# Farctional Calculus Operators of a new Class of Univalent Functions With Negative Coefficients of Complex Order

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## Abstract:-

In the present paper, we introduce a new class of univalent function with negative coefficients of complex order, by using fractional differ – integral operators studied recently by authors. Coefficient characterization, growth and distortion, radii of starlikeness and convexity , closure theorems of this class of functions are studied. We also study some integral operators on our class $G^{\mu,\nu,\eta}(\tau, A, B)$.

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## 1-Introduction :-

Let S($n$) denote the class of functions of the from :

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, n \in IN = \{1,2,3,...\}. \qquad (1.1)$$

That are analytic in the unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$, let W($n$) denot the subclass of a consisting of analytic and univalent functions $f(z)$ in the unit disk $U$. Further T($n$) denote subclass of S($n$) consisting of functions $f(z)$ of the form :

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, (a_k \geq o, n \in IN = \{1,2,3,...\}). \qquad (1.2)$$

A functions of $f(z) \in$ T($n$) is said to be in the class $G^{\mu,\nu,\eta}(\tau, A, B)$ if and only if

$$\left| \frac{(H_{0,z}^{\mu,\nu,\eta} f(z))' - 1}{(A-B)\tau + B((H_{0,z}^{\mu,\nu,\eta} f(z))' - 1} \right| < 1. \qquad (1.3)$$

For $-1 \leq B < A \leq 1, \tau \in C/\{0\}, z \in U, -\infty < \mu < 1, \nu < 2, \eta \in IR$

**<u>KeyWords:</u> Fractional Calculus Operator, Univalent Function, Coefficient Characterization, Distortion Theorem, Radii of Starlikeness, Closure Theorem , Integral Operator.**

and $H_{0,z}^{\mu,v,\eta}$ is the fractional differintegral operator of order $\mu$ $(-\infty < \mu < 1)$ see [1]. For this operator

if $H_{0,z}^{\mu,v,\eta} : W(n) \to W(n), then$ (1.4)

$$H_{0,z}^{\mu,v,\eta} f(z) = z - \sum_{k=n+1}^{\infty} R_k(\mu,v,\eta) a_k z^k, \tag{1.5}$$

$(a_k \geq 0, n \in IN = \{1,2,3,...\}, z \in U),$

where $R_k(\mu,v,\eta) = G(\mu,v,\eta) M(\mu,v,\eta,k)$ (1.6)

and

$$G(\mu,v,\eta) = \frac{(1-v)(1-\mu+\eta)}{(1-v+\eta)}, M(\mu,v,\eta,k) = \frac{\Gamma(k+1)(1-v+\eta)_k}{(1-v)_k (1-\mu+\eta)_k} \tag{1.7}$$

Throughout the paper

$$(a)_n = \prod_{k=1}^{n} (a+k-1) \text{ or } = a(a+1)(a+2)...(a+n-1) \tag{1.8}$$

is the factorial function , or if a > 0 , then

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad \text{(where } \Gamma \text{ is Euler`s Gamma function).} \tag{1.9}$$

For $z \neq 0$ , (1.5) may be expressed as

$$H_{0,z}^{\mu,v,\eta} f(z) = \begin{cases} \frac{\Gamma(2-v)\Gamma(2-\mu+n)}{\Gamma(2-v+\eta)} z^v J_{0,z}^{\mu,v,\eta} f(z); 0 \leq \mu < 1 \\ \frac{\Gamma(2-v)\Gamma(2-\mu+n)}{\Gamma(2-v+\eta)} z^v I_{0,z}^{-\mu,v,\eta} f(z); -\infty < \mu < 0 \end{cases} \tag{1.10}$$

where $J_{0,z}^{\mu,v,\eta} f(z)$ is the fractional derivative operator of order $\mu$ $(0 \leq \mu < 1)$ , while $I_{0,z}^{-\mu,v,\eta} f(z)$ is the fractional integral operator of order - $\mu$ $(-\infty < \mu < 0)$ introduced and studied by ([5],[6]).

It may be worth noting that, by choosing $-\infty < \mu = v < 1$ the operator $H_{0,z}^{\mu,v,\eta} f(z)$ becomes

$$H_{0,z}^{\mu,v,\eta} f(z) = H_z^{\mu} f(z) = \Gamma(2-\mu) z^{\mu} D_z^{\mu} f(z) \tag{1.11}$$

Where $D_z^{\mu} f(z)$ is respectively, the fractional integral operator of order $-\mu$ $(-\infty < \mu < 0)$ and fractional derivative operator of order $\mu(0 \leq \mu < 1)$ considered by [4] and defined by [3]. Further if $\mu = v = 0$, then

$$H_{0,z}^{0,0,\eta} f(z) = f(z) \tag{1.12}$$

and for $\mu \to 1^-$ and $v = 1$

$$\lim_{\mu \to 1^-} H_{0,z}^{\mu,1,\eta} f(z) = zf'(z) \tag{1.13}$$

## 2. Coefficient characterization Theorem:

We now investigate the coefficient characterization theorem for the function *f*(*z*) to belong to the class $G^{\mu,v,\eta}(\tau, A, B)$, there by , obtaining the coefficient bounds.

Theorem 1: A function $f(z)$ defined by (1.2) is in the class $G^{\mu,v,\eta}(\tau,A,B)$ if and only if

$$\sum_{k=n+1}^{\infty} k(1+B)R_k(\mu,v,\eta)a_k \le (A-B)|\tau| \qquad (2.1)$$

under the parametric restraints given by (1.3). The result is sharp.

Proof: Let the inequality (2.1) holds true. For $|z|=1$ , we have

$$\left|\left(H_{0,z}^{\mu,v,\eta} f(z)\right)' -1\right| - \left|(A-B)\tau + B\left(\left(H_{0,z}^{\mu,v,\eta} f(z)\right)' -1\right)\right|$$

$$= \left|-\sum_{k=n+1}^{\infty} kR_k(\mu,v,\eta)a_k z^{k-1}\right| - \left|(A-B)\tau + B\left(\sum_{k=n+1}^{\infty} kR_k(\mu,v,\eta)a_k z^{k-1}\right)\right|$$

$$\le \sum_{k=n+1}^{\infty} kR_k(\mu,v,\eta)a_k - (A-B)|\tau| + \sum_{k=n+1}^{\infty} BkR_k(\mu,v,\eta)a_k$$

$$= \sum_{k=n+1}^{\infty} k(1+B)R_k(\mu,v,\eta)a_k - (A-B)|\tau| \le 0. \qquad (by(2.1))$$

Hence by the principle of maximum modulus , $f(z)$ belongs to the class $G^{\mu,v,\eta}(\tau,A,B)$. Conversely, assume that $f(z)$ is defined by (2.1) and $f(z)\in G^{\mu,v,\eta}(\tau,A,B)$. Then by using (1.5) in (1.3), we get

$$\left|\frac{(H_{0,z}^{\mu,v,\eta} f(z))' -1}{(A-B)\tau - B\left(\left(H_{0,z}^{\mu,v,\eta} f(z)\right)' -1\right)}\right| = \left|\frac{-\sum_{k=n+1}^{\infty} kR_k(\mu,v,\eta)a_k z^{k-1}}{(A-B)\tau - \sum_{k=n+1}^{\infty} BkR_k(\mu,v,\eta)a_k z^{k-1}}\right|$$

Since $|\mathrm{Re}(z)| \le |z|$, therefore , we have

$$\mathrm{Re}\left\{\frac{\sum_{k=n+1}^{\infty} kR_k(\mu,v,\eta)a_k z^{k-1}}{(A-B)|\tau| - \sum_{k=n+1}^{\infty} BkR_k(\mu,v,\eta)a_k z^{k-1}}\right\}$$

$$\le \mathrm{Re}\left\{\frac{\sum_{k=n+1}^{\infty} kR_k(\mu,v,\eta)a_k z^{k-1}}{(A-B)\tau - \sum_{k=n+1}^{\infty} BkR_k(\mu,v,\eta)a_k z^{k-1}}\right\} < 1 \qquad (2.2)$$

Now, letting $z\to 1^-$, though real values in (2.2), we at once obtain (2.1) and theorem is completely proved.

## 3. Growth and Distortion Theorem:

In the next theorem , we concentrate upon getting growth and distortion theorem for the fractional differintegral operator of order $\mu$ of the function $f(H_{0,z}^{\mu,v,\eta} f(z))$.

Theorem 2: Let $f(z)\in G^{\mu,v,\eta}(\tau,A,B)$. Then for $|z|\le r<1$, we have

$$r-r^{n+1}\frac{(A-B)|\tau|}{(n+1)(1+B)}\le\left|H_{0,z}^{\mu,v\eta}f(z)\right|\le r+r^{n+1}\frac{(A+B)|\tau|}{(n+1)(1+B)} \quad (3.1)$$

$$1-r^{n}\frac{(A-B)|\tau|}{(1+B)}\le\left|\left(H_{0,z}^{\mu,v,\eta}f(z)\right)'\right|\le 1+r^{n}\frac{(A-B)|\tau|}{(1+B)} \quad . \quad (3.2)$$

The bounds (3.1) and (3.2) are sharp.

Proof : Under the assumption and conditions of validity for the theorem it is obvious that the function

$$R_k(\mu,v,\eta)=\frac{(1-v)(1-\mu+\eta)\Gamma(k+1)(1-v+n)_k}{(1-v+\eta)(1-v)_k(1-\mu+\eta)_k}$$

Is increasing for $k\ge n+1$ . Indeed

$$R_{k+1}(\mu,v,\eta)-R_k(\mu,v,\eta)=R_n(\mu,v,\eta)\left\{\frac{(k+1)(1-v+\eta+k)}{(1-v+k)(1-\mu+\eta+k)}-1\right\}$$

$$=R_k(\mu,v,\eta)\left\{\frac{\mu(k+1)-v(\mu-\eta)}{(1-v+k)(1-\mu+\eta+k)}\right\}>0$$

or $R_{n+2}(\mu,v,\eta)-R_{n+1}(\mu,v,\eta)$

$$=R_{n+1}(\mu,v,\eta)\left\{\frac{\mu(n+2)-v(\mu-\eta)}{(2-v+n)(2-\mu+\eta+n)}\right\}>0 .$$

Thus on account of (2.1) , we get

$$(n+1)(1+B)R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty}a_k\le\sum_{k=n+1}^{\infty}k(1+B)R_k(\mu,v,\eta)a_k\le(A-B)|\tau|,$$

then $\sum_{k=n+1}^{\infty}a_k\le\dfrac{(A-B)|\tau|}{(n+1)(1+B)R_{n+1}(\mu,v,\eta)}$.

Hence $\left|H_{0,z}^{\mu,v,\eta}f(z)\right|\le|z|+|z|^{n+1}R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty}a_k$

$$\le r+r^{n+1}R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty}a_k$$

$$\le r+r^{n+1}\frac{(A-B)|\tau|}{(n+1)(1+B)}$$

and

$$\left|H_{0,z}^{\mu,v,\eta}f(z)\right|\ge|z|-|z|^{n+1}R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty}a_k$$

$$\ge r-r^{n+1}R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty}a_k$$

$$\ge r-r^{n+1}\frac{(A-B)|\tau|}{(n+1)(1+B)} ,$$

thus (3.1) is true. Further

$$\left|\left(H_{0,z}^{\mu,v,\eta} f(z)\right)'\right| \le 1+(n+1)r^n R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty} a_k$$
$$\le 1+r^n \frac{(A-B)|\tau|}{(1+B)} \quad ,$$

and also

$$\left|\left(H_{0,z}^{\mu,v,\eta} f(z)\right)'\right| \ge 1-(n+1)r^n R_{n+1}(\mu,v,\eta)\sum_{k=n+1}^{\infty} a_k$$
$$\ge 1-r^n \frac{(A-B)|\tau|}{(1+B)} \quad .$$

Finally , we can prove that the bounds in (3.1) and (3.2)are sharp by taking the function

$$f(z)=z-\frac{(A-B)|\tau|}{(n+1)(1+B)R_{n+1}(\mu,v,\eta)} z^{n+1} \quad .$$

This complete the proof of the theorem .

## 4. Radii of Starlikeness and Convexity :

In the next theorems , we have studied the radii of starlikeness and convexity for the class $G^{\mu,v,\eta}(\tau,A,B)$.

Theorem 3 : Let $f(z)$ be in the class $G^{\mu,v,\eta}(\tau,A,B)$. Then $H_{0,z}^{\mu,v,\eta} f(z)$ is starlike of order $\varepsilon(0\le\varepsilon<1)$ in $|z|<r_1(\tau,A,B,\varepsilon)$ , where

$$r_1\ (\tau,A,B,\varepsilon)=\inf_k\left\{\frac{(1-\varepsilon)k(1+B)}{(k-\varepsilon)(A-B)|\tau|}\right\}^{\frac{1}{k-1}}, k\ge n+1. \qquad (4.1)$$

Proof: It is sufficient to show that $\left|\frac{z\left(H_{0,z}^{\mu,v,\eta} f(z)\right)'}{H_{0,z}^{\mu,v,\eta} f(z)}-1\right|\le 1-\varepsilon$

for $|z|<r_1\ (\tau,A,B,\varepsilon)$, but we have

$$\left|\frac{z(H_{0,z}^{\mu,v,\eta} f(z))'}{H_{0,z}^{\mu,v,\eta} f(z)}-1\right|=\left|\frac{-\sum_{k=n+1}^{\infty}(k-1)R_k(\mu,v,\eta)a_k z^k}{z-\sum_{k=n+1}^{\infty}R_k(\mu,v,\eta)a_k z^k}\right|\le\frac{\sum_{k=n+1}^{\infty}(k-1)R_k(\mu,v,\eta)a_k|z|^k}{z-\sum_{k=n+1}^{\infty}R_k(\mu,v,\eta)a_k|z|^k}.$$

Thus $\left|\frac{z(H_{0,z}^{\mu,v,\eta} f(z))'}{H_{0,z}^{\mu,v,\eta} f(z)}-1\right|\le 1-\varepsilon$ holds true if

$$\sum_{k=n+1}^{\infty}\frac{(k-\varepsilon)R_k(\mu,v,\eta)a_k|z|^{n-1}}{(1-\varepsilon)}\le 1. \qquad (4.2)$$

From Theorem 1 , since f(z) $\in G^{\mu,v,\eta}(\tau,A,B)$, we obtain

$$\sum_{k=n+1}^{\infty}\frac{k(1+B)R_k(\mu,v,\eta)a_k}{(A-B)|\tau|}\le 1. \qquad (4.3)$$

By using (4.3), so (4.2) will be true if

$$\frac{(k-\varepsilon)R_n(\mu,v,\eta)a_k|z|^{k-1}}{1-\varepsilon}\le\frac{k(1+B)R_k(\mu,v,\eta)a_k}{(A-B)|\tau|}$$

or equivalently $|z|^{k-1}\le\frac{(1-\varepsilon)k(1+B)}{(k-\varepsilon)(A-B)|\tau|}$ .

Hence

$$|z|\le\left\{\frac{(1-\varepsilon)k(1+B)}{(k-\varepsilon)(A-B)|\tau|}\right\}^{\frac{1}{k-1}},k\ge n+1$$

and this complete the proof.

Theorem 4: Let $f(z)$ be in the class $G^{\mu,v,\eta}(\tau,A,B)$. Then $H_{0,z}^{\mu,v,\eta}f(z)$ is convex of order $\varepsilon$ for $o\le\varepsilon<1$ in $|z|<r_2(\tau,A,B,\varepsilon)$ , where

$$r_2(\tau,A,B,\varepsilon)=\inf_k\left\{\frac{(1-\varepsilon)(1+B)}{(k-\varepsilon)(A-B)|\tau|}\right\}^{\frac{1}{k-1}},k\ge n+1. \qquad (4.4)$$

Proof of Theorem 4 is similar to that the Theorem 3 and hence details are omitted.

5.Clousre Theorems :

Theorem 5: Let $f_1(z), f_2(z),..., f_m(z)$ defined by

$$f_j(z)=z-\sum_{k=n+1}^{\infty}a_{k,j}z^k,(j=1,2,...,m)$$

Be in the class $G^{\mu,v,\eta}(\tau,A,B)$ .Then arithmetic mean of $f_j(z)$ ($j$=1,2,…,$m$) defined by

$$h(z)=\frac{1}{m}\sum_{j=1}^{m}f_j(z) \qquad (5.1)$$

is also in the class $G^{\mu,v,\eta}(\tau,A,B)$ .

Proof: By (5.1) we can write

$$h(z)=\frac{1}{m}\sum_{j=1}^{m}\left(z-\sum_{k=n+1}^{\infty}a_{k,j}z^k\right)=z-\sum_{k=n+1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}a_{k,j}\right)z^k.$$

Since $f_j(z)\in G^{\mu,v,\eta}(\tau,A,B)$ for every $j$=1,2,…,$m$, so by using Theorem 1, we have

$$\sum_{k=n+1}^{\infty}k(1+B)R_k(\mu,v,\eta)\left(\frac{1}{m}\sum_{j=1}^{m}a_{k,j}\right)$$

$$=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=n+1}^{\infty}k(1+B)R_k(\mu,v,\eta)a_{k,j}\right)$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\left((A-B)|\tau|\right)=(A-B)|\tau| \ .$$

The proof is complete.

In the next theorem, we have studied the convex linear combination property for the class $G^{\mu,v,\eta}(\tau, A, B)$ .

**Theorem 6:** The class $G^{\mu,v,\eta}(\tau, A, B)$ is closed under convex linear combination.

Proof: Let the function f and g be in the class $G^{\mu,v,\eta}(\tau, A, B)$ ,where

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k,\ g(z)=z-\sum_{k=n+1}^{\infty}b_k z^k.$$

It is sufficient to show that the function $h(z)$ defined by
$h(z)=\lambda f(z)+(1-\lambda)g(z),\ 0\leq\lambda\leq 1$

is the class $G^{\mu,v,\eta}(\tau, A, B)$ , therefore $o\leq\lambda\leq 1$, we have

$$h(z)=z-\sum_{k=n+1}^{\infty}\left[\lambda a_k+(1-\lambda)b_k\right]\ z^k.$$

By Theorem 1 , we have

$$\sum_{k=n+1}^{\infty}\frac{k(1+B)R_k(\mu,v,\eta)\left[\lambda a_k+(1-\lambda)b_k\right]}{(A-B)|\tau|}$$

$$=\lambda\sum_{k=n+1}^{\infty}\frac{k(1+B)R_k(\mu,v,\eta)a_k}{(A-B)|\tau|}+(1-\lambda)\sum_{k=n+1}^{\infty}\frac{k(1+B)R_k(\mu,v,\eta)b_k}{(A-B)|\tau|}\leq 1,$$

which implies that h(z) is the class $G^{\mu,v,\eta}(\tau, A, B)$ and this completes the proof.

In the next theorem , we obtain extreme points for the class $G^{\mu,v,\eta}(\tau, A, B)$.

Theorem 7: Let $f_n(z)=z$ and $f_k(z)=z-\dfrac{(A-B)|\tau|}{k(1+B)R_k(\mu,v,\eta)}z^k$,

where $k\geq n+1, n\in IN, -1\leq B<A\leq 1, \tau\in C\ \backslash\{0\}, z\in U, -\infty<\mu<1,$ $v<2, \eta\in IR.$ Then $f(z)$ is in the class $G^{\mu,v,\eta}(\tau, A, B)$ if and only if it can be expressed in the from $f(z)=\sum_{k=n}^{\infty}\sigma_k f_k(z),$ where $\sigma_k\geq o$ and $\sum_{k=n}^{\infty}\sigma_k=1$ or $\sigma_n+\sum_{k=n+1}^{\infty}\sigma_k=1.$

Proof: Let us express f as in the above theorem, therefore, we can write

$$f(z)=\sum_{k=n}^{\infty}\sigma_k f_k(z)=\sigma_n f_n(z)+\sum_{k=n+1}^{\infty}\sigma_k\left[z-\frac{(A-B)|\tau|}{k(1+B)R_k(\mu,v,\eta)}z^k\right]$$

$$=z\left(\sigma_n+\sum_{k=n+1}^{\infty}\sigma_k\right)-\sum_{k=n+1}^{\infty}\frac{(A-B)|\tau|}{k(1+B)R_k(\mu,v,\eta)}\sigma_k z^k$$

$$=z-\sum_{k=n+1}^{\infty}P_n z^n,$$

where $P_n=\frac{(A-B)|\tau|\sigma_k}{k(1+B)R_k(\mu,v,\eta)}$. Therefore $f\in G^{\mu,v,\eta}(\tau,A,B)$,

since

$$\sum_{k=n+1}^{\infty}\frac{P_n k(1+B)R_k(\mu,v,\eta)}{(A-B)|\tau|}=\sum_{k=n+1}^{\infty}\sigma_k=1-\sigma_n<1.$$

Conversely, assume that $f\in G^{\mu,v,\eta}(\tau,A,B)$. Then by (2.1), we may set

$$\sigma_k=\frac{k(1+B)R_k(\mu,v,\eta)a_k}{(A-B)|\tau|},k\geq n+1 \quad \text{and } 1-\sum_{k=n+1}^{\infty}\sigma_k=\sigma_n.$$

Then

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k=z-\sum_{k=n+1}^{\infty}\frac{(A-B)|\tau|\sigma_k}{k(1+B)R_k(\mu,v,\eta)}z^k$$

$$=z-\sum_{k=n+1}^{\infty}\sigma_k\left(z-f_k(z)\right)$$

$$=z\left(1-\sum_{k=n+1}^{\infty}\sigma_k\right)+\sum_{k=n+1}^{\infty}\sigma_k f_k(z)$$

$$=\sigma_n z+\sum_{k=n+1}^{\infty}\sigma_k f_k(z)=\sum_{k=n}^{\infty}\sigma_k f_k(z).$$

This completes the proof .

## 6. Integral Operators

Definition (6.1) : Let c be a real number such that c>-1 and let $f\in G^{\mu,v,\eta}(\tau,A,B)$, komato operator in [2] be defined by

$$r(z)=\int_0^1\frac{(c+1)^{\theta}}{\Gamma(\theta)}t^c\left(\log\frac{1}{t}\right)^{\theta-1}\frac{f(tz)}{t}dt, c>-1, \theta\geq 0 \; . \qquad (6.1)$$

Theorem 8: $r(z)$ defined in (6.1) be in the class $G^{\mu,v,\eta}(\tau,A,B)$.

Proof: Since

$$\int_0^1 t^c \left(\log\frac{1}{t}\right)^{\theta-1} dt = \frac{\Gamma(\theta)}{(c+1)^\theta}$$

and

$$\int_0^1 t^{k+c-1} \left(\log\frac{1}{t}\right)^{\theta-1} dt = \frac{\Gamma(\theta)}{(c+K)^\theta}, k \geq n+1.$$

Therefore, we obtain

$$r(z) = \frac{(c+1)^\theta}{\Gamma(\theta)}\left[\int_0^1 \left(t^c z\left(\log\frac{1}{t}\right)^{\theta-1} - \sum_{k=n+1}^{\infty}\left(\log\frac{1}{t}\right)^{\theta-1} t^{k+c-1} a_k z^k\right)dt\right]$$

$$= z - \sum_{k=n+1}^{\infty}\left(\frac{c+1}{c+k}\right)^\theta a_k z^k. \qquad (6.2)$$

Therefore and with use of Theorem 1 and $\frac{c+1}{c+k} < 1$ for $k \geq n+1$, we can write

$$\sum_{k=n+1}^{\infty} k(1+B)R_k(\mu,v,\eta)\left(\frac{c+1}{c+k}\right)^\theta a_k \leq (A-B)|\tau|. \qquad (6.3)$$

So $r(z) \in G^{\mu,v,\eta}(\tau, A, B)$.

Theorem 9: The function r(z) defined by (6.1) is starlike of order $\varepsilon (0 \leq \varepsilon < 1)$ in $|z| < r_1 = r_1(\tau, A, B, c, \varepsilon)$,
where

$$r_1(\tau, A, B, c, \varepsilon) = \inf_{k \geq n+1}\left\{\frac{k(1+B)R_k(\mu,v,\eta)}{(A-B)|\tau|}\left(\frac{1-\varepsilon}{k-\varepsilon}\right)\left(\frac{c+k}{c+1}\right)^\theta\right\}^{\frac{1}{k-1}}$$

Proof: We must show that

$$\left|\frac{zr'(z)}{r(z)} - 1\right| < 1 - \varepsilon$$

or we must show

$$\left|\frac{zr'(z)}{r(z)}-1\right| \le \frac{\sum_{k=n+1}^{\infty}\left(\frac{c+1}{c+k}\right)^{\theta}(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}\left(\frac{c+1}{c+k}\right)^{\theta}a_k|z|^{k-1}} < 1-\varepsilon .$$

The last inequality holds if

$$\sum_{k=n+1}^{\infty}\left(\frac{c+1}{c+k}\right)^{\theta}\frac{k-\varepsilon}{1-\varepsilon}a_k|z|^{k-1} \le 1.$$

Now in view of (6.2) , (6.3) the last inequality holds if

$$|z|^{k-1} \le \frac{k(1+B)R_k(\mu,v,\eta)}{(A-B)|\tau|}\left(\frac{1-\varepsilon}{k-\varepsilon}\right)\left(\frac{c+k}{c+1}\right)^{\theta}, k \ge n+1$$

and this gives the require result.

<u>Theorem 10:</u> Let the function *r*(*z*) be defined in (6.1). Then *r*(*z*) is convex of order $\varepsilon$ $(0 \le \varepsilon < 1)$ in $|z| < r_2(\tau, A, B, c, \varepsilon)$, where

$$r_2(\tau, A, B, c, \varepsilon) = \inf_{k \ge n+1}\left\{\frac{k(1+B)R_k(\mu,v,\eta)}{(A-B)|\tau|}\left(\frac{1-\varepsilon}{k(k-\varepsilon)}\right)\left(\frac{c+k}{c+1}\right)^{\theta}\right\}^{\frac{1}{k-1}} .$$

Proof of Theorem 10 is similar to that to Theorem 9 and hence details are omitted.

<u>Definition 2:</u> Let $f \in G^{\mu,v,\eta}(\tau, A, B)$. We define the function $f_{\ell}(z)$ as

$$F_{\ell}(z) = (1-\ell)z + \ell\int_0^z \frac{f(t)}{t}dt, \quad 0 \le \ell < 1, z \in U . \tag{6.4}$$

<u>Theorem 11:</u> Let the function $F_{\ell}(z)$ be defined in (6.4). Then $F_{\ell}(z) \in G^{\mu,v,\eta}(\tau, A, B)$ for $0 \le \ell < 1$.

<u>Proof :</u> Let $f(z) \in G^{\mu,v,\eta}(\tau, A, B)$ and is of the form (2.1) so

$$F_{\ell}(z) = (1-\ell)z + \ell\left(\int_o^z\left(1-\sum_{k=n+1}^{\infty}a_k t^{k-1}\right)dt\right)$$

$$= z - \sum_{k=n+1}^{\infty}\frac{\ell}{k}a_k z^k,$$

By Theorem 1, we must show

$$\sum_{k=n+1}^{\infty} k(1+B)R_k(\mu,\nu,\eta)\frac{\ell}{k}a_k$$
$$=\sum_{k=n+1}^{\infty} \ell(1+B)R_k(\mu,\nu,\eta)a_k$$
$$\le \sum_{k=n+1}^{\infty} (1+B)R_k(\mu,\nu,\eta)a_k \le (A-B)|\tau|.$$

So $F_\ell(z) \in G^{\mu,\nu,\eta}(\tau, A, B)$.

Remark 1: By the similar method which we applied for Theorems 9 and 10, we obtain the radii of starlikeness and convexity of order $\varepsilon(0 \le \varepsilon < 1)$ for $F_\ell(z)$ respectively as following :

$$r_1(\tau, A, B, \ell, \varepsilon) = \inf_{k \ge n+1}\left\{\frac{k(1+B)R_k(\mu,\nu,\eta)}{(A-B)|\tau|}\left(\frac{1-\varepsilon}{k-\varepsilon}\right)\left(\frac{k}{\ell}\right)\right\}^{\frac{1}{k-1}}$$

and

$$r_2(\tau, A, B, \ell, \varepsilon) = \inf_{k \ge n+1}\left\{\frac{k(1+B)R_k(\mu,\nu,\eta)}{(A-B)|\tau|}\left(\frac{1-\varepsilon}{\ell(k-\varepsilon)}\right)\right\}^{\frac{1}{k-1}},$$

where $0 \le \ell < 1$.

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# مؤثرات الحساب ألكسوري لصنف جديد من الدوال أحادية التكافؤ ذات المعاملات السالبة من الرتبة المعقدة

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**الخلاصة:-**

قدمنا في بحثنا الحالي صنف جديد من الدوال أحادية التكافؤ ذات العاملات السالبة من الرتبة المعقدة باستخدام المؤثرات التفاضلية التكاملية ألكسوري والتي درست حديثا من قبل الباحثين . ودرسنا تمثيل المعامل بالنسبة لهذه الدوال , النمو و التشويه , إنصاف الأقطار النجمية والمحدبة , نظريات الانغلاق لهذا الصنف من الدوال وكذالك درسنا بعض المؤثرات التكاملية على صنفنا $G^{\mu,\nu,\eta}(\tau, A, B)$ .