S^*- (o)-converges in Orlicz lattice

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Abstract:-

In this paper, we shall review some of the definitions and propositions which are needed in our work. Also we have proved that $\|x_n-x\|_F$ (o)-convergence to zero if and only if $F(x_n-x)$ (o)-convergence to zero.

1. Introduction:-

In this work a series of known notions, notations and facts of the theory of Boolean algebra, vector lattices [1,2,3], the integration theory for measures with values in semi-field [4,5,6,7] is cited. Suppose that R is the set of real numbers and E is a partially ordered set ($E \subseteq R$) .The main results in this work is the following :

Proposition I: If $x\in X$, $\|x\|_F \le \hat{\imath}$ (F is S^*- Orlicz modular) and e={ $\|x\|_F= 1$ } = 0, then $F(x) \le \|x\|_F$ ($\hat{\imath}$ is a Freudenthal unit, i.e for all $x\in X$, $x \wedge \hat{\imath} = 0$, that $x = 0$).

Proposition II: Suppose that $(X, \|.\|_F)$ be S^*- Orlicz Lattice, $x_n, x \in X$. Then, $\|x_n-x\|_F \xrightarrow{(o)} 0$ if and only if $F(x_n-x) \xrightarrow{(o)} 0$ ($\xrightarrow{(o)}$ i.e (o)- converges).

2. **The Basic Concepts**

In this section, we shall review some of the definitions and propositions which are needed in our work.

2.**1. DEFINITION [8]**

A vector space X equipped with a partial order "$\le$" is called a vector lattice, if for each pair x,y in X :

i. there is the smallest element z (denoted by $x\vee y$) for which $x\le z$ and $y\le z$.

ii. there is the largest element w (denoted by $x\wedge y$) for which $w\le x$ and $w\le y$.

iii. if $x \le y$, then $x + z \le y + z$ for all $x,y,z\in X$.

iv. if $x \le y$ and $c\in R+$, then $c x \le c y$.

2.2**. DEFINITION [8]**

Suppose that X is a S^* - vector lattice. A mapping $F: X \to S^*$ (S^* The ring of all measurable functions on [0,1]) is called an S^*- Orlicz modular if:

1. $F(x) \ge 0$ for all $x \in X$ and $F(x) = 0$ if and only if $x = 0$.
2. $F(x) \le F(y)$, if $|x| \le |y|$ for all $x,y\in X$.
3. $F((\propto x + (\hat{\imath}-\propto) y) \le \propto F(x) + (\hat{\imath}-\propto) F(y)$ for all $x,y\in X$, $\propto\in S^*$ and $0\le\propto\le \hat{\imath}$.
4. $F(2 x) \le c F(x)$ for all $x\in X$ and $c >0$ where c is constant.
5. $F(x + y) = F(x) + F(y)$, if $x,y\in X$ and $x \wedge y = 0$ ($x \wedge y$ i.e x infimum of y).
6. $F(e x) = e F(x)$ for all $x\in X$ and $e\in\nabla(S^*)$.

From the definition, we get that $F(x) = F(|x|)$ and $F(\propto x) \le \propto F(x)$ for all $x\in X$, $\propto\in S^*$ and $0 \le \propto \le \hat{\imath}$.

2.3. Note [8]

If $(L_M, \|.\|_{(M)})$ is an S*-Orlicz space generated by the Orlicz function M(u) with the Δ_2-condition (i.e the N-function M(u) satisfies the Δ_2-condition, if there exist $k>0$ and $uo \geq 0$ such that $M(2u) \leq k\, M(u)$ for any $u \geq uo$) then $F(x)=\mu(M(|x|))$ is an S*-Orlicz modular on L_M (μ An Lebesgue integral in $C_\infty(Q(\nabla))$ where $C_\infty(Q(\nabla))$ is a Stone compact set).

Suppose that X is an S^*- Orlicz modular on a S^*-vector lattice X. For all $x \in X$, we set $B(x) = \{\lambda \in S_+^* : F(\lambda^{-1}x) \leq \hat{\imath}, \lambda$ is invertible$\}$.

If $x \in X$, $\lambda = F(x) + \hat{\imath}$, then $F(\lambda^{-1} x) \leq \lambda^{-1} F(x) \leq \hat{\imath}$, it means, that $B(x) \neq \emptyset$. For each $x \in X$, we set $\|x\|_F = \inf \{\lambda : \lambda \in B(x)\}$ [8].

2.4. **proposition [8]**

An S^*-Orlicz Lattice $(X, \|.\|_F)$ is a complete Lattice and an S^*-norm $\|.\|_F$ is order continuous.

2.5. **Proposition [8]**

Suppose that $\|.\|_{(M)}$ is an S^*-norm on L_M. Furthermore, it follows from $|x| \leq |z|$, $x,z \in L_M$, that $\|x\|_{(M)} \leq \|z\|_{(M)}$, thus $(L_M, \|.\|_{(M)})$ is a normed S^*-vector lattice.

2.6. **proposition [9]**

Suppose $x_n, x \in C_\infty(Q(\nabla))$, $0 \leq x_n \uparrow x$, then $M(x_n) \uparrow M(x)$ ($\{x_n\}$ is increasing (decreasing) then, we write $x_n \uparrow x$, (respectively, $x_n \downarrow x$)) .

2.7. **proposition [8]**

Let $\|.\|_{(M)}$ be an S^*-norm on L_M, then $(L_M, \|.\|_{(M)})$ is a Banach S^*-vector lattice.

3. The main result In this section, we shall prove an important propositions.

3.1. **Proposition**

If $x \in X$, $\|x\|_F \leq \hat{\imath}$ and $e=\{ \|x\|_F = 1 \} = 0$, then $F(x) \leq \|x\|_F$.

Proof: Put $B(x)= \{\lambda \in S_+ : F(\lambda^{-1}x) \leq \hat{\imath} : \lambda$ is invertible$\}$, we have choose $\lambda_n \in B(x)$: $\lambda_n \downarrow \|x\|_F$.

Let $\lambda_1, \lambda_2 \in B(x)$, $e=\{\lambda_1 \leq \lambda_2\}$, then $\beta=\lambda_1 \wedge \lambda_2 = \lambda_1 e + \lambda_2(\hat{\imath} - e) \in S^+$ and β is invertible.

$F(\beta^{-1}x)= F(\lambda_1^{-1} e+ \lambda_2^{-1} (\hat{\imath}-e)x) = e\, F(\lambda_1^{-1} xe)+(\hat{\imath} - e)\, F(\lambda_2^{-1} x(\hat{\imath}-e)) \leq e+(\hat{\imath}-e)= \hat{\imath}$.

i.e $\beta \in B(x)$. Using the method of mathematical induced, we get $\bigwedge_{i=1}^{n} \lambda_i \in B(x)$ for any finite subset $\{\lambda_1, \dots , \lambda_n\} \in B(x)$.

Since S^* is of countable type, there exists a subsequence $\{\lambda_n\} \subset B(x)$ such that $\lambda_n \downarrow \inf B(x) = \|x\|_F$.

Since $\beta_n= \lambda_n + 2^{-1} \varepsilon\, \hat{\imath} \in B(x)$ ($\varepsilon \in S_+$) and $\beta_n \downarrow \|x\|_F + 2^{-1} \varepsilon\, \hat{\imath}$ (by proposition 2.5.), we have $(|x|\beta_n^{-1}) \uparrow |x|(\|x\|_F + 2^{-1} \varepsilon\, \hat{\imath})^{-1}$.

Hence by proposition (2.5) then, $F((\|x\|_F + 2^{-1} \varepsilon\, \hat{\imath})^{-1}x) = \sup F(\beta_n^{-1} x) \leq \hat{\imath}$ then, $\|x\|_F \in B(x)=\{\lambda \in S_+^* : F(\lambda^{-1}x) \leq \hat{\imath}\}$.

Suppose that $\lambda \in S_+^*$, $\|x\|_F \leq \lambda \leq \hat{\imath}$ and $f=\{ \lambda=\|x\|_F\}=0$, i.e (λ is invertible). Set $f_n=\{\lambda < \lambda_n \}$, n= 1,2,… then $f_{n+1} \leq f_n$ (it is clear that).

$f_1=\{\lambda<\lambda_1\}$, $f_2=\{ \lambda<\lambda_2\}$, … , $f_n=\{ \lambda<\lambda_n\}$, if $n \neq k$, then $f_n . f_k=0$, $\sup_{n \geq 1} f_n= \hat{\imath}$.

By proposition 2.6 then, $F(x_n) \uparrow F(x)$.

$\|x\|_F=\inf \{\lambda : \lambda \in B(x) \}$, if $\|x\|_F=1$ and $\|x\|_F=\lambda$ then $\lambda=1$, therefore $F(\lambda^{-1}x) \leq \hat{\imath}$ then $F(x) \leq \hat{\imath} \leq 1=\|x\|_F$. Hence $F(x) \leq \|x\|_F$.

3.2. Proposition

Suppose that $(X, \|.\|_F)$ be S*- Orlicz Lattice, $x_n, x \in X$. Then, $\|x_n-x\|_F \xrightarrow{(o)} 0$ if and only if $F(x_n-x) \xrightarrow{(o)} 0$.

Proof: Suppose that $\|x_n-x\|_F \xrightarrow{(o)} 0$.

Let $e_n=\{ \|x_n-x\| \le 1\}$, n=1,2, … . Since $\|x_n-x\|_F \xrightarrow{(o)} 0$ then $\|x_n-x\|_F <1 \xrightarrow{(o)} \hat{\imath}$, $e_n \in S^*$ and $\|e_n(x_n-x)\|=e_n\|x_n-x\|_F \le \hat{\imath}$ and $\{e_n\|x_n-x\|= \hat{\imath}\}=0$.

By proposition (3.1), we have $F(e_n(x_n-x)) \le e_n\|x_n-x\|_F \le \|x_n-x\|_F$, hence $e_nF(x_n-x)= F(e_n(x_n-x)) \xrightarrow{(o)} 0$, Furthermore $(\hat{\imath} -e_n) \le (\hat{\imath} -e_n) \|x_n-x\|_F \le \|x_n-x\|_F$,

i.e. $(\hat{\imath} -e_n) \xrightarrow{(o)} 0$.

Put $f_n=\sup_{i\ge n} (\hat{\imath} - e_i)$. Then we have $(\hat{\imath} - e_n)\le f_n$, $f_n \downarrow 0$ and $(\hat{\imath} -e_n)F((x_n-x))\le f_nF(x_n-x)$.

Since the (o)- convergence in S^* is equivalent to the convergence almost everywhere with respect to Lebesgue measure [10], then it follows from $f_n \downarrow 0$ that $f_nF(x_n-x) \xrightarrow{(o)} o$, hence $(\hat{\imath} -e_n)F((x_n-x)) \xrightarrow{(o)} 0$, therefore

$F((x_n-x))= e_nF(x_n-x)+(\hat{\imath}-e_n)F((x_n-x)) \xrightarrow{(o)} 0$. Hence $F(x_n-x) \xrightarrow{(o)} 0$.

Suppose that $F(x_n-x) \xrightarrow{(o)} 0$.

By proposition (2.3), we get $F(2^i(x_n-x)) \xrightarrow{(o)} 0$. $(1\le i\le n)$.

From the previous assertion, the element $\lambda=F(x)+ \hat{\imath} \in B(x)$. Hence $\|x\|_F \le F(x)+ \hat{\imath}$ for all $x\in X$. Since $F(2y)\le kF(y)$ for any $y\in C_\infty(Q(\nabla))$, then $F(2^i(x_n-x))\le k^iF(x_n-x)$ for all i=1,2, … , From which, we get $F(2^i(x_n-x)) \xrightarrow{(o)} 0$ as $n\to\infty$ from any fixed i.

Thus, using Young's inquality, notice that, for any $y\in L_N$ with $F(y)\le \hat{\imath}$, the inequality $|2^i(x_n-x)y| \le F(2^i(x_n-x))+ \hat{\imath}$, take place.

Hence $\|2^i(x_n-x)\|_F \le F(2^i(x_n-x))+ \hat{\imath}$, and $\|x_n-x\|_F \le 2^{-1}F(2^i(x_n-x))+2^{-1}\hat{\imath}$. From this, we get (o)- $\overline{Lim}\|x_n-x\|_F = \bigwedge_{k=1}^{\infty} \bigvee_{n\ge k} \|x_n-x\|_F \le 2^{-1}\hat{\imath}$, for any i=1,2, … .

This means that (o)- $\overline{Lim}\|x_n-x\|_F =0$, i.e. $\|x_n-x\|_F \xrightarrow{(o)} 0$.

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الخلاصة :-
في هذا البحث نستعرض بعض التعاريف والمبرهنات التي نحتاجها في عملنا. كذلك سنبرهن بان $||x_n-x||_F$ يتقارب تقارب- (o) إلى الصفر أذا وفقط أذا كان $F(x_n-x)$ يتقارب تقارب- (o) إلى الصفر.