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S^*-submodule and a vector sublattice

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Abstract :-

In this paper, we review here some of the ideas we have encountered S^*-submodule and a vector sub lattice. We have proved that $N(y)$ be a complement N-function $M(u)$ which satisfies the Δ_2-condition, then L_N is a normal S^*-submodule and a vector sub lattice of $C_\infty(Q(\nabla))$.

1-Introduction:-

In this work a series of known notations, notations and facts of the theory of Boolean algebra, vector Lattice [2,8,6], the integration theory for measures with values in semi-field [2,3,4,5] is cited.

Suppose that R is the set of real numbers and E is a partially ordered set ($E \sqsubseteq R$). The main results in this work is the following:

Proposition I: Let $y_n, y \in C_\infty\big(Q(\nabla)\big), 0 < y_n \uparrow y,\ then\ \ N(y_n) \uparrow N(y)$.

Proposition II: If $y \in S^*\ then\ N(y) \in S^*\ .In\ particular, S^* \subset L_N$.

Proposition III: Suppose that $N(y)$ be a complement N-function $M(u)$ which satisfies the Δ_2-condition then L_N is a normal S^*-submodule and a vector sublattice of $C_\infty(Q(\nabla))$.

2- Definitions and Basic concepts

In this section, we shall review some of the definitions and propositions which are needed in our work.

2.1. DEFINITION [9]

Suppose that $M: I \to R$ is defined on some interval of the real line R. A function M is called convex if $M(\frac{u_1+u_2}{2}) \leq \frac{1}{2}(M(u_1)+M(u_2))$ for all $u_1, u_2 \in I$.

A function M is called convex if the following inequality satisfies for $0 \leq \propto \leq 1$,

$$M(\propto u_1 + (1-\propto)u_2) \leq \propto M(u_1) + (1-\propto)M(u_2), \text{ for all } u_1, u_2 \in I,$$

Which is called Jensseń s inequality [9], we can generalize the inequality for any $u_1, u_2, \dots, u_n$ by $M(\frac{u_1+u_2+\cdots+u_n}{n}) \leq \frac{1}{n}(M(u_1)+M(u_2)+\cdots+M(u_n))$.

2.2. DEFINTION [9]

Suppose $p(t)$ is positive, non decreasing and continuous from the right for $t \geq 0$, and satisfies the conditions :

$$p(0) = 0$$

$$p(\infty) = \lim_{t\to\infty} p(t) = \infty$$

Let us define $q(s)$ for $s \geq 0$ as $q(s) = \sup_{p(t)\leq s}$ t. Note that $q(s)$ is positive, non-decreasing and continuous from the right and satisfies $q(0) = 0$, $\lim_{s\to\infty} q(s) = \infty$. Also , we have q(p(t))≥t and p(q(s))≥s [7].

If $p(t)$ is continuous and increasing then $q(s)$ is equivalent to the inverse of $p(t)$. In general q is called the right inverse to p [7]. If q is the right inverse to p, then the right inverse to q is equivalent to p.

Now, $M(u) = \int_0^{|u|} p(t)dt,\ and\ \ N(v) = \int_0^{|v|} q(s)ds$ are N- functions and one complement each other. Now, recall the Young inequality [2], $uv \leq T + S = M(u) + N(v)$ where $T = M(u)\ and\ S = N(v)$.

2.3. DEFINITION [9]

We say that the N-function $M(u)$ satisfies the Δ_2-condition, if there exist $k > 0$ and $u_0 > 0$ such that $M(2u) \leq kM(u)$ for any $u \geq u_0$.

2.4. DEFINITION [1]

A bimodule X over S^* is called a normal S^*-module if:

1. λx=xλ for all x∈X, λ∈ S^*;

2. For any e∈∇ (S^*), e≠0, there exists x∈X such that x e≠0;

3. For any decomposition of the identity {e_i}⊂∇ (S^*) and for any {x_i}⊂X such that $xe_i=x_ie_i$, for all I;

4. for any x∈X and any sequence {e_n} of mutually disjoint elements from ∇ (S^*) it follows from the equalities $e_nx=0$, n=1,2,…, that $(\sup_{n\supseteq 1} e_n)x=0$.

2.5. Note [10]

Suppose that ∇ is an arbitrary σ-complete Boolean algebra, m is a strictly positive measure on ∇ with values in S^* (m is strictly positive, m(e)=0 for all e∈∇ , that e=0) .In this case ∇ is of a countable type, hence the Boolean algebra ∇ is complete. Let $C_\infty(Q(\nabla))$ be a complete vector lattice of all continuous functions on the stone compactum $Q(\nabla)$, which can take the values $\pm\infty$ on nowhere dense sets from $Q(\nabla)$. We denote by the $L_1(m)$ the set of all integrable by the measure m elements from $C_\infty(Q(\nabla))$, and by μ the integral constructed by the measure m .

The set $L_N(\nabla, m) = L_N = \{y \in C_\infty\big(Q(\nabla)\big) : N(y) \in L_1(m)\}$ is called S^*-orlicz class.

2.6. Proposition [2]

Suppose that L_N is a convex set .In addition, if $x \in L_N$, $y \in C_\infty\big(Q(\nabla)\big)$,

$|y| \le |x|$, then $y \in L_N$.

2.7. Proposition [2]

If a N-function $N(u)$ satisfies the Δ_2-condition, then L_N is a linear space.

3- The Maine results

In this section, we shall prove an important propositions related to the S^*-sub module and a vector sublattice .

3.1. Proposition :

Let $y_n, y \in C_\infty\big(Q(\nabla)\big), 0 < y_n \uparrow y \; then \; N(y_n) \uparrow N(y)$.

Proof:

Since $N(y_n) \le N(x_{n+1}) \le N(y)$, there exists in $C_\infty\big(Q(\nabla)\big)$ an element :

$y = \sup_{n\geq 1} N(y_n) \leq N(y)$.

The function $N^{-1}(u)$ is continuous, positive and monotonically increasing for $u > 0$. So $N^{-1}(N(y_n)) \leq N^{-1}(y)$. From this we get $x = \sup_{n\geq 1} x_n \leq N^{-1}(y)$.

Hence $N(y) \leq N(N^{-1}(y)) = y$. there for $N(y) = y = \sup_{n\geq 1} N(y_n)$. i.e. $N(y_n) \uparrow N(y)$.

3.2. Proposition :

If $y \in S^*$, then $N(y) \in S^*$. In particular , $S^* \subset L_N$.

Proof:

Choose for $y \in S^*$ a sequences of simple elements

$$y_n = \sum_{i=1}^{k(n)} \lambda_i e_i \in S^*, e_i . e_j = 0, i \neq j, \lambda_i > 0.$$

Such as $y_n \uparrow |y|$. then by proposition (3.1)

$$N(y) = N(|y|) = \sup_{n\geq 1} N(y_n) = \sup_{n\geq 1} \sum_{i=1}^{k(n)} N(\lambda_i) e_i .$$

Since $(\sum_{i=1}^{k(n)} N(\lambda_i)\, e_i) \in S^*$ and S^* is a regular sub lattice in $C_\infty(Q(\nabla))$, we have $N(y) \in S^* \subset L_1(N)$ In particular $y \subset L_N$

3.3. Proposition :

Suppose that $N(y)$ be a complement N-function $M(u)$ which satisfies the Δ_2-condition, then L_N is a normal S^*-submodule and a vector sublattice of $C_\infty(Q(\nabla))$.

Proof:

It follows from proposition (2.6) and (2.7) that. L_N is a vector sublattice of $C_\infty(Q(\nabla))$ and $x + y \in L_N$ from any $x, y \in L_N$ .Let $\alpha \in S^*, x \in L_N$.

We show that $\alpha x \in L_N$. since $N(u)$ satisfies the Δ_2-condition, then there exists $u_o > 0$. such that for any number $l \geq 1$ the inequality $N(lu) \leq k(l). N(u)$.

Takes place for every $u \geq u_0$ and some number $k(l) > 0$.

Let $e = \{|x| \leq u_o\}, g = \hat{1} - e$ clearly $N(\propto x) = N(\propto xe) + N(\propto xg)$.

since $|\propto xe| \leq u_0 \propto$ then, by proposition (2.6)

We have $0 \leq N(\propto xe) \leq N(u_o \propto) \in L_1(N). i.e. N(\propto xe) \in L_1(N)$.

Put $g_n = \{n-1 \leq |\propto| < n\}, n = 1,2,\ldots$. it is clear that $g_n \in S^*$,

$g_n \in S^*$, $g_n.g_k=0$, $n \neq k$, and $\sup_{n \supseteq 1} g_n = \hat{\imath}$.

We have then $0 \leq \mu\big(N(\alpha x g g_n)\big) = \mu\big(N(|\alpha|.|x|.g.g_n)\big) \leq \mu\big(N(n g_n g|x|)\big) = g_n \mu\big(N(n|x|g)\big) \leq k(n) g_n \mu\big(N(x)\big)$.

The elements $k(n) g_n \mu(N(x))$ are mutually disjoint in S^*. Hence the element

$Z = \sup_{n \geq 1}(k(n)\ {}_n g \mu(N(x))$ exists in S^*.

In addition, $\mu(\sum_{n=1}^{k} N(\propto x g g_n)) \leq \sum_{n=1}^{k} k(n) g_n\, \mu(N(x)) \leq Z$ for all $k = 1,2,\ldots$.
We get from Levi's theorem that

$$N(\alpha x g) = \sup_{n \geq 1}(g_n N(\alpha x g)) = \sup_{k \geq 1} \sum_{n=1}^{k} g_n N(\alpha x g) = (\sup_{k \geq 1} \sum_{n=1}^{k} N(\alpha x g\, g_n)) \in L_1(N).$$

There fore $N(\propto x) = N(\propto xe) + N(\propto xg) \in L_1(N)$.

Let $\{x_i\} \subset L_N$ and let x be an element of $C_\infty\big(Q(\nabla)\big)$ such that $xe_i = x_i e_i$. Then $|x|e_i = |x_i|e_i$ and $|x| = \sup_{n \geq 1} \sum_{i=1}^{n} |x_i| e_i$.

By proposition (3.1), we have

$$N(x) = N(|x|) = \sup_{n \geq 1} N(\sum_{i=1}^{n} |x_i| e_i) = \sup_{n \geq 1} \sum_{i=1}^{n} N(|x_i| e_i).$$

Beside $\mu(\sum_{i=1}^{n} N(|x_i| e_i)) = \sum_{i=1}^{n} e_i \mu(N(|x_i| e_i))$.

The elements $e_i\mu(N(|x_i|e_i\ ))$ are mutually disjoint in S^* , so the element $a = \sup_{n\geq 1} \mu(\sum_{i=1}^{n} N(|\,x_i\,|\,e_i))$ exists in S^*. It follows from Levi's theorem for the integral $\mu(x)$that $N(x) \in L_1(N)$. Thus , L_N Is a normal S^*-module.

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S^*- متجهات الحزم الجزئية والموديول الجزئي

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الخلاصة :-

في هذا البحث نستعرض بعض المفاهيم التي تواجهنا في موديول جزئي-S^* وحزم المتجه الجزئي. فاننا سنبرهن اذا هذا كانت $N(y)$ هي دالة مكملة لـ دالة- N $M(u)$ التي تحقق شرط -Δ_2 فان L_N يكون موديول جزئي- S^* المعياري وحزم المتجه الجزئي لـ $C_\infty(Q(\nabla))$.