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Some Results on FeeblEy Multiplication Modules

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Abstract:

Let R be a commutative ring with non- zero identity. An R- module M is called weekly multiplication if every prime submodule of M is multiplication submodule of M. In this paper, we define the notion of feebly multiplication module and then we obtain some related result. In particular we prove that the tensor product of two feebly multiplication modules is a feebly multiplication module.

1. Introduction

In this paper all rings are commutative rings with non-zero identity and all modules are unital. A submodule N of an R- module M is called prime (resp. semiprime) if for any $r \in R$ and $m \in M$ such that $rm \in N$ (resp. $r^2m \in N$) either $m \in N$ or $rM \subseteq N$ (resp. $rm \in N$). So every prime submodule is a semiprime submodule. Note that in this definition

We do not require that N is a proper submodule of M as it was define in [7].
An ideal I of a ring R is semiprime if whenever $r^2 \in I$ for some $r \in R$ then $r \in I$. [5, p.243] A submodule N of an R-module M is called a multiplication submodule if for each submodule K of N, there exists an ideal I of R such that K = IN. In this case we can take
$I = (K: N) = \{r \in R: rN \subseteq K\}$. A module M is called multiplication module if every submodule of M is multiplication submodule of M [8]. As a generalization of multiplication module, Jain in [8] introduced the concept of weak multiplication module as follows:
An R-module M is said to be a weak multiplication module if every prime submodule of M is a multiplication submodule of M.
In this paper, we introduce the notion of feebly multiplication module as another generalization of multiplication module. A number of results concerning feebly module are given.

2. Feebley multiplication modules

In this section, we introduce the concept of feebly multiplication module and we prove some of its properties.
Definition (2.1): An R-module M is said to be feebly multiplication module if every semiprime submodule of M is a multiplication

submodule of M.
Remark (2.2): multiplication module $\Rightarrow$ feebly multiplication module $\Rightarrow$ weak multiplication module.

Keywords: Multiplication module, Weak multiplication, Semiprime submodule, feebly multiplication module.

The following proposition shows that a homomorphic image of a feebly multiplication module is feebly multiplication module. But before that we need the following lemma which appears in [6].

Lemma (2.3): Let $\phi = M_1 \to M_2$ be an epimorphism. If K is a semiprime submodule of M_2, then ϕ^{-1} (K) is a semiprime submodule of M

Proof: Clear.

Proposition (2.4): Let $\psi = M_1 \to M_2$ be an epimorphism. If M_1 is feebly multiplication module, then so is M_2

Proof: Let K be a semiprime submodule of M_2 and L be a submodule of M_2 such that $L \subseteq K \subseteq M_2$, it is clear that $\psi^{-1}(L) \subseteq \psi^{-1}(K) \subseteq M_1$.

But M_1 is feebly multiplication module, and by Lemma (2.3) $\psi^{-1}(K)$ is a semiprime submodule of M_1, thus $\psi^{-1}(K)$ is a multiplication submodule of M_1, and hence there exists an ideal I of R such that $\psi^{-1}(L) = I\,\psi^{-1}(K)$

Now, $\psi(\psi^{-1}(L)) = \psi(I\,\psi^{-1}(K))$. But ψ is an epimorphism, then L = IK. Therefore K is a multiplication submodule of M_2, and hence M_2 is a feebly multiplication module.

3. The tensor product of feebly multiplication modules.

In this section, we prove that the tensor product of two feebly multiplication modules is a feebly multiplication modules. Let us state the following known results which are needed later.

Proposition (3.1) [3, Theorem (2.3)]: Let each of M and N be R-modules. If M is a multiplication submodule of M and N is a multiplication submodule of N, then $M \otimes N$ is a multiplication submodule of $M \otimes N$.

Proposition (3.2) [2, Proposition (1.16)]: Let each of M and N be R–modules. If N is a multiplication submodule of N and M is a multiplication module, then $M \otimes N$ is a multiplication module.

From [2, Proposition (2.5)] and [4, Proposition 5] we have the following proposition.

Proposition (3.3): Let N be a submodule of an R-module M. if M is a multiplication submodule of M. then the following are equivalent:

1. N is a semiprime submodule of M
2. [N: M] is a semiprime ideal of R.
3. N=AM for some semiprime ideal A of R with Ann (M) $\subseteq$ A.

The following proposition show that $M \otimes N$ is a feebly multiplication module under certain conditions

Proposition (3.4): Let each of M and N be R-modules. If M is a feebly multiplication module and N is a multiplication submodule of N, then $M \otimes N$ is a feebly multiplication module.

Proof: Let K be a semiprime submodule of $M \otimes N$. Since M is a feebly multiplication module, than M is a multiplication submodule of M, and hence $M \otimes N$ is a multiplication submodule of $M \otimes N$ by (3.1).

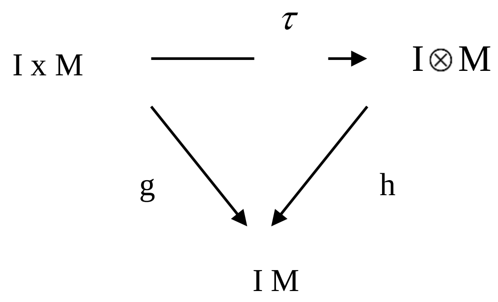
Thus K= (K: $M \otimes N$) ($M \otimes N$) = [(K: $M \otimes N$) M] $\otimes$ N. [3, Theorem (2.3)]

Since K is a semiprime submodule of $M \otimes N$, then (K: $M \otimes N$) is a semiPrime ideal in R by (3.3). But, clearly Ann (M) $\subseteq$ (K: $M \otimes N$), thus again by (3.3) (K: $M \otimes N$) M is a semiprime submodule of M. and hence

(K: $M \otimes N$) M is a multiplication submodule of M.

Therefore by (3.1) K= [(K: M$\otimes$N) M] $\otimes$N is a multiplication submodule of M$\otimes$N. Thus M$\otimes$N is a feebly multiplication module.

Corollary (3.5): If each of M and N is feebly multiplication modules, then M$\otimes$N is feebly multiplication module.
Corollary (3.6): Let M be an R- module. If M is a multiplication submodule of M and I is a feebly multiplication ideal then IM is a feebly multiplication module.
Proof: Consider the following diagram



Where τ the tensor map and g is the map defined by g(r, m) = rm $\forall r \in I$, $\forall m \in M$. It is clear that g is a bilinear map and hence there is a homomorphism h from I $\otimes$M into IM such that ho τ = g. It can be easily checked that h is an epimorphism. But I $\otimes$ M is a feebly multiplication module by (3.4), hence IM is a feebly multiplication module by (2.4).

4. The module of homomorphisms of feebly multiplication modules.

In this section, we prove that if M is a finitely generated feebly multiplication module and N is a multiplication submodule of N such that Ann M $\subseteq$ Ann N, then Hom (M,N) is a feebly multiplication module.

Let as state the following known proposition
Proposition (4.1) [3, Theorem (3.4)]: Let each of M and N be an R-module. If M is a finitely generated multiplication submodule of M and N is multiplication submodule of N such that Ann M$\subseteq$ Ann N, then
Hom (M, N) is a multiplication submodule of Hom (M, N).
Definition (4.2) [1]: An R- module M is called a weak cancellation module whenever AM = BM for ideals A and B of R, then
A + Ann (M) = B + Ann (M).
The following proposition is needed later.
Proposition (4.3) [1, Theorem (6.6)]: Let M be a multiplication submodule of M. Then M is a weak cancellation module if and only if M is finitely generated.
Before, we state and prove our next result, we need the following:
Lemma (4.4): Let M and N be R- modules. If K is a submodule of
Hom (M, N), then Ann M$\subseteq$ (K: Hom (M, N)
Proof:
Let r $\in$ Ann (M) and f $\in$ Hom (M, N). We have to show that r f $\in$K

Now, (r f) (M) = f (r M) = f (0) =0, that is r f is the zero homomorphism and hence r f $\in$ K.

For feebly multiplication modules we have

Proposition (4.5): Let each of M and N be R-modules with M is a finitely generated feebly module. If N is a multiplication submodule of N such that Ann M $\subseteq$ Ann N, then Hom (M, N) is a feebly multiplication module.

Proof: Let K be a semiprime submodule of Hom (M,N) and L be a submodule of Hom (M,N) such that L$\subseteq$k. Then

(L: Hom (M, N)) $\subseteq$ (K: Hom (M, N) and thus

(L: Hom (M, N) M $\subseteq$(k: Hom (M, N) M

Since K is a semiprime submodule of Hom (M, N) then (K: Hom (M, N) is a semiprime ideal of R, by (3.3). But M is a feebly multiplication module and M is a semiprime submodule of M, so M is a multiplication submodule of M. By Lemma (4.4) and proposition (3.3) we have

(K: Hom (M, N) M is a semiprime submodule of M. Thus there exists an ideal I in R such that (L: Hom (M, N)) M = I (K: Hom (M, N) M.

Now since M is finitely generated and multiplication submodule of M, then M has the weak cancellation property by (4.3), and hence

(L: Hom (M, N)) + Ann M = I (K: Hom(M, N) + Ann M. Thus

[(L:Hom(M,N))+AnnM]Hom(M,N)=[I(K:Hom(M,N)+AnnM)Hom(M,N)Since Ann (M) $\subseteq$ Ann (Hom (M, N), then

(L: Hom (M, N) Hom(M, N) = I (K: Hom(M, N)) Hom (M, N).

But, Hom(M, N) is a multiplication submodule of Hom (M, N), by (4.1). Therefore L = I K, and hence K is a multiplication submodule of

Hom (M, N) and thus Hom (M, N) is a feebly multiplication module,

Corollary (4.6): If M is finitely generated and feebly multiplication module, then Hom (M, M) is a feebly multiplication module.

Corollary (4.7): If M is faithful, finitely generated and feebly multiplication module, then M^* = Hom (M, R) is a feebly multiplication module.

The following proposition is a partial converse, of (4.5)

Proposition (4.8): Let each of M and N be R- modules, If M is a multiplication submodule of M such that Ann M = Ann Hom (M, N) and Hom (M, N) is finitely generated and feebly multiplication module, then M is a feebly multiplication module.

Proof: Let K be a semiprime submdule of M and L be a submodule of M such that L $\subseteq$K. then (L: M) $\subseteq$ (K: M) and hence

(L: M) Hom (M, N) $\subseteq$(K: M) Hom (M, N). Since K is a semiprime submodule of M, then (K: M) is a semiprime ideal in R by (3.3) But

Hom (M, N) is a multiplication submodule of Hom (M, N), and hence by the previous similar argument we have, (K:M) Hom (M,N) is a semiprime submodule of Hom(M,N), and therefor (K:M) Hom(M,N) is a multiplication submodule of Hom(M,N), This implies the existence of an ideal I in R such that (L:M) Hom(M,N) = I (K:M) Hom (M,N).

Now, Hom (M, N) is finitely generated and multiplication submodule of M, so Hom (M, N) has the weak cancellation property by [4.3]. That is
(L: M) + Ann Hom (M, N) = I (K: M) + Ann Hom (M, N)
Thus [(L: M) + Ann Hom (M, N)] M= [I (k: M) + Ann Hom(M, N)]M.
But Ann M = Ann Hom(M, N).Therefore (L: M) M= I (k: M) M.
Also, M is a multiplication submodule of M, so L= I K, then K is a multiplication submodule of M. Therefore M is a feebly multiplication module.

We end this paper by the following two corollaries
Corollary (4.6): If M is a multiplication submodule of M such that
Ann (M) = Ann (Hom (M, M)) and Hom (M, M) is a finitely generated, feebly multiplication module then M is an feebly multiplication module.

Corollary (4.7): Let M be a multiplication submodule of M such that
Ann M = Ann M^*. If M^* is a finitely generated feebly multiplication module, then M is a feebly multiplication module.

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بعض النتائج في المقاسات الواهنة الجداء

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الخلاصة :-

تكن R حلقة ابدالية ذات عنصر محايد $0 \neq I$ يقال لمقاس M معرف على الحلقة R بأنه ضعيف الجداء اذا كان كل مقاس جزئي اولي منه مقاس جزئي جدائي. عرفنا في هذا البحث مفهوم المقاس الواهن الجداء وحصلنا على بعض النتائج ومنها برهنا على ان الجداء التنسوري لمقاسين كل منها واهن الجداء هو مقاس واهن الجداء.