-\frac{3}{2}+1)} t^{5-\frac{3}{2}} = \frac{5!}{\Gamma(\frac{9}{2})} t^{7/2} = \frac{1920}{105\sqrt{\pi}} t^{7/2}$$

The third case: when $p = 6$ and $\alpha = \frac{1}{2}$

$$D_*^{1/2} t^6 = \frac{\Gamma(6+1)}{\Gamma(6-\frac{1}{2}+1)} t^{6-\frac{1}{2}} = \frac{6!}{\Gamma(\frac{13}{2})} t^{11/2} = \frac{46080}{10395\sqrt{\pi}} t^{11/2}$$

The fourth case: when $p = 7$ and $\alpha = \frac{1}{2}$

$$D_*^{1/2} t^7 = \frac{\Gamma(7+1)}{\Gamma(7-\frac{1}{2}+1)} t^{7-\frac{1}{2}} = \frac{7!}{\Gamma(\frac{15}{2})} t^{13/2} = \frac{465120}{135135\sqrt{\pi}} t^{13/2}$$

4.3 Examples:

In this example we will apply the form $D_*^{-\alpha} t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\nu+\mu+1)} t^{\mu+\nu}$, on the result of the previous four cases in example 4.2, respectively, then we will getting the original $f(t)$, as follows:

First case

$$\begin{aligned} D_*^{-3/2} \frac{192}{15\sqrt{\pi}} t^{5/2} &= \frac{192}{15\sqrt{\pi}} \left(\frac{\Gamma(\frac{5}{2}+1)}{\Gamma(\frac{5}{2}+\frac{3}{2}+1)} \right) t^{\frac{5}{2}+\frac{3}{2}} \\ &= \frac{192}{15\sqrt{\pi}} \left(\frac{\Gamma(\frac{7}{2})}{\Gamma(5)} \right) t^4 = \frac{192}{15\sqrt{\pi}} \left(\frac{15\sqrt{\pi}}{192} \right) t^4 = t^4 = f(t) \end{aligned}$$

Second case

$$\begin{aligned} D_*^{-3/2} \frac{1920}{105\sqrt{\pi}} t^{5/2} &= \frac{1920}{105\sqrt{\pi}} \left(\frac{\Gamma(\frac{7}{2}+1)}{\Gamma(\frac{7}{2}+\frac{3}{2}+1)} \right) t^{\frac{7}{2}+\frac{3}{2}} \\ &= \frac{1920}{105\sqrt{\pi}} \left(\frac{\Gamma(\frac{9}{2})}{\Gamma(6)} \right) t^5 = \frac{1920}{105\sqrt{\pi}} \left(\frac{105\sqrt{\pi}}{1920} \right) t^5 = t^5 = f(t) \end{aligned}$$

The third case

$$D_*^{-1/2} \frac{46080}{10395\sqrt{\pi}} t^{11/2} = \frac{46080}{10395\sqrt{\pi}} \left(\frac{\Gamma(\frac{11}{2}+1)}{\Gamma(\frac{11}{2}+\frac{1}{2}+1)} \right) t^{\frac{11}{2}+\frac{1}{2}}$$

$$= \frac{46080}{10395\sqrt{\pi}} \left(\frac{\Gamma\left(\frac{13}{2}\right)}{\Gamma(7)} \right) t^6 = \frac{46080}{10395\sqrt{\pi}} \left(\frac{10395\sqrt{\pi}}{46080} \right) t^6 = t^6 = f(t).$$

The fourth case

$$D_*^{-1/2} \frac{645120}{135135\sqrt{\pi}} t^{13/2} = \frac{645120}{135135\sqrt{\pi}} \left(\frac{\Gamma\left(\frac{13}{2} + 1\right)}{\Gamma\left(\frac{13}{2} + \frac{1}{2} + 1\right)} \right) t^{\frac{13}{2} + \frac{1}{2}}$$

$$= \frac{645120}{135135\sqrt{\pi}} \left(\frac{\Gamma\left(\frac{15}{2}\right)}{\Gamma(8)} \right) t^7 = \frac{645120}{135135\sqrt{\pi}} \left(\frac{135135\sqrt{\pi}}{645120} \right) t^7 = t^7 = f(t).$$

So, from the above four cases we conclude that

$$D_*^{-\alpha} D_*^{\alpha} f(t) = D_*^0 f(t) = f(t), \text{ i.e., } D_*^0 f(t) = I \text{ (I is the identity operat)} \quad (15)$$

4.4 Example:

Suppose that $f(t) = t^{\mu}$, when $n - 1 < \alpha < n, \mu > n - 1, \mu \notin \{-1, -2, -3, \dots\}, t > 0$, (α is the order of fractional integral) three case are considered, namely fractional integral of the functions t^3, t^5 and t^7 , i.e., $\mu = 3, \mu = 5$ and $\mu = 7$ respectively. by applying the form

$$D_*^{-\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \alpha > 0, \text{ we will obtain:}$$

The first case: when $\mu = 3$, and $\alpha = \frac{1}{2}$,

$$D_*^{-\frac{1}{2}} t^5 = \frac{\Gamma(5+1)}{\Gamma\left(5 + \frac{1}{2} + 1\right)} t^{5+\frac{1}{2}} = \frac{5!}{\Gamma\left(\frac{13}{2}\right)} t^{\frac{11}{2}} = \frac{7680}{10395\sqrt{\pi}} t^{\frac{11}{2}}$$

The third case: when $\mu = 7$, and $\alpha = \frac{1}{2}$,

$$D_*^{-\frac{1}{2}} t^7 = \frac{\Gamma(7+1)}{\Gamma\left(7 + \frac{1}{2} + 1\right)} t^{7+\frac{1}{2}} = \frac{7!}{\Gamma\left(\frac{17}{2}\right)} t^{\frac{15}{2}} = \frac{1290240}{2027025\sqrt{\pi}} t^{\frac{15}{2}}$$

4.5 Example:

In this example we will apply the form $D_*^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}$, on the result of the previous three cases in Example (4.4), respectively then we will getting the original $f(t)$, as follows:

The first case: when $p = \frac{7}{2}$, and $\alpha = \frac{1}{2}$,

$$D_*^{\frac{1}{2}} \frac{96}{105\sqrt{\pi}} t^{\frac{7}{2}} = \frac{96}{105\sqrt{\pi}} \left(\frac{\Gamma(\frac{7}{2} + 1)}{\Gamma(\frac{7}{2} - \frac{1}{2} + 1)} \right) t^{\frac{7}{2} - \frac{1}{2}} = \frac{96}{105\sqrt{\pi}} \left(\frac{\Gamma(\frac{9}{2})}{\Gamma(4)} \right) t^3 = \frac{96}{105\sqrt{\pi}} \left(\frac{105\sqrt{\pi}}{96} \right) t^3 = t^3 = f(t)$$

The second case: when $p = \frac{11}{2}$, and $\alpha = \frac{1}{2}$,

$$D_*^{\frac{1}{2}} \frac{7680}{10395\sqrt{\pi}} t^{\frac{11}{2}} = \frac{7680}{10395\sqrt{\pi}} \left(\frac{\Gamma(\frac{11}{2} + 1)}{\Gamma(\frac{11}{2} - \frac{1}{2} + 1)} \right) t^{\frac{11}{2} - \frac{1}{2}} = \frac{7680}{10395\sqrt{\pi}} \left(\frac{\Gamma(\frac{13}{2})}{\Gamma(6)} \right) t^5$$

$$= \frac{7680}{10395\sqrt{\pi}} \left(\frac{10395\sqrt{\pi}}{7680} \right) t^5 = t^5 = f(t).$$

The third case: when $p = \frac{15}{2}$, and $\alpha = \frac{1}{2}$,

$$D_*^{\frac{1}{2}} \frac{1290240}{2027025\sqrt{\pi}} t^{\frac{15}{2}} = \frac{1290240}{2027025\sqrt{\pi}} \left(\frac{\Gamma(\frac{15}{2} + 1)}{\Gamma(\frac{15}{2} - \frac{1}{2} + 1)} \right) t^{\frac{15}{2} - \frac{1}{2}} = \frac{1290240}{2027025\sqrt{\pi}} \left(\frac{\Gamma(\frac{17}{2})}{\Gamma(8)} \right) t^7$$

$$= \frac{1290240}{2027025\sqrt{\pi}} \left(\frac{2027025\sqrt{\pi}}{1290240} \right) t^7 = t^7 = f(t).$$

So, from the above three cases we conclude that

$$D_*^\alpha D_*^{-\alpha} f(t) = D_*^0 f(t) = f(t), \text{ i.e., } D_*^0 f(t) = I(I \text{ is the identity operat}) \quad (16)$$

5. Caputo Expansion Formula

In this section we present, a new definition of fractional derivatives Caputo Expansion formula.

5.1 Definition: The basic form of Caputo Expansion formula for fractional derivative of the power function can be written as follows:

$$D_*^\alpha t^\mu = \begin{cases} \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, & n-1 < \alpha < n, \quad \mu > n-1, \mu \in R, t > 0, \\ \frac{\Gamma(|-\mu+1|)}{\Gamma(|-\mu-\alpha+1|)} t^{-\mu-\alpha}, & t > 0, n-1 < \alpha < n, \mu < n-1, \mu \in \{-2, -3, -4, \dots\} \\ 0, & n-1 < \alpha < n, \mu = -1, t > 0, \end{cases}$$

5.2 Definition: The basic form of Caputo Expansion formula for fractional integral of the power function can be written as follows:

$$D_*^{-\alpha} t^\mu = \begin{cases} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, & n-1 < \alpha < n, \quad \mu > n-1, \mu \in R, t, \alpha > 0, \\ \frac{\Gamma(|-\mu+1|)}{\Gamma(|-\mu+\alpha+1|)} t^{-\mu+\alpha}, & t, \alpha > 0, n-1 < \alpha < n, \mu < n-1, \mu \in \{-2, -3, -4, \dots\} \\ 0, & n-1 < \alpha < n, \quad \mu = -1, t, \alpha > 0, \end{cases}$$

6. Application and Result :

In this subsection, in order to show the high importance of Caputo Expansion formula, we apply it to solve fractional derivative and fractional integral when the order of the functions is negative integer and we have obtained approximate values for derivatives and integrations: as in the examples below

6.1 Example:

Suppose that $f(t) = t^{-\mu}$, when $n-1 < \alpha < n, \mu < n-1, \mu \in \{-2, -3, -4, \dots\}$, (α is the order of differentiation) three case are considered, namely fractional derivatives of the functions t^{-2}, t^{-5} and t^{-7} , i.e., $\mu = -2, \mu = -5$ and $\mu = -7$ respectively. by applying the new form $D_*^\alpha t^{-\mu} = \frac{\Gamma(|-\mu+1|)}{\Gamma(|-\mu-\alpha+1|)} t^{-\mu-\alpha}, t > 0$, we will obtain:

The first case: when $\mu = -2$, and $\alpha = \frac{3}{2}$,

$$D_*^{\frac{3}{2}} t^{-2} \cong \frac{\Gamma(|-2+1|)}{\Gamma(|-2-\frac{3}{2}+1|)} t^{-2-\frac{3}{2}} \cong \frac{\Gamma(1)}{\Gamma(\frac{5}{2})} t^{-\frac{7}{2}} \cong \frac{4}{3\sqrt{\pi}} t^{-\frac{7}{2}}$$

The second case: when $\mu = -5$, and $\alpha = \frac{3}{2}$,

$$D_*^{\frac{3}{2}} t^{-5} \cong \frac{\Gamma(|-5+1|)}{\Gamma(|-5-\frac{3}{2}+1|)} t^{-5-\frac{3}{2}} \cong \frac{\Gamma(4)}{\Gamma(\frac{11}{2})} t^{-\frac{13}{2}} \cong \frac{192}{945\sqrt{\pi}} t^{-\frac{13}{2}}$$

The third case: when $\mu = -7$, and $\alpha = \frac{3}{2}$,

$$D_*^{\frac{3}{2}} t^{-7} \cong \frac{\Gamma(|-7+1|)}{\Gamma(|-7-\frac{3}{2}+1|)} t^{-7-\frac{3}{2}} \cong \frac{\Gamma(6)}{\Gamma(\frac{15}{2})} t^{-\frac{17}{2}} \cong \frac{15360}{135135\sqrt{\pi}} t^{-\frac{17}{2}}$$

6.2 Example:

In this example we will apply the new form of integral $D_*^{-\alpha} t^{-\mu} = \frac{\Gamma(|-\mu+1|)}{\Gamma(|-\mu+\alpha+1|)} t^{-\mu+\alpha}$, on the result of the previous three cases in Example (5.1), respectively, then we will getting the original $f(t)$, as follows:

The first case: when $\mu = -\frac{7}{2}$, and $\alpha = \frac{3}{2}$, $t > 0$,

$$D_*^{\frac{-3}{2}} \frac{4}{3\sqrt{\pi}} t^{-\frac{7}{2}} \cong \frac{4}{3\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{7}{2}+1|)}{\Gamma(|-\frac{7}{2}+\frac{3}{2}+1|)} \right) t^{-\frac{7}{2}+\frac{3}{2}} \cong \frac{4}{3\sqrt{\pi}} \left(\frac{\Gamma(\frac{5}{2})}{\Gamma(1)} \right)$$

$$\cong \frac{4}{3\sqrt{\pi}} \left(\frac{3\sqrt{\pi}}{4} \right) t^{-2} = t^{-2} = f(t)$$

The second case: when $\mu = -\frac{13}{2}$, and $\alpha = \frac{3}{2}$, $t > 0$,

$$D_*^{\frac{-3}{2}} \frac{192}{945\sqrt{\pi}} t^{-\frac{13}{2}} \cong \frac{192}{945\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{13}{2}+1|)}{\Gamma(|-\frac{13}{2}+\frac{3}{2}+1|)} \right) t^{-\frac{13}{2}+\frac{3}{2}} \cong \frac{192}{945\sqrt{\pi}} \left(\frac{\Gamma(\frac{11}{2})}{\Gamma(4)} \right) t^{-5}$$

$$\cong \frac{192}{945\sqrt{\pi}} \left(\frac{945\sqrt{\pi}}{192} \right) t^{-5} = t^{-5} = f(t)$$

The third case: when $\mu = -\frac{17}{2}$, and $\alpha = \frac{3}{2}$, $t > 0$,

$$D_*^{\frac{-3}{2}} \frac{15360}{135135\sqrt{\pi}} t^{-\frac{17}{2}} \cong \frac{15360}{135135\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{17}{2}+1|)}{\Gamma(|-\frac{17}{2}+\frac{3}{2}+1|)} \right) t^{-\frac{17}{2}+\frac{3}{2}}$$

$$\cong \frac{15360}{135135\sqrt{\pi}} \left(\frac{\Gamma(\frac{15}{2})}{\Gamma(6)} \right) t^{-7} \cong \frac{15360}{135135\sqrt{\pi}} \left(\frac{135135\sqrt{\pi}}{15360} \right) t^{-7} = t^{-7} = f(t)$$

6.3 Example:

Suppose that $f(t) = t^{-\mu}$, when $n - 1 < \alpha < n, \mu < n - 1, \mu \in \{-2, -3, -4, \dots\}, t > 0$, (α is the order of fractional integral) four cases are considered, namely fractional integral of the functions t^{-2}, t^{-3}, t^{-4} and t^{-5} , i.e., $\mu = -2, \mu = -3, \mu = -4$ and $\mu = -5$ respectively. by applying the new integral form, $D_*^{-\alpha} t^{-\mu} = \frac{\Gamma(-\mu+1)}{\Gamma(-\mu+\alpha+1)} t^{-\mu+\alpha}, \alpha > 0$, we will obtain:

The first case: when $\mu = -2$, and $\alpha = \frac{1}{2}, t > 0$,

$$D_*^{-\frac{1}{2}} t^{-2} \cong \frac{\Gamma(-2+1)}{\Gamma\left(-2+\frac{1}{2}+1\right)} t^{-2+\frac{1}{2}} \cong \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} t^{-\frac{3}{2}} \cong \frac{1}{\sqrt{\pi}} t^{-\frac{3}{2}}$$

The second case: when $\mu = -3$, and $\alpha = \frac{1}{2}, t > 0$,

$$D_*^{-\frac{1}{2}} t^{-3} \cong \frac{\Gamma(-3+1)}{\Gamma\left(-3+\frac{1}{2}+1\right)} t^{-3+\frac{1}{2}} \cong \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} t^{-\frac{5}{2}} \cong \frac{2}{\sqrt{\pi}} t^{-\frac{5}{2}}$$

The third case: when $\mu = -4$, and $\alpha = \frac{1}{2}, t > 0$,

$$D_*^{-\frac{1}{2}} t^{-4} \cong \frac{\Gamma(-4+1)}{\Gamma\left(-4+\frac{1}{2}+1\right)} t^{-4+\frac{1}{2}} \cong \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} t^{-\frac{7}{2}} \cong \frac{8}{3\sqrt{\pi}} t^{-\frac{7}{2}}$$

The fourth case: when $\mu = -5$, and $\alpha = \frac{1}{2}, t > 0$,

$$D_*^{-\frac{1}{2}} t^{-5} \cong \frac{\Gamma(-5+1)}{\Gamma\left(-5+\frac{1}{2}+1\right)} t^{-5+\frac{1}{2}} \cong \frac{\Gamma(4)}{\Gamma\left(\frac{7}{2}\right)} t^{-\frac{9}{2}} \cong \frac{48}{15\sqrt{\pi}} t^{-\frac{9}{2}}$$

6.4 Example:

In this example we will apply the form $D_*^{\alpha} t^{-\mu} = \frac{\Gamma(-\mu+1)}{\Gamma(-\mu-\alpha+1)} t^{-\mu-\alpha}, t > 0$, on the result of the previous four cases in Example (5.3), respectively then we will get the original $f(t)$, as follows:

The first case: when $\mu = -\frac{3}{2}$, and $\alpha = \frac{1}{2}, t > 0$,

$$D_*^{\frac{1}{2}} \frac{1}{\sqrt{\pi}} t^{-\frac{3}{2}} \cong \frac{1}{\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{3}{2}+1|)}{\Gamma(|-\frac{3}{2}-\frac{1}{2}+1|)} \right) t^{-\frac{3}{2}-\frac{1}{2}} \cong \frac{1}{\sqrt{\pi}} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \right) t^{-2} \cong \frac{1}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{1} \right) t^{-2} = t^{-2} = f(t),$$

The second case: when $\mu = -\frac{5}{2}$, and $\alpha = \frac{1}{2}$, $t > 0$,

$$D_*^{\frac{1}{2}} \frac{2}{\sqrt{\pi}} t^{-\frac{5}{2}} \cong \frac{2}{\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{5}{2}+1|)}{\Gamma(|-\frac{5}{2}-\frac{1}{2}+1|)} \right) t^{-\frac{5}{2}-\frac{1}{2}} \cong \frac{2}{\sqrt{\pi}} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2)} \right) t^{-3} \cong \frac{2}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} \right) t^{-3} = t^{-3} = f(t),$$

The third case: when $\mu = -\frac{7}{2}$, and $\alpha = \frac{1}{2}$, $t > 0$

$$D_*^{\frac{1}{2}} \frac{8}{3\sqrt{\pi}} t^{-\frac{7}{2}} \cong \frac{8}{3\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{7}{2}+1|)}{\Gamma(|-\frac{7}{2}-\frac{1}{2}+1|)} \right) t^{-\frac{7}{2}-\frac{1}{2}}$$

$$\cong \frac{8}{3\sqrt{\pi}} \left(\frac{\Gamma(\frac{5}{2})}{\Gamma(3)} \right) t^{-4} \cong \frac{8}{3\sqrt{\pi}} \left(\frac{3\sqrt{\pi}}{8} \right) t^{-4} = t^{-4} = f(t)$$

The fourth case: when $\mu = -\frac{9}{2}$, and $\alpha = \frac{1}{2}$, $t > 0$

$$D_*^{\frac{1}{2}} \frac{48}{15\sqrt{\pi}} t^{-\frac{9}{2}} \cong \frac{48}{15\sqrt{\pi}} \left(\frac{\Gamma(|-\frac{9}{2}+1|)}{\Gamma(|-\frac{9}{2}-\frac{1}{2}+1|)} \right) t^{-\frac{9}{2}-\frac{1}{2}}$$

$$\cong \frac{48}{15\sqrt{\pi}} \left(\frac{\Gamma(\frac{7}{2})}{\Gamma(4)} \right) t^{-5} \cong \frac{48}{15\sqrt{\pi}} \left(\frac{15\sqrt{\pi}}{48} \right) t^{-5} = t^{-5} = f(t)$$

6. Conclusion

The new formula, Caputo Expansion formula has proven as an efficient technique to obtain approximate values for derivatives and integrations, when the order of the functions is negative integer. Compared with the previous definitions of Riemann-Liouville and Caputo Fractional derivatives the result of the derivation is zero, when the order of the functions is negative. In order to achieve our goal, we presented some important example in fractional derivative and fractional integral, So the goal has been achieved successfully. As well as we proved, $D_*^\alpha D_*^{-\alpha} f(t) = D_*^{-\alpha} D_*^\alpha f(t) = D_*^0 f(t) = f(t)$ when $(0 < \alpha < 1)$.

References

- [1] Loverro A, "*Fractional calculus: history definition and applications for the engineer*", Rapport technique, University of Norte Dame: Department of Aerospace and Mechanical Engineering, 3, USA (2004).
- [2] Podlubny I., "*Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*", Academic press, 198, 10 (1998).
- [3] Ishteva M. K., M.Sc. Thesis, "*Properties and Applications of the Caputo Fractional Operator*", University of Karlsruhe (TH), 7, Bulgaria (2005).
- [4] Gorenflo R., Mainardi F., "*Fractional Calculus: Integral and Differential Equations of Fractional Order*", International Centre for Mechanical Sciences, Springer Verlag, 378, 224 (1997).
- [5] Mladenov V., Mastorakis N., "*Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling*", World Scientific and Engineering Academy and Society, 4 (2014).
- [6] El-Ajou, A., Abu Arqub, O. A., Zhou, Z. A., Momani, S., "*New results on fractional power series: theories and applications*", Entropy, 15(12), 5305 (2013).
- [7] Öğrekçi S., "*Generalized Taylor series method for solving nonlinear fractional differential equations with modified Riemann-Liouville derivative*", Advances in Mathematical Physics, Hindawi Publishing Corporation, 2015, 1 (2015).
- [8] Mekkaoui T., Hammouch Z., "*Approximate analytical solutions to the Bagley-Torvik equation by the fractional iteration method*", Annals of the University of Craiova-Mathematics and Computer Science Series, 39(2), 251 (2012).
- [9] Caputo M., "*Linear Model of Dissipation Whose Q Is Almost Frequency Independent-II*", Geophysical Journal International, 13(5), 529 (1967).
- [10] Caputo M., Fabrizio M., "*A New Definition of fractional Derivative without Singular Kernel*", Progress in Fractional Differentiation and Applications, 1(2), 74 (2015).
- [11] Podlubny I., "*Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation*", Fractional Calculus and Applied Analysis, 5(4), 367 (2002).
- [12] Hilfer R., "*Applications of Fractional Calculus in Physics*", World Scientific Publishing, Singapore, 128, 20 (2000).



-
- [13] Kilbas A. A. A., Srivastava H.M.,Trajillo J. J., "*Theory and Application of fractional Differential Equations*", Elsevier, 204, 45 (2006).
- [14] Mainardi F, "*Fractional Calculus and Waves in Linear Viscoelasticity*", An Introduction to Mathematical Models, World Scientific, 3 (2010).
- [15] Baleanu D., Muslish S. I., Tas K., "*Fractional Hamiltonian Analysis of Higher Order Derivatives Systems*", Journal of Mathematical Physics, 147(10), 103503, (2006).
- [16] Baleanu D., "*Hamiltonian Formulation of Systems with Linear Velocities within Riemann-Liouville Fractional Derivatives*", Journal of Mathematics Analysis and Applications, 304(2), 599 (2005).