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Anti Fuzzy Bi- Γ- Ideals Of Γ- Semigroups

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Abstract:-

In this paper , we defined the algebraic structures of anti fuzzy bi-Γ-ideal , anti Q- fuzzy bi-Γ-ideal and studied some basic properties .

Introduction:-

In 1965 Zadeh introduced the concepts of fuzzy set [6]. In 1980 Kuroki introduced and studied fuzzy ideals and fuzzy bi-ideals in semigroup [8]. In 1986 Sen and Saha gave a characterization of a bi- Γ-semigroup [7] .In 2007 Chinram and Jirojkal introduced the concept of a bi- Γ-ideal in a Γ-semigroup [9]. In 2009 Williams, Latha and Chandrasekeran introduced a notion of fuzzy bi- Γ-ideal of a Γ-semigroup [3].

In this paper we define anti fuzzy subsemigroup , anti fuzzy bi-ideal , anti fuzzy bi-Γ-ideal , anti fuzzy bi- Γ-ideal of Γ- semigroup and study some of their related properties .

1-Preliminaries

In this section we review some basic definitions which will be used in this paper .

Definition (1.1) [6]: A function μ from a non empty set X to the interval [0,1] is called a fuzzy set of X and the complement of μ is denoted μ^c is a fuzzy set of X defined by $\mu^c(x)=1-\mu(x)$ $\forall x\in X$.

Definition (1.2) [4]: Let f be a function from a set X to a set Y while μ is a fuzzy set of X then the image $f(\mu)$ of μ is the fuzzy set $f(\mu)$: $Y\to[0,1]$, defined by

$$f(\mu(y)) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

Definition (1.3) [4]: Let f be a function from a set X to a set Y while μ is fuzzy set of Y then the inverse image $f^{-1}(\mu)$ of μ under f is the fuzzy set $f^{-1}(\mu)$: X→ [0,1], defined by $f^{-1}(\mu)(x)= \mu(f(x))$.

Definition (1.4) [4]: Let σ and μ be two fuzzy set of a sets X then the union of μ and σ denoted by μ ∪ σ is a fuzzy set of X defined by (μ ∪ σ)(x)= max {μ(x) , σ(x)} and the intersection of μ and σ denoted by μ ∩ σ is a fuzzy set of X defined by (μ ∩ σ)(x)= min {μ(x) , σ(x)} .

Definition (1.5) [4]: Let μ be a fuzzy set of a set X and t ∈[0,1]. The subset

μ_t of X define by μ_t= {x ∈X | μ(x)≥ t} is called a level set of X .

Definition (1.6) [4]: Let μ be a fuzzy set of a set X and t ∈[0,1]. The subset L(μ,t) of X define by L(μ,t) ={x ∈X | μ(x)≤ t} is called a lower level set of X .

Definition (1.7) [2]: A fuzzy set μ of a set X is said to have the inf property if for any subset A of X there is a_o ∈A such that $\mu(a_o)$= inf { μ(a) | a ∈A} .

Definition (1.8) [11]: Let μ and σ be two fuzzy set of a set X .The Cartesian product of μ and σ denoted by μ × σ is fuzzy set defined by

(μ ×σ)(x,y)=min {μ(x), σ(y)} .

Definition (1.9) [8]: A fuzzy subsemigroup μ of a semigroup S is called a fuzzy bi-ideal of S if $\mu(xyz) \geq \min \{ \mu(x) , \mu(z)\}$, $\forall$ x,y,z $\in$S .

Definition (1.10) [7]: Let M={x,y,z,…} and Γ={α,β,γ,…} be two non empty sets then M is called a Γ-semigroup if it satisfies

i- $x\gamma y \in M$ ($M\Gamma M \subseteq M$) , $\forall$ x,y$\in$M and $\gamma \in \Gamma$

ii- $(x\beta y)\gamma z = x\beta(y\gamma z)$, $\forall$ x,y,z$\in$M and $\gamma,\beta \in \Gamma$.

Definition (1.11) [9]: A non empty subset A of a Γ-semigroup M is called a Γ- ideal of M if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition (1.12) [9]: A non empty subset A of a Γ-semigroup M is called a bi -Γ- ideal of M if $A\Gamma M\Gamma A \subseteq A$.

Definition (1.13) [3]: Let M be a Γ-semigroup and M_1 a Γ_1-semigroup. A pair of mappings f_1: $M \rightarrow M_1$ and f_2: $\Gamma \rightarrow \Gamma_1$ is said to be a homomorphism from (M , Γ) to (M_1, Γ_1) if $f_1(a\alpha b)=f_1(a)f_2(\alpha)f_1(b)$, $\forall$ a,b $\in$M , $\alpha \in \Gamma$.

Definition (1.14) [3]: A fuzzy set μ of a Γ-semigroup M is called a fuzzy Γ-subsemigroup of M if $\mu(x\gamma y) \geq \min \{ \mu(x) , \mu(y)\}$,$\forall$ x,y$\in$M and $\gamma \in \Gamma$.

Definition (1.15) [3]: A fuzzy set μ of a Γ-semigroup M is called a fuzzy Γ- ideal of M if $\mu(x\alpha y\beta z) \geq \mu(z)$ and $\mu(x\alpha y\beta z) \geq \mu(x)$,$\forall$ x,y,z$\in$M and $\alpha,\beta \in \Gamma$.

Definition (1.16) [3]: A fuzzy set μ of a Γ-semigroup M is called a fuzzy bi -Γ- ideal of M if $\forall$ x,y,z$\in$M and $\alpha,\gamma,\beta \in \Gamma$

1- $\mu(x\gamma y) \geq \min \{ \mu(x) , \mu(y)\}$
2- $\mu(x\alpha y\beta z) \geq \min \{ \mu(x) , \mu(z)\}$

Definition (1.17) [5]: Let S and S' are any two semigroups then $f : S \to S'$ is called anti-homomorphism if $f(xy)=f(y)f(x)$ $\forall$ $x,y \in S$.

Definition (1.18) [10]: Let Q and G be any two sets . A function μ is a fuzzy set on $G\times Q$ is called a Q-fuzzy set in G .

2-The Main Results:-

Definition (2.1): A fuzzy set μ of a semigroup S is called an anti fuzzy subsemigroup if $\mu(xy) \leq \max \{ \mu(x) , \mu(y)\}$, $\forall$ $x,y\in S$.

Definition (2.2): An anti fuzzy subsemigroup μ of a semigroup S is called anti fuzzy bi-ideal of S if $\mu(xyz) \leq \max \{ \mu(x) , \mu(z)\}$, $\forall$ $x,y,z\in S$.

Definition (2.3) : A fuzzy set μ of a Γ -semigroup is called an anti fuzzy Γ-subsemigroup of M if $\mu(x\gamma y) \leq \max \{ \mu(x) , \mu(y)\}$, $\forall$ $x,y\in M$ and $\gamma\in\Gamma$.

Definition (2.4): A fuzzy set μ of a Γ-semigroup M is called an anti fuzzy Γ- ideal of M if $\mu(x\alpha y\beta z) \leq \mu(z)$ and $\mu(x\alpha y\beta z) \leq \mu(x)$,$\forall$ $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.

Definition (2.5): A fuzzy set μ of a Γ-semigroup M is called anti fuzzy bi -Γ-ideal of M if

1-$\mu(x\gamma y) \leq \max \{ \mu(x) , \mu(y)\}$, $\forall$ $x,y\in M$ and $\gamma\in\Gamma$.

2-$\mu(x\alpha y\beta z) \leq \max \{ \mu(x) , \mu(z)\}$, $\forall$ $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.

Example (2.6): For $a,b \in[0,1]$, let $M=[0,a]$ and $\Gamma=[0,b]$. Clearly M is a Γ-semigroup under usual multiplication. Moreover the fuzzy set $\mu : M\to[0,1]$ defined by

$$\mu(x) = \begin{cases} 0.5 & \text{if } x \in Q \\ 0.1 & \text{if } x \notin Q \end{cases}$$

is anti fuzzy bi -Γ-ideal of M .

Lemma (2.7): Let B be a bi-Γ-ideal of a Γ-semigroup M then for t∈(0,1) there exists an anti fuzzy bi-Γ-ideal of M such that L(μ ,t)=M .

Proof :

Let μ : M→ [0,1] defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases} \qquad \text{where } t\in(0,1) .$$

For all x ∈M , either x ∈B then μ(x)= t

or x ∉B then μ(x)= 0 < t

therefore L(μ ,t)= M

for all x,y∈B and γ∈Γ such that xγy∈B ,we have

μ(xγy) = t = max { μ(x) , μ(y)}

Also , ∀ x,y,z∈B and γ,β∈Γ such that xαyβz∈B ,we have

μ(xαyβz) = t = max { μ(x) , μ(z)} .

Hence μ is anti fuzzy bi-Γ-ideal of M . ■

Lemma (2.8): Let B be a bi-Γ-ideal of a Γ-semigroup M then χ_B is an anti fuzzy bi-Γ-ideal of M .

Proof :

$$\chi_B(x)=\begin{cases}1 & \text{if } x\in B\\ 0 & \text{if } x\notin B\end{cases}$$

Since B is Γ-ideal of a Γ-semigroup M then $B\Gamma M\subseteq B$ and $M\Gamma B\subseteq B$

also , since B is bi-Γ-ideal of a Γ-semigroup M then $B\Gamma M\Gamma B\subseteq B$

i- Let $x,y\in M$ and $\gamma\in\Gamma$.

1- if $x,y\in B$ then $x\gamma y\in B\Gamma B\subseteq B\Gamma M\subseteq B$ and $\chi_B(x)=\chi_B(y)=\chi_B(x\gamma y)=1$

hence $\chi_B(x\gamma y)\leq \max\{\chi_B(x),\chi_B(y)\}$.

2- if $x\notin B, y\in B$ then $x\gamma y\in M\Gamma B\subseteq B$ and $\chi_B(x)=0$, $\chi_B(y)=\chi_B(x\gamma y)=1$

hence $\chi_B(x\gamma y)\leq \max\{\chi_B(x),\chi_B(y)\}$.

3- if $x\in B, y\notin B$ then $x\gamma y\in B\Gamma M\subseteq B$ and $\chi_B(x)=\chi_B(x\gamma y)=1$, $\chi_B(y)=0$

hence $\chi_B(x\gamma y)\leq \max\{\chi_B(x),\chi_B(y)\}$.

4- if $x\notin B, y\notin B$ then $x\gamma y\in M\Gamma M\not\subset B$ and $\chi_B(x)=\chi_B(y)=\chi_B(x\gamma y)=0$

hence $\chi_B(x\gamma y)\leq \max\{\chi_B(x),\chi_B(y)\}$.

ii- Let $x,y,z\in M$ and $\alpha,\beta\in\Gamma$

1- if $x,z\in B$ then $x\alpha y\beta z\in B\Gamma M\Gamma B\subseteq B$ and $\chi_B(x)=\chi_B(z)=\chi_B(x\alpha y\beta z)=1$

hence $\chi_B(x\alpha y\beta z)\leq \max\{\chi_B(x),\chi_B(z)\}$.

2- if $x\notin B, z\in B$ then $x\alpha y\beta z\in M\Gamma M\Gamma B\not\subset B$ and $\chi_B(x)=\chi_B(x\alpha y\beta z)=0$, $\chi_B(z)=1$

hence $\chi_B(x\alpha y\beta z)\leq \max\{\chi_B(x),\chi_B(z)\}$.

3- if $x\in B, z\notin B$ then $x\alpha y\beta z\in B\Gamma M\Gamma M\not\subset B$ and $\chi_B(x)=1$, $\chi_B(z)=\chi_B(x\alpha y\beta z)=0$

hence $\chi_B(x\alpha y\beta z)\leq \max\{\chi_B(x),\chi_B(z)\}$.

4- if $x\notin, z\notin B$ then $x\alpha y\beta z\in M\Gamma M\Gamma M\not\subset B$ and $\chi_B(x)=\chi_B(z)=\chi_B(x\alpha y\beta z)=0$

hence $\chi_B(x\alpha y\beta z)\leq \max\{\chi_B(x),\chi_B(z)\}$.

Thus, χ_B is anti fuzzy bi- Γ-ideal of M . ■

Theorem (2.9): If $\{\mu_j : j \in \Lambda\}$ is a family of anti fuzzy bi- Γ-ideal of a Γ-semigroup M then $\bigcup_{j\in\Lambda}\mu_j$ is an anti fuzzy bi- Γ-ideal of M.

Proof : $\bigcup_{j\in\Lambda}\mu_j = \sup_{j\in\Lambda}\{\mu_j(x) : x \in M\}$

i- Let x,y∈B and γ∈Γ,

$$\bigcup_{j\in\Lambda}\mu_j(x\gamma y) = \sup_{j\in\Lambda}\{\mu_j(x\gamma y)\}$$
$$\leq \sup_{j\in\Lambda}\{\max\{\mu_j(x), \mu_j(y)\}$$
$$= \max\{\sup_{j\in\Lambda}(\mu_j(x)), \sup_{j\in\Lambda}(\mu_j(y))\}$$
$$= \max\{\{\sup_{j\in\Lambda}(\mu_j(x)) : x \in M\}, \{\sup_{j\in\Lambda}(\mu_j(y)) : y \in M\}\}$$
$$= \max\{\bigcup_{j\in\Lambda}\mu_j(x), \bigcup_{j\in\Lambda}\mu_j(y)\}$$

ii- Let x,y,z∈B and α,β∈Γ

$$\bigcup_{j\in\Lambda}\mu_j(x\alpha y\beta z) = \sup_{j\in\Lambda}\{\mu_j(x\alpha y\beta z)\}$$
$$\leq \sup_{j\in\Lambda}\{\max\{\mu_j(x), \mu_j(z)\}$$
$$= \max\{\sup_{j\in\Lambda}(\mu_j(x)), \sup_{j\in\Lambda}(\mu_j(z))\}$$
$$= \max\{\{\sup_{j\in\Lambda}(\mu_j(x)) : x \in M\}, \{\sup_{j\in\Lambda}(\mu_j(z)) : z \in M\}\}$$
$$= \max\{\bigcup_{j\in\Lambda}\mu_j(x), \bigcup_{j\in\Lambda}\mu_j(z)\}$$

Hence $\bigcup_{j\in\Lambda}\mu_j$ is an anti fuzzy bi-Γ-ideal of M . ■

Theorem (2.10): Let μ be a fuzzy set of a Γ-semigroup M then μ is a fuzzy bi-Γ-ideal of M if and only if μ^c is an anti fuzzy bi-Γ-ideal of M.

Proof :

suppose μ is fuzzy bi-Γ-ideal of M

i- Let $x,y\in M$ and $\gamma\in\Gamma$,

$\mu(x\gamma y) \geq \min \{\mu(x) , \mu(y) \}$

$1-\mu^c(x\gamma y) \geq \min \{1-\mu^c(x) , 1-\mu^c(y) \}$

$\mu^c(x\gamma y) \leq 1- \min \{1-\mu^c(x) , 1-\mu^c(y)\}$

$\mu^c(x\gamma y) \geq \max \{\mu^c(x) , \mu^c(y)\}$

ii-Let $x,y,z\in M$ and and $\alpha,\beta\in\Gamma$

$\mu(x\alpha y\beta z) \geq \min \{\mu(x) , \mu(z)\}$

$1-\mu^c(x\alpha y\beta z) \geq \min \{1-\mu^c(x) , 1-\mu^c(z)\}$

$\mu^c(x\alpha y\beta z) \leq 1- \min \{1-\mu^c(x) , 1-\mu^c(z)\}$

$\mu^c(x\alpha y\beta z) \leq \max \{\mu^c(x) , \mu^c(z)\}$

Hence μ^c is an anti fuzzy bi-Γ-ideal of M.

Conversely suppose μ^c is an anti fuzzy bi-Γ-ideal of M .

i- Let $x,y\in M$ and $\gamma\in\Gamma$,

$\mu^c(x\gamma y) \geq \max \{\mu^c(x) , \mu^c(y) \}$

$1-\mu(x\gamma y) \geq \max \{1-\mu(x) , 1-\mu(y) \}$

$\mu(x\gamma y) \leq 1- \max \{1-\mu(x) , 1-\mu(y)\}$

$\mu(x\gamma y) \leq \min \{\mu(x) , \mu(y)\}$

ii-Let $\forall$ $x,y,z\in M$ and $\alpha,\beta\in\Gamma$

$\mu^c(x\alpha y\beta z) \geq \max \{\mu^c(x) , \mu^c(z)\}$

$1-\mu(x\alpha y\beta z) \geq \max \{1-\mu(x) , 1-\mu(z)\}$

$\mu(x\alpha y\beta z) \leq 1- \max \{1-\mu(x) , 1-\mu(z)\}$

$\mu(x\alpha y\beta z) \leq \min \{\mu(x) , \mu(z)\}$

Hence μ is a fuzzy bi-Γ-ideal of M . ■

Theorem (2.11): A fuzzy set μ in a Γ-ideal of a Γ-semigroup M is an anti fuzzy bi-Γ-ideal of M if and only if the lower level set $L(\mu,t)$ is a bi-Γ-ideal in M when it is non empty .

Proof :

Suppose μ is an anti fuzzy bi-Γ-ideal of M

i- Let $x,y\in L(\mu,t)$ and $\gamma\in\Gamma$,

$\mu(x) \leq t$, $\mu(y)\leq t$ then

$\mu(x\gamma y) \leq \max\{ \mu(x) , \mu(y)\} \leq t$

$x\gamma y\in L(\mu,t)$, therefore

$L(\mu,t)\ \Gamma L(\mu,t)\subseteq L(\mu,t)$

Hence $L(\mu,t)$ is a sub Γ- semigroup

ii- Let $x,z\in L(\mu,t)$, $y\in M$ and $\alpha,\beta\in\Gamma$

$\mu(x) \leq t$ and $\mu(z) \leq t$

$\mu(x\alpha y\beta z) \leq \max\{ \mu(x) , \mu(z)\} \leq t$

$x\alpha y\beta z \in L(\mu,t)$

Hence, $L(\mu,t)$ is a bi-Γ- ideal in M

Conversely, suppose $L(\mu,t)$ is a bi-Γ- ideal in M

i- Let $x,y\in M$ and $\gamma\in\Gamma$,and $t=\max\{\mu(x) , \mu(y)\}$ then

$x,y \in L(\mu,t)$ and $x\gamma y \in L(\mu,t)$, $\mu(x\gamma y) \leq t = \max\{ \mu(x) , \mu(y)\}$

ii-Let $x,y,z\in M$ and $\alpha,\beta\in\Gamma$ and $t= \max\{\mu(x) , \mu(z)\}$ then

$x,y,z\in L(\mu,t)$, $x\alpha y\beta z\in L(\mu,t)$

$\mu(x\alpha y\beta z) \leq t= \max\{ \mu(x) , \mu(z)\}$

Therefore $L(\mu,t)\ \Gamma M\ \Gamma L(\mu,t)\subseteq L(\mu,t)$

hence , μ is an anti fuzzy bi-Γ-ideal of M . ■

Theorem (2.12): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup and the pair of mapping f: $M_1 \rightarrow M_2$, h : $\Gamma_1 \rightarrow \Gamma_2$ be a homomorphism if μ is an anti fuzzy bi-Γ_2-ideal of M_2,then $f^{-1}(\mu)$ is an anti fuzzy bi-Γ_1-ideal of M_1 .

Proof :

i- Let $x,y \in M_1$ and $\gamma \in \Gamma_1$,

$$f^{-1}(\mu)(x\gamma y)= \mu(f(x\gamma y))$$

$$= \mu(f(x)h(\gamma)f(y))$$

$$\leq \max\{\mu(f(x) , \mu(f(y))\}$$

$$= \max\{f^{-1}(\mu(x)), f^{-1}(\mu(y))\}$$

ii- Let $x,y,z \in M_1$ and $\alpha,\beta \in \Gamma_1$

$$f^{-1}(\mu)(x\alpha y\beta z) = \mu(f(x\alpha y\beta z))$$

$$= \mu(f(x)h(\alpha)f(y)h(\beta)f(z))$$

$$\leq \max \{\mu(f(x)) , \mu(f(z))\}$$

$$= \max \{f^{-1}(\mu(x)) ,f^{-1}(\mu(z))\}$$

Hence $f^{-1}(\mu)$ is an anti fuzzy bi-Γ_1-ideal of M_1 . ■

Definition (2.13): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup . The pair of mapping f: $M_1 \rightarrow M_2$, h : $\Gamma_1 \rightarrow \Gamma_2$ is said to be an anti homomorphism from (M_1,Γ_1) to(M_2,Γ_2) if $f(x\gamma y)=f(y)h(\gamma)f(x)$ $\forall$ $x,y \in M_1$ and $\gamma \in \Gamma_1$.

Theorem (2.14): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup and the pair of mapping f: $M_1 \rightarrow M_2$, h : $\Gamma_1 \rightarrow \Gamma_2$ be an anti homomorphism if μ is an anti fuzzy bi Γ_2-ideal of M_2,then $f^{-1}(\mu)$ is an anti fuzzy bi-Γ_1-ideal of M_1 .

Proof :

i- Let $x,y\in M_1$ and $\gamma\in\Gamma_1$,

$f^{-1}(\mu)(x\gamma y)= \mu(f(x\gamma y))$

$= \mu(f(y)h(\gamma)f(x))$

$\leq \max\{\mu(f(y) , \mu(f(x))\}$

$= \max\{f^{-1}(\mu(y)), f^{-1}(\mu(x))\}$

ii- Let $x,y,z\in M_1$ and $\alpha,\beta\in\Gamma_1$

$f^{-1}(\mu)(x\alpha y\beta z) = \mu(f(x\alpha y\beta z))$

$= \mu(f(z)h(\alpha)f(y)h(\beta)f(x))$

$\leq \max \{\mu(f(z)) , \mu(f(x))\}$

$= \max \{f^{-1}(\mu(z)) ,f^{-1}(\mu(x))\}$

Hence $f^{-1}(\mu)$ is an anti fuzzy bi-Γ_1-ideal of M_1 . ■

Theorem (2.15): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup and the pair of mapping f: $M_1\rightarrow M_2$, h : $\Gamma_1\rightarrow\Gamma_2$ be a homomorphism if μ is an anti fuzzy bi -Γ_2- ideal of M_2 and $L(\mu,t)$ is bi -Γ_2- ideal in M_2 , then $L(f^{-1}(\mu),t)$ is bi -Γ_1- ideal in M_1

Proof :

i- Let $x,y\in L(f^{-1}(\mu),t)$ and $\gamma\in\Gamma_1$,

$f^{-1}(\mu)(x)= \mu(f(x)) \leq t$, $f^{-1}(\mu)(y)= \mu(f(y)) \leq t$

$f^{-1}(\mu)(x\gamma y)= \mu(f(x\gamma y))$

$= \mu(f(y)h(\gamma)f(x))$

$\leq \max\{\mu(f(y) , \mu(f(x))\} \leq t$

$x\gamma y \in L(f^{-1}(\mu),t)$

ii- Let $x,y,z \in L(f^{-1}(\mu),t)$ and $\alpha,\beta \in \Gamma_1$,

$f^{-1}(\mu)(x)= \mu(f(x)) \leq t$, $f^{-1}(\mu)(z)= \mu(f(z)) \leq t$

$f^{-1}(\mu)(x\alpha y\beta z)= \mu(f(x\alpha y\beta z))$

$= \mu(f(x)h(\alpha)f(y)h(\beta)f(z))$

$\leq \max\{\mu(f(x) , \mu(f(z))\} \leq t$

$x\alpha y\beta z \in L(f^{-1}(\mu),t)$

Hence, $L(f^{-1}(\mu),t)$ is bi -Γ_1-ideal in M_1 . ■

We can prove the following theorem in a similar manner

Theorem (2.16): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup and the pair of mapping f: $M_1 \rightarrow M_2$, h : $\Gamma_1 \rightarrow \Gamma_2$ be an anti homomorphism if μ is an anti fuzzy bi -Γ_2- ideal of M_2 and $L(\mu,t)$ is bi -Γ_2- ideal in M_2 , then $L(f^{-1}(\mu),t)$ is bi -Γ_1- ideal in M_1 .

Theorem (2.17): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup and the pair of mapping f: $M_1 \rightarrow M_2$, h : $\Gamma_1 \rightarrow \Gamma_2$ be a homomorphism if μ is an anti fuzzy bi -Γ_1- ideal of M_1 which has the inf property and $L(\mu,t)$ is bi -Γ_1- ideal in M_1 , then $L(f(\mu),t)$ is bi -Γ_2- ideal in $f(M_1)$.

Proof :

Let $f(x)$, $f(y)$, $f(z) \in L(f(\mu),t)$

$(f(\mu))(f(x)) \leq t$, $(f(\mu))(f(y)) \leq t$, $(f(\mu))(f(z)) \leq t$

let $x_o \in f^{-1}(f(x))$, $y_o \in f^{-1}(f(y))$, $z_o \in f^{-1}(f(z))$, be such that

$$\mu(x_0) = \inf_{g \in f^{-1}(f(x))} \mu(g), \ \mu(y_0) = \inf_{g \in f^{-1}(f(y))} \mu(g), \quad \mu(z_0) = \inf_{g \in f^{-1}(f(z))} \mu(g)$$

i- Let $\gamma \in \Gamma_2$

$$(f(\mu))(f(x)\gamma f(y)) = \inf_{w \in f^{-1}(f(x)\gamma f(y))} \mu(w)$$

$\leq \max \{ \mu(x_o) , \mu(y_o)\}$

$= \max \{(f(\mu))(f(x)) , (f(\mu))(f(y))\} \leq t$

ii- Let $\alpha,\beta\in\Gamma_2$

$$(f(\mu))(f(x)\alpha f(y)\beta f(z)) = \inf_{w\in f^{-1}(f(x)\alpha f(y)\beta f(z))} \mu(w)$$

$$\leq \max \{ \mu(x_o) , \mu(z_o)\}$$

$$= \max \{(f(\mu))(f(x)) , (f(\mu))(f(z))\} \leq t$$

Hence , $L(f(\mu),t)$ is bi-Γ_2-ideal of $f(\mu)$. ■

We can prove the following theorem in a similar manner

Theorem (2.18): Let M_1 be Γ_1-semigroup and M_2 be Γ_2-semigroup and the pair of mapping f: $M_1\rightarrow M_2$, h : $\Gamma_1\rightarrow\Gamma_2$ be an anti homomorphism if μ is an anti fuzzy bi -Γ_1- ideal of M_1 which has the inf property and $L(\mu,t)$ is bi -Γ_1- ideal in M_1 , then $L(f(\mu),t)$ is bi -Γ_2- ideal in $f(M_1)$.

Definition (2.19): A fuzzy set μ is called anti fuzzy interior Γ-ideal of a Γ-semigroup M if

$\mu(x\alpha y\beta z) \leq \mu(y)$ $\forall$ x,y,z$\in$M and $\alpha,\beta\in\Gamma$.

Theorem (2.20): Let A be an interior of a Γ-semigroup M then τ_A is anti fuzzy interior Γ- ideal of M , where τ_A defined by

$$\tau_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

Proof :

Let x,y,z $\in$A and $\alpha,\beta\in\Gamma$ then $x\alpha y\beta z\in M\Gamma A\Gamma M\subseteq A$

i-if $y\in A$ then $\tau_A(y) =0$

Thus , $\tau_A(x\alpha y\beta z) \leq \tau_A(y)$

ii- if $y\notin A$ then $\tau_A(y) =1$

Thus , $\tau_A(x\alpha y\beta z) \leq \tau_A(y)$

Hence , τ_A is anti fuzzy interior Γ- ideal of M . ■

Definition (2.21): Let μ and σ be two fuzzy sets of a nonempty set S then

$(\mu \times\sigma)(x,y) = \max \{\mu(x) , \sigma(y)\}\ \forall\ x,y \in S$ is called anti cartesian product of μ and σ .

Theorem (2.22): If μ and σ are anti fuzzy bi-ideal of a semigroups S_1 and S_2 respectively ,then $\mu \times\sigma$ is anti fuzzy bi-ideal of $S_1\times S_2$.

Proof :

Let (a,b) ,(c,d) and (e,f) $\in S_1\times S_2$

$(\mu \times\sigma)((a,b)(c,d)(e,f)) = (\mu \times\sigma)(ace,bdf)$

$= \max \{\mu (ace) ,\sigma(bdf)\}$

$\leq \max \{\max \{\mu (a) ,\mu(e)\} , \max\{ \sigma(b), \sigma(f)\}\}$

$\leq \max \{\max \{\mu (a) , \sigma(b) \} , \max\{\mu(e), \sigma(f)\}\}$

$= \max \{(\mu \times\sigma)(a,b) , (\mu \times\sigma)(e,f)\}$

Hence $\mu \times\sigma$ is anti fuzzy bi-ideal of $S_1\times S_2$. ■

Theorem (2.23): Let μ is an anti fuzzy bi-Γ_1-ideal of a Γ-semigroup M_1 and σ is an anti fuzzy bi-Γ_2-ideal of a Γ-semigroup M_2 then $\mu \times\sigma$ is anti fuzzy bi-$\Gamma_1\times\Gamma_2$ ideal of $M_1\times M_2$

Proof :

i- Let (a,b) ,(c,d) $\in M_1\times M_2$ and $(\gamma_1,\gamma_2) \in\Gamma_1\times\Gamma_2$

$(\mu \times\sigma)((a,b)(\gamma_1,\gamma_2)(c,d)) = (\mu \times\sigma)(a\, \gamma_1 c,b\gamma_2 d))$

$= \max \{\mu (a\, \gamma_1 c) ,\sigma(b\gamma_2 d)\}$

$\leq \max \{\max \{\mu(a) , \mu(c)\} , \max\{\sigma(b), \sigma(d)\}\}$

$\leq \max \{\max \{\mu(a) , \sigma(b)\} , \max\{\mu(c), \sigma(f)\}\}$

$= \max \{(\mu \times\sigma)(a,b) , (\mu \times\sigma)(c,d)\}$

ii- Let (a,b) ,(c,d),(e,f) $\in M_1\times M_2$ and (α_1,α_2) , $(\beta_1,\beta_2) \in\Gamma_1\times\Gamma_2$

$$(\mu\times\sigma)((a,b)(\alpha_1,\alpha_2)(c,d)(\beta_1,\beta_2)(e,f)) = (\mu\times\sigma)(a\alpha_1c\beta_1e,b\alpha_2d\beta_2f))$$

$$= \max\{\mu(a\alpha_1c\beta_1e),\sigma(b\alpha_2d\beta_2f)\}$$

$$\leq \max\{\max\{\mu(a),\mu(e)\},\max\{\sigma(b),\sigma(f)\}\}$$

$$\leq \max\{\max\{\mu(a),\sigma(b)\},\max\{\mu(e),\sigma(f)\}\}$$

$$= \max\{(\mu\times\sigma)(a,b),(\mu\times\sigma)(e,f)\}$$

Hence $\mu\times\sigma$ is anti fuzzy bi- $\Gamma_1\times\Gamma_2$ -ideal of $M_1\times M_2$. ■

Theorem (2.24): Let μ , σ be fuzzy subset of a non empty set M and $\mu\times\sigma$ is an anti fuzzy bi-ideal of M×M such that either $\mu(a)\leq\sigma(b)$ or $\sigma(b)\leq\mu(a)$ $\forall a,b\in M$ then either μ is anti fuzzy bi- ideal of M or σ anti fuzzy bi-ideal of M

Proof :

$(\mu\times\sigma)(x,y) = \max\{\mu(x),\sigma(y)\}$ and

$(\mu\times\sigma)((x,y)(w,z)) =(\mu\times\sigma)(xw,yz) = \max\{\mu(xw),\sigma(yz)\}$ ……….(1)

also

$(\mu\times\sigma)((x,y)(w,z)) \leq \max\{(\mu\times\sigma)(x,y),(\mu\times\sigma)(w,z)\}$

$= \max\{\max\{\mu(x),\sigma(y)\},\max\{\mu(w),\sigma(z)\}\}$ ………..(2)

From (1) and (2) ,we obtain

$\max\{\mu(xw),\sigma(yz)\} \leq \max\{\max\{\mu(x),\sigma(y)\},\max\{\mu(w),\sigma(z)\}\}$ ………..(3)

i- Suppose $\mu(a)\leq\sigma(b)$ $\forall a,b\in M$

From (3) we obtain

$\sigma(yz)\leq\max\{\sigma(y),\sigma(z)\}$ $\forall y,z\in M$

Hence σ anti fuzzy bi-ideal of M

ii- Suppose $\sigma(b) \leq \mu(a)\ \forall\ a,b \in M$

From (3) we obtain

$\mu(xw)$ max $\{\mu(x)\ ,\ \sigma(w)\}\ \forall\ x,w \in M$

Hence μ anti fuzzy bi-ideal of M . ■

Theorem (2.25): Let μ and σ be fuzzy subset of a non empty set M and

$\mu \times \sigma$ is an anti fuzzy bi-Γ- ideal of $M \times M$ such that either $\mu(a) \leq \sigma(b)$ or $\sigma(b) \leq \mu(a)\ \forall a,b \in M$ then either μ is anti fuzzy bi-Γ- ideal of M or σ anti fuzzy bi-Γ-ideal of M

Proof :

$(\mu \times \sigma)(x,y) = \max\{\mu(x)\ ,\ \sigma(y)\}$ and

$(\mu \times \sigma)((x,y)(\gamma_1,\gamma_2)(w,z)) = (\mu \times \sigma)(x\ \gamma_1 w, y\gamma_2 z))$

$= \max\{\mu\ (x\gamma_1 w)\ ,\sigma(y\gamma_2 z)\}$ …………(1)

also

$(\mu \times \sigma)((x,y)(\gamma_1,\gamma_2)(w,z)) \leq \max\{(\mu \times \sigma)(x\ ,y)\ ,\ (\mu \times \sigma)(w,z)$

$= \max\{\max\{\mu\ (x)\ ,\sigma(y)\}\ ,\ \max\{\mu\ (w)\ ,\sigma(z)\}$ ……..(2)

From (1) and (2) ,we obtain

$\max\{\mu\ (x\gamma_1 w)\ ,\sigma(y\gamma_2 z)\} \leq \max\{\max\{\mu\ (x)\ ,\sigma(y)\}\ ,\ \max\{\mu\ (w)\ ,\sigma(z)\}$ ………(3)

$(\mu \times \sigma)((x,y)\ (\alpha_1,\alpha_2)(w,z)(\beta_1,\beta_2)\ (g,h)) = (\mu \times \sigma)(x\alpha_1 w\beta_1 g\ ,\ y\alpha_2 z\beta_2 h))$

$= \max\{\mu\ (x\alpha_1 w\beta_1 g)\ ,\sigma(\ y\alpha_2 z\beta_2 h)\}$ …..(4)

also

$(\mu \times \sigma)((x,y)\ (\alpha_1,\alpha_2)(w,z)(\beta_1,\beta_2)\ (g,h)) \leq \max\{(\mu \times \sigma)(x,\ y)\ ,\ (\mu \times \sigma)(g,\ h)\}$

$= \max\{\max\{\mu(x)\ ,\ \sigma(y)\}\ ,\ \max\{\mu(g),\ \sigma(h)\}\}$ …………..(5)

From (4) and (5) ,we obtain

$\max\{\mu(x\alpha_1 w\beta_1 g),\sigma(y\alpha_2 z\beta_2 h)\} \leq \max\{\max\{\mu(x),\sigma(y)\},\ \max\{\mu(g),\sigma(h)\}\}$ ..….(6)

Because M is a Γ-semigroup then $x\alpha_1 w$, $y\alpha_2 z \in M$ and $x\alpha_1 w\beta_1 g$, $y\alpha_2 z\beta_2 h \in M$ i- Suppose $\mu(a)\leq \sigma(b)\ \forall\ a,b\in M$

From (3) we obtain

$\sigma(y\gamma_2 z) \leq \max \{\sigma (y) , \sigma(z)\}\ \forall\ y,z \in M , \gamma\in\Gamma$

Also from (6), we obtain

$\sigma(y\alpha_2 z\beta_2 h) \leq \max \{\sigma (y) , \sigma(h)\}\ \forall\ y,z,h \in M , \alpha_2,\beta_2\in\Gamma$

Hence σ anti fuzzy bi-Γ-ideal of M

ii- Suppose $\sigma(b)\leq \mu(a)\ \forall\ a,b\in M$

From (3) we obtain

$\mu(x\gamma_1 w) \max \{\mu(x) , \mu (w)\}\ \forall\ x,w\in M , \gamma_1\in\Gamma$

Also from (6), we obtain

$\mu(x\alpha_1 w\beta_1 g) \leq \max\{\max\{\mu(x),\mu(g)\}\ \forall\ x,w,g \in M , \alpha_1,\beta_1\in\Gamma$

Hence μ anti fuzzy bi-Γ-ideal of M . ■

Definition (2.26): A Q-fuzzy set μ of a semigroup S is called an anti Q-fuzzy subsemigroup if $\mu(xy,q) \leq \max \{ \mu(x,q) , \mu(y,q)\}$, $\forall\ x,y\in S$, $q\in Q$.

Definition (2.27): An anti Q-fuzzy subsemigroup μ of a semigroup S is called anti Q-fuzzy bi-ideal of S if $\mu(xyz,q) \leq \max \{ \mu(x,q) , \mu(z,q)\}$, $\forall\ x,y,z\in S$, $q\in Q$.

Definition (2.28) : A Q-fuzzy set μ of a Γ -semigroup is called an anti Q-fuzzy Γ-subsemigroup of M if $\mu(x\gamma y,q) \leq \max \{ \mu(x,q) , \mu(y,q)\}$, $\forall\ x,y\in M$, $q\in Q$ and $\gamma\in\Gamma$.

Definition (2.29): A Q-fuzzy set μ of a Γ-semigroup M is called anti Q-fuzzy Γ- ideal of M if $\mu(x\alpha y\beta z,q) \leq \mu(z,q)$ and $\mu(x\alpha y\beta z,q) \leq \mu(x,q)$,$\forall\ x,y,z\in M$, $q\in Q$ and $\alpha,\beta\in\Gamma$. [3]

Definition (2.30): A Q-fuzzy set μ of a Γ-semigroup M is called anti Q-fuzzy bi -Γ-ideal of M if

1-$\mu(x\gamma y,q) \leq \max\{\mu(x,q), \mu(y)\}$, $\forall$ $x,y\in M$, $q\in Q$ and $\gamma\in\Gamma$.

2-$\mu(x\alpha y\beta z,q) \leq \max\{\mu(x,q), \mu(z,q)\}$, $\forall$ $x,y,z\in M$, $q\in Q$ and $\alpha,\beta\in\Gamma$.

Theorem (2.31): Let μ be Q- fuzzy set of a Γ-semigroup M then μ is Q- fuzzy bi-Γ-ideal of M if and only if μ^c is an anti Q-fuzzy bi-Γ-ideal of M.

Proof :

suppose μ is Q- fuzzy bi-Γ-ideal of M

i- Let $x,y\in M$, $q\in Q$ and $\gamma\in\Gamma$,

$\mu(x\gamma y,q) \geq \min\{\mu(x,q), \mu(y,q)\}$

$1-\mu^c(x\gamma y,q) \geq \min\{1-\mu^c(x,q), 1-\mu^c(y,q)\}$

$\mu^c(x\gamma y,q) \leq 1-\min\{1-\mu^c(x,q), 1-\mu^c(y,q)\}$

$\mu^c(x\gamma y,q) \geq \max\{\mu^c(x,q), \mu^c(y,q)\}$

ii-Let $x,y,z\in M$, $q\in Q$ and $\alpha,\beta\in\Gamma$

$\mu(x\alpha y\beta z,q) \geq \min\{\mu(x,q), \mu(z,q)\}$

$1-\mu^c(x\alpha y\beta z,q) \geq \min\{1-\mu^c(x,q), 1-\mu^c(z,q)\}$

$\mu^c(x\alpha y\beta z,q) \leq 1-\min\{1-\mu^c(x,q), 1-\mu^c(z,q)\}$

$\mu^c(x\alpha y\beta z,q) \leq \max\{\mu^c(x,q), \mu^c(z,q)\}$

Hence μ^c is an anti fuzzy bi-Γ-ideal of M.

Conversely suppose μ^c is an anti fuzzy bi-Γ-ideal of M .

i- Let $x,y\in M$, $q\in Q$ and $\gamma\in\Gamma$,

$\mu^c(x\gamma y,q) \geq \max\{\mu^c(x,q), \mu^c(y,q)\}$

$1-\mu(x\gamma y,q) \geq \max\{1-\mu(x,q), 1-\mu(y,q)\}$

$\mu(x\gamma y,q) \leq 1-\max\{1-\mu(x,q), 1-\mu(y,q)\}$

$\mu(x\gamma y,q) \leq \min\{\mu(x,q), \mu(y,q)\}$

ii-Let $\forall$ $x,y,z\in M$, $q\in Q$ and $\alpha,\beta\in\Gamma$

$\mu^c(x\alpha y\beta z,q) \geq \max\{\mu^c(x,q), \mu^c(z,q)\}$

$1-\mu(x\alpha y\beta z,q) \geq \max\{1-\mu(x,q), 1-\mu(z,q)\}$

$\mu(x\alpha y\beta z,q) \leq 1- \max\{1-\mu(x,q), 1-\mu(z,q)\}$

$\mu(x\alpha y\beta z,q) \leq \min\{\mu(x,q), \mu(z,q)\}$

Hence μ is a Q-fuzzy bi-Γ-ideal of M . ■

Theorem (2.32): A Q-fuzzy set μ in a Γ-ideal of a Γ-semigroup M is an anti Q- fuzzy bi-Γ-ideal of M if and only if the lower level set L(μ,t) is a bi-Γ-ideal in M when it is non empty .

Proof :

Suppose μ is an Q-anti fuzzy bi-Γ-ideal of M

i- Let $x,y\in L(\mu,t)$, $q\in Q$ and $\gamma\in\Gamma$,

$\mu(x,q) \leq t$, $\mu(y,q)\leq t$

$\mu(x\gamma y,q) \leq \max\{\mu(x,q), \mu(y,q)\} \leq t$

$x\gamma y\in L(\mu,t)$

ii- Let $x,y,z\in L(\mu,t)$, $q\in Q$ and $\alpha,\beta\in\Gamma$

$\mu(x,q) \leq t$, $\mu(y,q)\leq t$ and $\mu(z,q) \leq t$

$\mu(x\alpha y\beta z,q) \leq \max\{\mu(x,q), \mu(z,q)\} \leq t$

$x\alpha y\beta z \in L(\mu,t)$

Hence, L(μ,t) is a bi-Γ- ideal in M

Conversely, suppose L(μ,t) is a bi-Γ- ideal in M

i- Let $x,y\in M$, $q\in Q$ and $\gamma\in\Gamma$,and $t=\max\{\mu(x,q), \mu(y,q)\}$ then

$x,y \in L(\mu,t)$ and $x\gamma y \in L(\mu,t)$, $\mu(x\gamma y,q) \leq t = \max\{\mu(x,q), \mu(y,q)\}$

ii-Let $x,y,z\in M$, $q\in Q$ and $\alpha,\beta\in\Gamma$ and $t= \max\{\mu(x,q), \mu(z,q)\}$ then

$x,y,z\in L(\mu,t)$, $x\alpha y\beta z\in L(\mu,t)$

$\mu(x\alpha y\beta z,q) \le t = \max\{\mu(x,q), \mu(z,q)\}$

hence , μ is an anti Q-fuzzy bi-Γ-ideal of M . ■

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ضد مثاليات كاما الثنائية الضبابية في شبه الزمرة كاما

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الخلاصه :-

في هذا البحث عرفنا البنى الجبرية ضد مثاليات كاما الثنائية الضبابية و ضد مثاليات كاما الثنائية الضبابية من النوع Q ودراسة بعض الخواص الأساسية لهما .