Exact Probability Equation for Friction Piles in Clay

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الخلاصة:

يعتمد البحث على تطبيق نظرية الاحتمالات لاشتقاق معادلة رياضية تستخدم كملحق في التقريب التقليدي للتحليل وكمساعد لتقييم معاملات الأمان المختارة لكل حد من حدود معادلة التحمل (لكل طبقة) في ركائز الاحتكاك في التربة الطينية. مقترح البحث يكمن في اشتقاق نموذج احتمالي قابل للتطبيق لكل طبقة من طبقات التربة الطينية على حدة, كل حد من معادلة تحمل الركيزة (كل طبقة من التربة) يحتوي على مقدار معين من الشك نتيجة للتغاير في معاملات التربة ولإلغاء هذا التغاير ولجعل احتمالية الفشل ضمن حد معين يتطلب استخدام معامل آمان جزئي لكل طبقة. أسلوب تعيين قيمة جزئية لمعاملات الأمان لكل طبقة باستخدام موديل كمي احتمالي بدلا من افتراض اعتباطي (غير كمي) لقيمة مفردة تستخدم لكل الطبقات سيساهم في تحسين الجانب الاقتصادي إضافة إلى جانب الأمان في تصميم وتنفيذ ركائز الاحتكاك. معادلة الاحتمالية الدقيقة قد تم اشتقاقها رياضياً على أساس المتغيرات الموروثة لمعاملات التربة (معدل مقاومة القص وسمك الطبقة الطينية). تم التحقق من تصرف المعادلة المشتقة (توزيعها الاحتمالي وقيم احتمالية الفشل) باستخدام طريقة المحاكاة (مونت كارلو) وأظهرت النتائج توافقاً دقيقاً. قد تم دراسة العلاقة بين معامل الأمان واحتمالية الفشل إضافة إلى حساسية المعادلة للتغاير في معاملات التربة ومن خلال نموذج توضيحي.

Abstract:

This paper presents exact probabilistic model as a complementary mathematical base for the traditional deterministic approach to quantify the selection of a factor of safety for each term of the load equation of friction piles in clay. The procedure of assigning a partial value of factor of safety for each clay layer using a quantified probabilistic model instead of the use of a single global factor of safety for all layers that based on arbitrarily judgments seems to introduce an enhancement to both economical and safety consideration in the design procedure of the friction piles. it is suggested in this paper to derive probabilistic equation that describe each layer of the problem individually, each term of the pile load equation (clay layers) consists on a certain amount of uncertainty and each request assigning a certain value of factor of safety to eliminate this variability and to keep the probability of failure (which is more reliable risk index) at certain level. Exact probability equation is mathematically derived on the bases of the variability inherited in soil parameters (average un-drained shear strength and thickness of each clay layer inserted). The equation was verified using Monte carol simulation method and results indicate excellent agreement in both, probability distribution shape and calculated failure values. The relationship between factor of safety and probability of failure produced from the derived equation was inspected in addition to the sensitivity of the equation to the change of the variability of input parameters through a reference example.

Keywords: *Friction Pile, Load, Probability, factor of safety*

Introduction

In clayey soils, single pile is supported by adhesion between pile surface and surrounding soil, such pile is known as friction pile and its ultimate load can be represented approximately as (13):

$$q_u = \sum_{i=1}^{N} Ca_i . As_i \qquad \text{................} \quad (1)$$

Where:

q_u = Ultimate load of the friction pile (kN).

Ca = Unit adhesion (kN/m^2).

As = Embedded area of pile (m^2).

N = Number of clay layers.

The unit adhesion Ca is expressed by ($Ca = \alpha .Cu$) where α is the adhesion factor and the value vary with the consistency of the soil and pile material (*see table* (1)), and Cu is the average undrained shear strength of a layer. (As), is defined as the multiplication of the layer thickness (L) that pile inserted by the circumference R of the pile section.

Table (1): Adhesion Factor, (13).

Material of pile	Consistency	*Cu* (Ton/m^2)	α Adhesion Factor
Timber & Concrete	Soft	0 – 3.75	1 – 0.90
	Medium	3.75 – 7.5	0.90 – 0.6
	Stiff	7.5 – 15.0	0.60 – 0.45
Steel	Soft	0 – 3.75	1.0 – 0.80
	Medium	3.75 – 7.5	0.10 – 0.50
	Stiff	7.5 – 15.0	0.50 - < 0.50

Traditional calculation of the allowable load capacity in friction piles is usually based on deterministic approach, such approach accounts for all uncertainty in the problem by using single factor of safety, uncertainty in friction pile problem arise from high variation in (*Cu*) and in the thickness of each clay layer (*L*) in multiple layers problem. *FOS* method has been successfully used over the past decades despite its deficiency to reflect clearly the quantity of uncertainty that eliminated and the risk incorporated, instead of searching for a replacement, this deficiency should be handled to improve the design output.

In friction piles, *FOS* selection is based on experience and judgment and such arbitrarily judgment may reflect only the degree of conservatism which considered as a poor guide of optimum design (11). Now let the allowable load capacity ($q_{all} = q_u / FOS$) to be re-arranged as:

$$q_{all} = \frac{(q_{ui} + q_{u(i+1)} \ldots\ldots + q_{un})}{FOS} = (\frac{q_{ui}}{FOS}) + (\frac{q_{u(i+1)}}{FOS}) \mathbf{K}\,\mathbf{K} + (\frac{q_{un}}{FOS}) \qquad \ldots\ldots \; (2)$$

As can be noticed, equation (2) reflects that the determination of the allowable load capacity is usually assessed by applying equally single *FOS* for all layers to eliminate the uncertainty in the resistance equation, and this might be a major problem since degree of uncertainty is not equally same in all layers to be eliminated by a fixed equally *FOS*, some layers might be satisfied with the suggested safety value but the others might not, they may need higher or lower values. An argument may arise to select a value based on the higher uncertainty; this still not quantified and based on arbitrarily judgment. Thus equation (2) is suggested to be written as:

$$q_{all} = (\frac{q_{ui}}{FOS_i}) + (\frac{q_{u(i+1)}}{FOS_{i+1}})\mathbf{K}\mathbf{K} + (\frac{q_{un}}{FOS_n}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

FOS_i , is introduced in above equation as a partial factor of safety for each clay layer (*i*). The fundamental weakness of applying different partial factor of safety concept in the past is the inability to explicitly account for uncertainty in each layer and the need to repeat judgment step several times to select several values comparing with simplicity of selecting single *FOS* based on highest uncertainty amount from all layers. Probabilistic approach through probability of failure (which is more reliable failure index and widely adopted in geotechnical engineering in recent years, (11)) can submit a powerful model to aid assessment of selection partial safety factors based on the uncertainty and probability of failure desired in each layer.

The suggestion of using probabilistic approach in this research is not a replacement for the deterministic approach but as a complementary tool to submit better understanding and more realistic results for the design. Design philosophy suggested in this research is based on dealing with each layer in spread way to calculate partial allowable load capacity through selecting partial factor of safety. Through fixing the level of risk in each clay layer a corresponding *FOS* is assessed (different values are expected due to difference in uncertainty inherited (4), partial allowable load capacity is then calculated and the summation based on eq. (3) reflects the global allowable design load to be applied.

Probabilistic Methods:

Several methods have been proposed in literature of geotechnical engineering(5), (A) Moment Generation Method, (B) Monte Carlo Simulation and (C) Exact method.

First method (Moment Generation Method) is based on generating statistical quantities (mean and standard deviation) for the problem equation and (based on Taylor series) and through assuming a probabilistic distribution (probability density function) the probability of failure can be assessed (8), usually arbitrary assumptions are adopted due to lack of information. For example, its highly accepted in geotechnical field to adopt normal distribution for the input component random variables but its not necessary that the global distribution of the equation incorporating theses normal variables to be identical with normal distribution shape; summation and multiplication process between variables in the equation governs and may produce another distribution shape and also depending on the linearity or non-linearity of the equation. Despite this fact and under lack of statistical information, normal, lognormal and other distributions are adopted arbitrarily for the global equation in this model and as can be concluded such arbitrarily assumption might be the fundamental weakness in this method because different distribution shapes gives different values of probability of failure.

Monte Carlo Simulation Method is a synthetic sampling tool that generating samples with aid of computer to be used in the probabilistic evaluation of the statistical quantities and the distribution function of the global equation incorporated sub variables (7). The accuracy of this method is more than other methods since its deal directly with the distribution function of the component variables in addition to there statistical quantities. Accuracy of this method depends upon number of iterations, the increase of generated iteration increasing the improvement of accuracy. It is widely accepted to reliable on this method as verification tool for assumed or derived probabilistic models.

Exact (closed form solution) method is the most accurate but incorporated on certain level of difficulties in mathematical derivation and its required statistical background knowledge ((2) and (5)). This method is seems to be applicable in friction pile problems according to the nature of the parameters involved and its general behavior.

Proposed Exact Probabilistic Model:

The proposed model to evaluate probability of failure is a closed-form (exact) solution of two independent component variables, adhesion (Ca) and embedded area surface of pile in clay (As). The statistical quantities (mean, standard deviation) and the probability distributions (probability density functions) of later variables will be defined through the derivation of the model suggested as below.

Using mathematical and statistical concepts of deriving mean (μ) and standard deviation (σ) from single component variable (12), here for (Ca) and (As) from components (Cu) and (L) defined in eq. (1) and considering α and R as deterministic variables resulting:

Equation Parameter	Mean (μ)	Standard deviation (σ)
Ca	$= \mu_{Cu} .\alpha$	$= \sigma_{Cu}.\alpha$
As	$= \mu_{As}.R$	$= \sigma_{As}.R$

For any layer, and based on eq (1), the partial value of ultimate load capacity which represents a part of the global load resistance in multiple clay layer problems is:

$$q_u = Ca.As \qquad \text{…………………. (4)}$$

Using the general expression of the probability density function of a product of two independent random variables (1), and based on eq. (4), resulting:

$$h[q_u]_i = \int_0^{\infty} \left[\frac{\partial As}{\partial q_u} \right] .\ f_{Ca}(Ca).\ f_{As}(f_{As(q_u)})\ \ dCa \qquad \text{………………. (5)}$$

h[.] is the probability density function of q_u for certain layer. f_{Ca}(.) and f_{As}(.) are the probability density functions for *Ca*, and *As*, respectively.

Since: $f_{As(q_u)} = \frac{q_u}{Ca}$ Is the inverse function of $q_u = Ca.As$ with respect to *As*,

Resulting:

$$h[q_u]_i = \int_0^{\infty} (\frac{1}{Ca})\ .\ f_{Ca}(Ca).\ f_{As}(\frac{q_u}{Ca})\ dCa \qquad \text{……………….(6)}$$

At this level the distribution types (probability density functions) is required for *Ca*, and *As*. Since *Ca* = *Cu*.α and as can be seen *Ca* is a linear function of *Cu* therefore the distributions of *Ca* and *Cu* are identical, same concept on *As* = *L*.*R* also shows that *As* and *L* are identical in distribution shape. Both *Cu and L* will be assumed to follow normal distribution (the use of such distribution to describe the basic random variable herein is highly accepted in geotechnical engineering ((6) and (9)). According to such assumption and discussed linearity between equation variables, f_{Ca}(.) and f_{As}(.) can be assumed to follow normal distribution.

The definition of the probability density function of a normal distributed variable is well defined in several text statistical literatures (12), Therefore, eq. (6) can be simplified to:

$$h[q_u] = \frac{1}{2p s_{Ca} s_{As}} \int_0^{\infty} (\frac{1}{Ca}) \, . \, \exp\left[-0.5 \left[\left(\frac{Ca - m_{Ca}}{s_{Ca}} \right)^2 + \left(\frac{\frac{q_u}{Ca} - m_{As}}{s_{As}} \right)^2 \right] \right] dCa \quad ... (7)$$

Through integration (accumulation of the derived probability density function), the probability of failure for any layer can be determined from eq. (7) as:

$$Pf = \int_0^{q_u/FOS} h[q_u] \, dq_u \qquad \text{.............. (8)}$$

From eq. (8); *Pf* against *FOS* can be investigated and can be solved using numerical integration, MathCAD program offer powerful tool to solve such level of numerical problems and it's adopted in this research.

Verification Exact of Exact Probabilistic Equation:

As a verification example, assume soil profile in Fig (1) which inserted by a square (0.3 by 0.3m) Concrete driven pile, the statistical information for all input variable (mean, coefficient of variation and the standard deviation) as well as output results are shown in table (2). The ratio of (σ/μ) is well known as the coefficient of variation (*COV)* and its used herein to estimate the standard deviation since it a better guide to the deviation of the data (8).

Fig (2) represents a sample of generated distributions using the derived equation (8) and simulation for hard clay layer. As can be observed, probability distribution of Exact closed solution shows excellent agreement with the Simulated Monte Carlo Distribution. Both, soft and medium layers show same behavior and reflect same excellent agreement level.

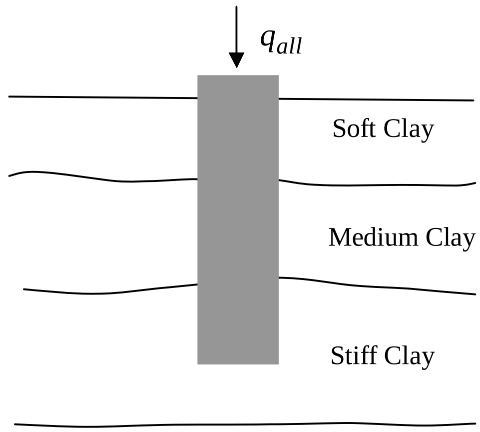


Figure (1): Verification example

Table (2): Input Parameters and output Results of Verification Example Using MathCAD, (General Mathematical Environment Program).

	Description	Soft Clay			Medium Clay			Stiff Clay		
		μ	***COV* %**	**σ**	**μ**	***COV* %**	**σ**	**μ**	***COV* %**	**σ**
Input Parameters	*Cu (kN/m²)*	30	10	3	50	15	7.5	80	15	12
	α	0.9	-	-	0.73	-	-	0.55	-	-
	L (m)	5	15	0.75	6	10	0.6	4	20	0.8
	Ca = α .*Cu*	27	10	2.7	36.5	15	5.48	44	15	6.6
	As = 4*0.3**L*	6	15	0.9	7.2	10	0.72	4.8	20	0.96
	FOS	2			2			2		
Output Results	q_u *(kN)*	162.0			262.8			211.2		
	q_{all} *(kN)*	81.0			131.4			105.6		
	Pf -Exact	0.0043376			0.0010558			0.0135798		
	Pf -Simulated	0.0042690			0.0010240			0.0135740		

Probability of failure (*Pf*), defined as the probability of ultimate load capacity to be equal or less an allowable value and its represent the shaded area in Fig (2). Under this definition, an argument might arise which is that the curve distribution of the left tail in the exact solution of Fig (2) might be of most importance than the accuracy of general shape, to investigate this point, both the exact solution and Monte Carlo (*Pf*), values are generated and results reflects excellent agreement as can be observed in Table (2) adopting equally partial *FOS* (= 2) for each of the different consistency clay layers. MathCAD, which is a powerful general mathematical computer environment was used to solve numerically the derived exact equation and to generate the Simulation by Monte Carlo method, complete details about simulation mechanism was prepared and involved (random number generation based on normally distributed soil parameters, sample size, frequency generation and visual statistical analysis). Iteration numbers in simulation are fixed on 1 million (10^6 tests) to give high accuracy and under higher present computer performance it's consumes only few seconds to generate a solution. The difference between resulting values by simulation comparing with exact solution seems very small and in the fourth and fifth digit which reflect the adequacy of the derived exact equation.

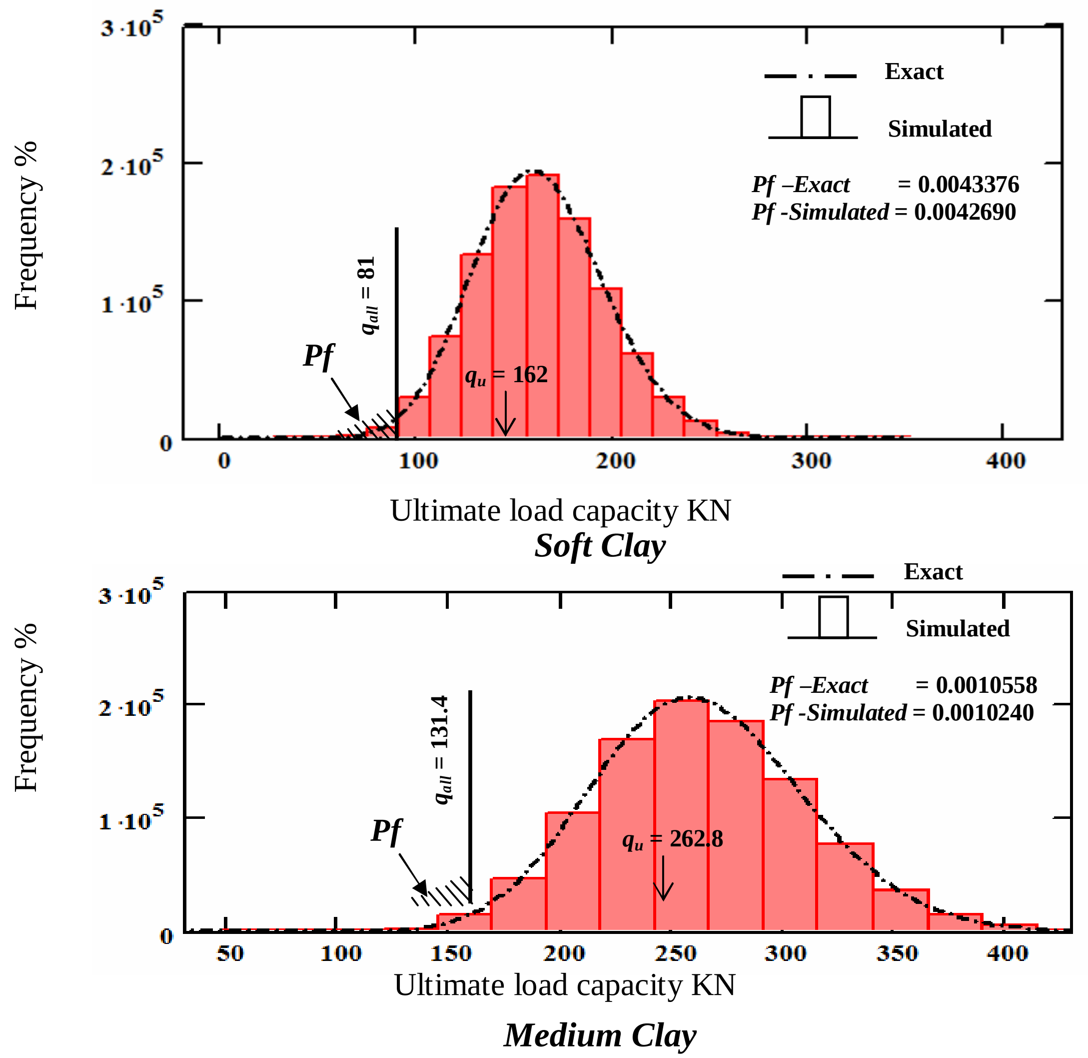
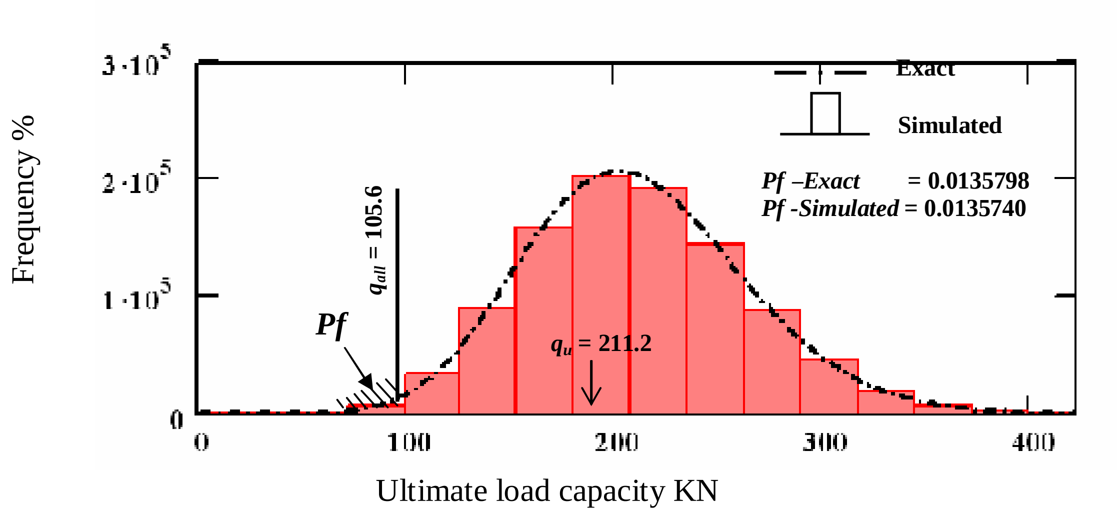


Fig (2): Probability distribution of ultimate load capacity equation of Piles (Exact solution and simulation



Stiff Clay

Fig (2):Continued

Results in Table (2) confirm the expectation that each layer is consist on certain risk depending on its inherited uncertainty, and it is more reliable in the design procedure to fix acceptable risk level (fix the value of *Pf*) than fixing *FOS* values. The example assumes q_{all} to be deterministic to focus on the variability effect of soil parameters in the calculations.

Application and Sensitivity:

Based on the inputs of the reference example illustrated in Table (2) and exact equation (8), factors of safety was determined for each layer in table (3) based on varied values of probability of failure. Further, the global *FOS* of the overall problem is then calculated and for each risk level

Table (3): Judgment step using probability model to select factor of safety for each layer of the reference example based on different risk levels.

	$q_{ui\ (KN)}$ eq. (4)	*FOSi* ; Calculated eq. (8)		
		Pf = 0.01	0.001	0.0001
Soft	162	1.812	2.379	3.180
Medium	262.8	1.641	2.009	2.442
Hard	211.2	2.093	2.944	4.316
$q_{all} = \sum_{i=1}^{N} [q_{u_i} / FOS_i]$		350.46	270.65	207.49
Global $FOS = \left[\sum_{i=1}^{N} q_{u_i}\right] / q_{all}$		1.815	2.350	3.065

Results indicate that the relation between *FOS* and *Pf* are very sensitive for any change, the soft clay layer request a *FOS* (= 2.379) against *Pf* (= 0.001) level and degreasing the *Pf* to (0.0001) reflect the need to increase the *FOS* to (3.180). The values of the global *FOS* indicates the reliability of using suggested model as a base for the judgment step; all values are more than (1) and again increased logically with decreasing of *Pf* level.

Considering integration limit [0 to ∞] into eq (8), a useful curve will be produced and which mathematically known as the (accumulative probability

density function), the value of each point on this curve represent the accumulative area (= *Pf)* under the probability density function (Fig (2)), total area will be equal to 1. However, accumulative probability density function was generated (sample of layer 3) in Fig (3) considering different *COV* values of *Ca* and *As* to help to investigate the effect of latest *COV* values on the *Pf* and *FOS*.

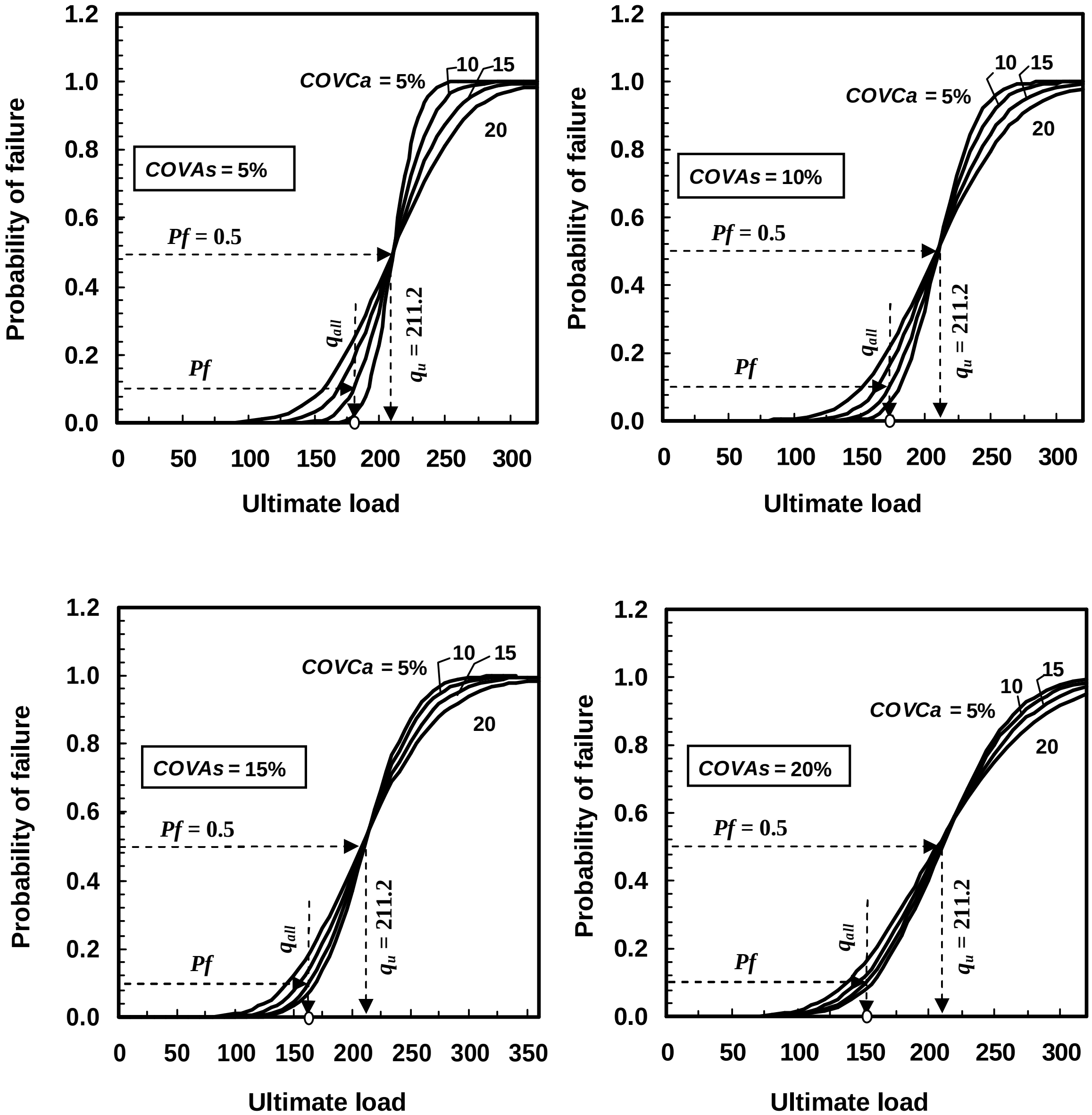


Fig (3): Effect of data scattering *COV* on the probability of failure (sample of stiff clay layer).

Results indicate that the increase in the *COV* values will force the generated curves to be more flat producing (at any *Pf* level) lower values of *qall* and as a result higher *FOS* values. It is worth to notice that the gap between the generated curves is reduced gradually and then intersected at (Pf= 0.5), the coordinate of this point of intersection are the same for all figures ($\boldsymbol{q = q_u}$ and $\boldsymbol{Pf = 0.5}$).

Conclusion:

Exact probabilistic equation was derived and introduced as a complementary to the factor of safety approach to provide more consistency in the calculation of the allowable load that to be applied in the friction pile system.

1. The variability in the soil design parameters is not equally inherited in all successive layers and applying equally single value of factor of safety to eliminate such variability without a quantitative base might consist on high risk or un-optimum design in friction pile system.
2. Results indicate that the estimation of the factor of safety for each layer in the pile system and based on certain probability of failure is appear to be more reliable than the traditional arbitrarily Judgments or even those based on experience alone.
3. Verification results indicate the adequacy of adopting the derived exact probability equation. Solving the equation can be performed using computers or even modern scientific programming hand calculator.
4. Since any failure in any layer can cause global system failure it is recommended in design of piles to fix a single value of Pf for all layers.
5. Parametric study on the reference example reflects that the exact equation is sensitive for the size of data scattering (Coefficient of variation) of the input variables, reliable input values should be considered in using the derived exact probability equation .

References:

1. Ang, H., and W.H. Tang (1984), *"Probability Concepts in Engineering Planning and Design:, Vol. 2, John Wiley and Sons,* NY, 1984.
2. Cherubini, C., (1990), *"A Closed Form Solution for Evaluating the Bearing Capacity of Shallow Foundations"* Canadian Geotechnical Journal. Vol. 27, pp. 526-529.
3. Christian, J. T., Ladd, C. C. and Beacher, G. B., (1994), "*Reliability Applied to Slope Stability Analysis*", ASCE Journal, Vol. 120, No. 12, pp. 2180-*2207.*
4. Duncan, J. M., (2000), "*Factor of Safety and Reliability in Geotechnical Engineering*", Journal of Geotechnical and Geo-environmental Engineering. Vol. 126, No. 4, pp. 307-316.
5. Easa, S. M, (1992), *"Exact Probabilistic Solution of Two-Parameter Bearing Capacity of Shallow Foundations*", Canadian Geotechnical Journal. Vol. 29, pp. 867-870.
6. Ferregut, C. and Picornell M. (1991),*"Reliability Analysis of Drilled Piers in Expansive Soils"*, Can. Geotech., J.28, PP. 834 - 842.
7. Hahn G. T. and Shapiro, S. S., (1967),*"Statistical Models in engineering"*. John Wiley and Sons, Inc.
8. Harr, M. E., (1977),),*"Mechanics of Particular Media: A Probabilistic Approch"* McGraw-Hill Book Co., London.
9. Lee, I. K., White, W. and Ingles, O. G., (1983), "*Geotechnical Engineering* ", Boston.
10. Lumb, P., (1974), "*Application of Statistics in Soil Mechanics*", chapter 3 in "Soil Mechanics, New Horizons" edited by I. K. Lee, Butterworth, London, pp.44-111.
11. National Reserch Council, (1995), "*Probabilistic Methods in Geotechnical Engineering"*, Committee on Reliability Methods for Risk Mitigation in Geotechnical Engineering.
12. SABAH, S. (1989),*" Introduction in Statistical Mathematic"*, Collage of science, University of Basra.
13. V.N.S. Murthy (1988), *"Soil Mechanics and Foundation Engineering", Dhanpat Rai & Sons.*