TWO FIXED POINT THEOREMS IN ORBITALLY COMPLETE GENERALIZED METRIC SPACE

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Abstract:

The aim of this paper is to prove two results about the existence of unique fixed point for self mappings defined on an orbitally complete, (or, on an orbitally complete chainable) generalized metric space.These results based up on general contraction mapping and locally general contraction mapping which include some known results as corollaries. Therefore, our results unify and extend the results in Das[1], Das and Day[2] and C'iri'c[4].

Key words: ***generalized metric space, fixed point, orbitally complete, locally contractive mapping, ϵ -chainabl***

1. Introduction and preliminaries:

For generalizing the notion of metric spaces, Branciari [5] introduced a general metric space by replacing the triangular inequality of a metric space by a general one which is rectangular inequality .Also he proved a version of Banach's contraction principle in general metric space. And then, Das[1] proved the existence of the unique fixed point for Kannan mapping. Recently Das and Day[2] proved a fixed point theorem for uniformly locally contractive mapping. Here, we prove two general theorems.The first one for contraction mapping which defined on the orbitally generalized metric space and locally contraction mapping.

Firstly, we denoted R^+ is the set of all non-negative real numbers and N is the set of all positive integers. Now we recall the definition of general metric spaces:

Definition 1. 1[6]: ***Let X be a nonempty set. Suppose that the mapping $\rho: X\times X \to R^+$ such that for all $x , y \in X$ and for all distinct points $z ,v \in X\setminus \{x, y\}$, satisfies:***

1. *$\rho(x, y) = 0$ if and only if $x = y$,*
2. *$\rho(x, y) = \rho(y, x)$,*
3. *$\rho(x, y) \leq \rho(x, z) + \rho(z, v) + \rho(v, y)$, (rectangular property),*

Then the ordered pair (X, ρ) is called a generalized metric space (or shortly G.M.S.).

Note that, any metric space is general metric space but the converse is not true, for examples,

Example1.2[6]**:** ***Let $X=\{a,b,c,d,\}$. Define $\rho : X \times X \to R$ by***

$\rho(a, b)= \rho(b, a)= 3$, $\rho(b, c)= \rho(c, b)= \rho(a, c)= \rho(c, a)=1$,

$\rho(a, d)= \rho(d, a)= \rho(b, d)= \rho(d, b)= \rho(c, d)= \rho(d, c)=4$.

It is easily to show that (X, ρ) is generalized metric space and it is not metric space, since

$$\rho(a,b) > \rho(a,c)+ \rho(c,b)$$

$$3 > 1 + 1$$

Example1.3: ***Consider $X=R$, $\mu : X \times X \to R$ and $\mu(x,y)= (x-y)^2$,***

Clearly μ is not generalized metric space and so is not metric space since, for $x=2$, $y=0$, $z=1$ and $v=1/2$. We have

$$\mu(2, 0) > \mu(2, 1)+ \mu(1, 1/2) + \mu(1/2, 0)$$

Example1.4: ***Let $\rho: R \times R \to R^+$ be a mapping such that***

$$\rho(x,y)=\max\{\mu(x,z), \mu(z,v), \mu(v,y)\},$$

where μ as in example above, then ρ is generalized metric space. Therefore, general metric space is a proper extension of a metric space.

Also, one can generate many generalization metric spaces by usual sense, such as:

Remark1.5[5]: ***The generalized metric space is continues function on $X \times X$.***

Remark1.6 [3]: ***As in the usual metric space settings, a general metric space is a topological space with respect to the basis given by***

$B=\{B(x,r):x\in X,\ r\in R^+\}$,where $B(x,r)=\{y\in X:\ \rho(x,y)<r\}$ is open ball centered by x and with radius r.

Definition1.7: ***A point x in X is a fixed point of the map $T : X \to X$ if $Tx = x$.***

Definition1.8 [3]: ***Let (X, ρ) be a G.M.S. A sequence $\{x_n\}$ in X is said to to be a Cauchy sequence if for any $\varepsilon > 0$ there exists n_ε in N such that for all $m, n \in N$ with $n \geq n_\varepsilon$, one has $\rho(x_n, x_{n+m}) < \varepsilon$. And the G.M.S. (X,ρ) is called complete if every Cauchy sequence in X is convergent.***

Definition1.9: ***Let T be a self mapping on X. Let $x_0 \in X$. A sequence $\{T^n x\}$ in X is said to be an orbit of x by T and denoted by $O(x, n)= \{x,Tx,T^2x,\ldots,T^nx\}$, for all $n \in N$. Also, $O(x, \infty) =\{x,Tx,T^2x,\ldots\}$.***

Definition 1.10: ***Let T be mapping on a G.M.S. (X,ρ) into itself. (X, ρ) is said to be T-orbitally complete if and only if every Cauchy sequence in $O(x, \infty)$ converges in X, for some $x \in X$.***

Definition 1.11: ***Let (X, ρ) be a G.M.S. and A be a nonempty subset of X. We define the diameter of A as $\delta(A) = Sup\{ \rho(x, y) : x, y \in A\}$.***

2 – Main results

Let (X,ρ) be generalized metric spaces and T be a self mapping on X .We recall the following three contractive conditions[7]:

i. For all x,y in X, $\exists\ \lambda$, $0 \leq \lambda < 1$ such that $\rho(Tx,Ty) \leq \lambda\, \rho(x,y)$. (Banach principle)

ii. For all x,y in X, $\exists\ \beta$, $0 \leq \beta < ½$ such that $\rho(Tx,Ty) \leq \beta\, [\rho(x,Tx)+\rho(y,Ty)]$. (Kanann principle)

iii. For all x,y in X, $\exists\ \alpha$, $0 \leq \alpha < ½$ such that $\rho(Tx,Ty) \leq \alpha\, [\rho(x,Ty)+ \rho(y,Tx)]$.(Chatterge principle)

The conditions in (i), (ii) and (iii) are independent since, if the mapping T satisfies (i)will be continuous but, if it satisfies (ii)or(iii) may be discontinuous .The following examples illustrate this facts :

Examples2.1:

1- ***T satisfies (i) is continuous.***
   Let $T:[0,1] \rightarrow [0,1]$, $Tx= x/3$ but not (ii) when $x=0$ and $y=1/3$.
2- ***T satisfies (ii) and T is discontinuous.***
   Let $T: [0, 1] \rightarrow [0,1]$ such that $Tx=x/4$ if $x \in [0, ½)$ and $Tx=x/5$ if $x \in [½,1]$.
3- ***T satisfies (ii) not satisfies (iii).***
   Let $T:R \rightarrow R$, $Tx= -x/2$ take $x=2$, $y= -2$.
4- ***T satisfies (iii) not satisfies (ii).***
   Let $T:[0,1] \rightarrow [0,1]$,$Tx =x/2$ if $x \in [0,1)$ and $Tx=0$ if $x=1$ take $x=½$, $y=0$.

Remark2.2 [7]:

1- ***The condition (i) ,(ii)and (iii) can be written in the following equivalent form***
   For all x,y in X, $0 \leq h < 1$
   $\rho(Tx,Ty) \leq h \max\{(\rho(x,Tx) + \rho(y,Ty))/2 , \rho(x,y), (\rho(x,Ty)+ \rho(y,Tx))/2\}$. ….. ***(2.1)***
2- ***The class of mapping in (1) above is a subclass of the mapping satisfying the follows :***
   For all x,y in X. $0 \leq \lambda < 1$.
   $\rho(Tx,Ty) \leq \lambda \max\{ \rho(x,Tx) , \rho(y,Ty), \rho(x,y), \rho(x,Ty), \rho(y,Tx)\}$. ….. ***(2,2)***

The following example illustrate the above remark

Example2.3 [8]: ***Consider $X = [0, \infty)$ with usual distance and $T: X \rightarrow X$, such that***

$Tx =x^2 /2(x+1)$, then for x,y in X , $\rho(Tx,Ty) \leq ½ \rho(x,y)$. So T satisfies condition (2.2).For $x \geq 1$, $\rho(Tx,T2x) = x^2(2x+3)/2(x+1)(2x+1)$

and $\max\{\rho(x,2x),[\rho(x,Tx)+\rho(2x,T2x)]/2,[\rho(x,T2x)+\rho(2x,Tx)]/2\} = x$.

Given any h satisfying $0< h <1$, one can find value of x large enough so that

$x^2 (2x+3)/2(x+1)(2x+1) > h$.

Such as $x = 1$, $h = 1/12$, therefore T does not satisfy (2.1).

As in the case of usual metric space (X, d) we can show the following proposition, for details see [4] ▪

Proposition2.4: *Let T be mapping of a G.M.S. (X, ρ) into itself satisfy (2) and $n \in N$, then*

1. $\forall x \in X$ and $\forall i, j \in \{1,2,\dots,n\} \Rightarrow \rho(T^i x, T^j x) \le q\, \delta(O(x, n))$.
2. $\forall n \in N, \exists k < n \ni \rho(x, T^k x) = q\, \delta(O(x, n))$.

Proposition2.5: *Let T be mapping of a G.M.S. (X, ρ) into itself satisfy (2.2), then*

$$\delta(O(x, \infty)) \le 1/(1+2q)\, \rho(x, Tx) \text{ holds } \forall x \in X.$$

Proof: Let $x \in X$, since $\delta(O(x, 1)) \le \delta(O(x, 2)) \le \dots$.

Therefore, $\delta(O(x, \infty)) = \sup\{ O(x, n) : n \in N\}$. The result will be true if we show that

$$\delta(O(x, n)) \le 1/(1+2q)\, \rho(x, Tx), \forall n \in N .$$

Let $\forall n \in N$ from2 in proposition 2.4, $\exists\, T^k x \in \delta(O(x, n))$, $1 \le k \le n$ such that

$$\rho(x, T^k x) = \delta(O(x, n)). \quad \dots\dots (2.3)$$

By 1 in proposition2.4 and (2.3)

$$\rho(x, T^k x) \le \rho(x, Tx) + \rho(Tx, T^2 x) + \rho(T^2 x, T^k x).$$

$$\delta(O(x, n)) \le \rho(x, Tx) + q\, \delta(O(x, n)) + q\, \delta(O(x, n))$$

$$\Rightarrow \delta(O(x, \infty)) \le 1/(1+2q)\, \rho(x, Tx),$$

This completes the proof ▪

Now, we will prove our first result

Theorem2.6: ***Let T be a self mapping satisfying (2.2) on X and X be T -orbitally complete generalized metric space. Then T has a unique fixed point in X.***

Proof: Let $x \in X$. To prove $\{T^n x\}$ is Cauchy sequence, for $n< m$, By 1 in proposition 2.4

$$\rho(T^n x,T^m x)= \rho(TT^{n-1}x,T^{m-n+1}T^{n-1}x)$$
$$\leq q\ \delta(O(T^{n-1}x,m-n+1))$$

By proposition2.4 $\exists k_1 ,1\leq k_1 \leq m-n+1 \ni$

$$\delta(O(T^{n-1}x,m-n+1))= \rho\ (T^{n-1}x,T^k\ T^{n-1}x)$$

Again by 1 in proposition2.4 we have

$$\rho\ (T^{n-1}x,T^k\ T^{n-1}x)= \rho\ (TT^{n-2}x,T^{k+}T^{n-2}x)$$

$$\leq q\ \delta(O(T^{n-2}x,k_1+1)\ )$$
$$\leq q\ \delta(O(T^{n-2}x,m-n+2))$$
$$\rho(T^n x,T^m x) \leq q\ \delta(O(T^{n-1}x,m-n+1)) \leq q\delta\ (O(T^{n-2}x,m-n+2)),$$

Continue in the manner we have

$$\rho\ (T^n x,T^m x) \leq q\ \delta(O(T^{n-1}x,m-n+1)) \leq \ldots.. \leq q^n\ \delta(\ O(x,n))$$

By proposition2.5

$$\rho(T^n x,T^m x) \leq q^n /(1+2q)\ \rho(x, Tx)\ ,$$

Since $\lim_{n\to\infty} q^n =0$,then $\{T^n x\}$is Cauchy sequence. Now since X is T-orbitally complete, then there is $u \in X$, $\lim T^n x=u$. To prove that $Tu=u$, let us consider the following

$$\rho(u, Tu) \leq \rho(u, T^n x)+ \rho(T^n x, T^{n+1} x)+ \rho(T^{n+1}x, Tu)$$

$$\leq \rho(u,T^n x)+\rho(T^n x,T^{n+1}x)+q\ \max\{\rho(T^n x, u),\rho(T^n xT^{n+1}x), \rho(u,Tu),\rho(T^n x.Tu),\rho(T^{n+1}x, u)\}$$

$\leq \rho(u, T^n x)+ \rho(T^n x, T^{n+1}x)+q[\rho(T^n x,T^{n+1}x)+ \rho(T^n x, u)+ \rho(u,Tu)+ \rho(T^{n+1}x.u)]$

Hence

$$\rho(u,Tu) \leq [(1+q) (\rho(u, T^n x)+ \rho(T^n x,T^{n+1}x))+q\, \rho(u, T^{n+1}x)] / (1-q)$$

Since $\lim_{n\to\infty} T^n x=u$ *then* $\rho(u,Tu)$ *must be zero ,therefore u fixed point*

The uniqueness follows from the condition 2 let v be another fixed point

$$\rho(u,v)= \rho(Tu,Tv) \leq q \max\{ \rho(u,v), \rho(u,Tu), \rho(v,Tv), \rho(u,Tv), \rho(v,Tu)\}$$

$$\leq q\, \rho(u,v), \text{ which is contradiction.}$$

So, u must be equal v .Then T has unique fixed point.

Consequently, the result in [2, Theorem 1],[5,Theorem3.1]and[4,Theorem1] as special cases of Theorem(2.6).Also , we have the following corollaries:

Corollary2.7: *If X as in Theorem 2.6 and T satisfies one of the following conditions*

1- $\rho (Tx,Ty) \leq q \max \{ \rho (x,Tx) , \rho(x,y)\}$
2- $\rho (Tx,Ty) \leq q \max \{ \rho (x,Tx) , \rho(y,Ty)\}$
3- $\rho (Tx,Ty) \leq \alpha \{ \rho (x,Ty)+ \rho (y,Tx) \}$
4- *condition (2.1)*

Then T has a unique fixed point

Finally, if we replace T in above by T^k*(for positive integer k) then T also has unique fixed point .*

To prove second result in locally case we need to define a chainable generalized metric space:

Definition2.8 [1] : *A G.M.S. X is said to be* ϵ*-chainable if for any two points a, b* $\in X$ *there exists a finite set of points* $a = x_0, x_1, \ldots , x_{n-1}, x_n = b$ *such that*

$$\rho (x_{i-1}, x_i) \leq \epsilon \text{ for } i = 1, 2, 3, \ldots , n \text{ where } \epsilon > 0$$

Theorem2.9: ***If T is self mapping defined on a T- orbitally complete, $\epsilon/2$ - chainable G.M.S. X satisfying the following conditions:***

1. ***If $\rho(x,y) < \delta \Rightarrow \rho(Tx,Ty) \le \lambda \max\{\rho(x,Tx),\rho(y,Ty),\rho(x,y),\rho(x,Ty),\rho(y,Tx)\}$, $0 \le \lambda < 1, \forall x,y \in X$.***
2. ***for all $x,y,z \in X$, $\rho(x,y) < \epsilon/2$ and $\rho(y,z) < \epsilon/2$ implies $\rho(x,z) < \epsilon$.***

Then T has a unique fixed point u in X. Moreover for some x in X, $\lim_{m\to\infty} T^m x = u$.

Proof: ***Let $x \in X$ and $M = \max\{\rho(x,Tx),\rho(y,Ty),\rho(x,y),\rho(x,Ty),\rho(y,Tx)\}$.*** *M give us fife probabilities as follows:*

I. ***If $M=\rho(x,Ty)$.***

To show that $\lim_{m\to\infty} T^m x = u$. Let $x \in X$ and $y = Tx$. Since X is $\epsilon/2$ - chainable, we can find finite number of points $x = x_0, x_1, x_2, \ldots, x_{1t-1}, x_{1t} = T^{m+1}x$ such that

$$\rho(x_{i-1},x_i) < \epsilon/2 \text{ for all } i = 1,2,\ldots,t_1. \quad \ldots\ldots (2.4)$$

Without any loss of generality suppose that $x \neq T^{m+1}x$ the points $x_1, x_2, \ldots, x_{1t-1}$ are distinct (and different from x and $T^{m+1}x$. we shall show that

$\rho(x, T^{m+1}x) < t_1\,\epsilon/2$, as follows:

- ***If $t_1=1$, from (2.4)***

$$\rho(x, T^{m+1}x) = \rho(x_0,x_1) < \epsilon/2. \quad \ldots\ldots (2.5)$$

- ***If $t_1=2$, from condition 2 and (2.4)***

$$\rho(x, T^{m+1}x) = \rho(x_0,x_2) < \epsilon \quad \ldots\ldots (2.6)$$

$$= 2\epsilon/2. \quad \ldots\ldots (2.7)$$

Now if $t_1>2$, consider two cases

Case1. First let t_1 be odd and $t_1 = 2j+1$, $j \ge 1$. Now

$$\rho(x, T^{m+1}x) \le \rho(x, x_1) + \rho(x_1,x_2) + \cdot\ \cdot + \rho(x_{2j}, x_{2j+1})$$
$$< (2j + 1)\, \epsilon/2 \quad [\text{from } (2.4)]$$
$$= t_1\, \epsilon/2. \quad \ldots.. (2.8)$$

Case 2. Let t_1 be even and $t_1= 2j$, $j \ge 2$. Then

$$\rho(x, T^{m+1}x) \le \rho(x, x_2) + \rho(x_2, x_3) + \cdot\ \cdot + \rho(x_{2j-1}, x_{2j})$$
$$< \epsilon + (2j - 2)\epsilon/2 \quad [\text{from } (2.4) \text{and } (2.6)]$$
$$= t_1\, \epsilon/2. \quad \ldots.. (2.9)$$

From (2.5),(2.7),(2.8)and (2.9), we obtain

$$\rho(x, T^{m+1}x) < t_1\epsilon/2, \quad \forall\ t_1. \quad \ldots.. (2.10)$$

By the same way

$$\rho(x, T^{m+2}x) \le t_2\, \epsilon/2. \quad \ldots.. (2.11)$$

We can find $\rho(x, T^{m+i} x) \le t_i\, \epsilon /2$, $\forall i \in N$ by the same way.
By induction,

$$\rho(T^m x, T^{m+1}x) < \lambda^m\, \rho(x, T^{m+1}x)$$
$$\rho(T^m x, T^{m+1}x) < \lambda^m\, t_1\epsilon/2 \quad \ldots.. (2.12)$$

and

$$\rho(T^m x,T^{m+i}x) \le \lambda^m\, \rho(x,T^{m+i} x)\ ,\ \forall i \in N$$
$$< \lambda^m\, t_i\, \epsilon /2. \quad \ldots.. (2.13)$$

Note that even if some of the points $T^m x_0,\ldots, T^m x_n$ are equal then the result is true.

$$\rho(T^m x, T^{m+1}x)= \rho(T^m x_0, T^m(Tx_0))=0$$

Now we shall first Note that if $T^m x=T^n x$ for some $m, n \in N$, $m>n$ then let $p= m-n$ and

$u=T^nx$ we have $T^p u=u$ and so $T^{kp} u = u$, $\forall k \in N$.if $u=x_0, x_1, x_2,..., x_{r-1},x_r= T^{kp+1} u$,

$$\rho(T^m u,T^{m+1} u)< \lambda^m r\epsilon/2, \forall m\in N \text{ for some fixed } r\in N. \text{ Then}$$

$$\rho(u, T u) = \rho(T^{kp} u, T^{kp+1} u) < \lambda^{kp} r\epsilon/2 \to 0 \text{ as } k\to\infty$$ ***and this implies $Tu=u$.***

If $T^m x \neq T^n x$, $\forall m, n \in N$.now we show that $\{T^n x\}$ is a Cauchy sequence in X.

Let $k\in N$, take $m (>k) \in N$. Again consider two cases.

Case 1: If n is odd, say $n = 2s + 1$, $s \geq 0$, by (2.12) we have

$$\rho(T^m x, T^{m+n} x)\leq \rho(T^mx,T^{m+1}x) + \rho(T^{m+1}x,T^{m+2}x) + \cdot\cdot\cdot + \rho(T^{m+2s}x,T^{m+2s+1}x)$$
$$< \lambda^m t_1 \epsilon /2+\lambda^{m+1} t_2 \epsilon /2+ \cdot\cdot +\lambda^{m+2s} t_n \epsilon /2$$
$$<\lambda^m a \epsilon /2 .$$

Case 2: If n is even, say $n = 2s$, $s \geq 1$, by (2.12) and (2.13),

$$\rho(T^mx, T^{m+n} x) \leq \rho(T^mx, T^{m+2}x) + \rho(T^{m+2}x,T^{m+3}x) + \cdot\cdot\cdot\cdot + \rho(T^{m+2s-1}x,T^{m+2s}x)$$
$$<\lambda^m t_2 \epsilon /2+\lambda^{m+2} t_3 \epsilon /2+ \cdot \cdot \cdot +\lambda^{m+2s-1} t_n \epsilon /2$$
$$<\lambda^m \epsilon (t_2 +\lambda^2 t_3 + \cdot \cdot \cdot +\lambda^{2s-1} t_n) /2$$
$$= \lambda^m b \epsilon /2$$

Where $a= t_1+\lambda t_2 + \cdot \cdot +\lambda^{2s} t_n$ and $b= t_2 +\lambda^2 t_3 + \cdot \cdot \cdot +\lambda^{2s-1} t_n$.

Therefore combining both the cases we have

$$\rho(T^m x ,T^{m+n} x) < \lambda^m z \epsilon /2,$$

where $z = \max\{ a , b\}$. Since $\lambda\in [0, 1)$, $\lambda^m \to 0$ as $m \to \infty$, this shows that $\{T^mx\}$ is a Cauchy sequence in X. Since X is T-orbitally complete,

$\{T^mx\}$ is convergent in X. Let

$$\lim_{m \to \infty} T^mx = u. \qquad \text{.....} \quad (2.14)$$

To show that $T(u) = u$ we need two cases.

i- if $T^mx\neq T(u),u$ for any $m\in N$,then by (2.14)

$$\rho(u,Tu) \leq \rho(u,T^m x) + \rho(T^m x,T^{m+1}x) + \rho(T^{m+1}x,Tu)$$
$$\leq \rho(u,T^m x) + \lambda^m \rho(x, T^{m+1}x) + \lambda^{m+1} \rho(x, Tu)$$
$$\leq \rho(u,T^m x) + \lambda^m [t_1\epsilon /2 + \lambda \rho(x, Tu)]$$

$$\lim_{m\to\infty}\rho(u,Tu) \leq \lim_{m\to\infty}\rho(u,T^m x) + \lim_{m\to\infty}\lambda^m[t_1\epsilon /2 + \lambda \rho(x, Tu)]$$

since $\lambda \in [0, 1)$, $\lambda^m \to 0$ as $m \to \infty$, thus $\lim_{m\to\infty}\rho(u,Tu) = 0$.Hence $u=Tu$.

ii- assume that $T^t x = u$, $T^t x = T^m x$ for some t . By (2.10)

$\rho(u, Tu) = \rho(T^t x , T^{t+1}x) \leq \lambda^t \rho(x,T^{t+1}x) \to 0$ as $t \to\infty$.Hence $u=Tu$.

To prove the uniqueness of the fixed point, let $v\in X$, $u\neq v$ and $Tv = v$. Since X is $\epsilon/2$ –chainable, we can find a finite chain $x_0, x_1, x_2,..., x_{r-1}, x_r$ By the same sense of (2.10)

$$\rho(u, T^m v) < r\, \epsilon/2.$$

$$\rho(T^m u, T^m v) < \lambda^m \rho(u, T^m v) \ , \forall m \in N.$$
$$< \lambda^m r\epsilon/2$$

Hence $\rho(u, v) = \rho(T^m u, T^m v) < \lambda^m r\epsilon/2 \to 0$ as $m \to \infty$, this implies u must be equal to v.

This completes the proof of (I).

II. If $M= \rho(x, y)$,

Proof of theorem 1 in[1].

III. If $M = \rho(x, Tx)$

Let $x \in X$ and $y= Tx$,

$$\Rightarrow \rho(Tx, T^2x) \leq \lambda \rho(x, Tx) \ ,0\leq\lambda<1$$

Step1: Similarly as in Das [1], we can show $\lim_{m\to\infty} T^m x = u$.

Step2:To show that $T(u) = u$ we need two cases.

i- if $T^n x \neq T(u), u$ for any $n \in N$, then

$$\rho(u,Tu) \leq \rho(u,T^n x) + \rho(T^n x, T^{n+1} x) + \rho(T^{n+1} x, Tu)$$
$$\leq \rho(u,T^n x) + \lambda^n \rho(x, Tx) + \lambda^{n+1} \rho(x, Tx)$$
$$\leq \rho(u,T^n x) + \lambda^n [1 + \lambda] \rho(x, Tx)$$
$$\lim_{n\to\infty} \rho(u,Tu) \leq \lim_{n\to\infty} \rho(u,T^n x) + \lim_{n\to\infty} \lambda^n [1 + \lambda]\, n\epsilon/2$$

since $\lambda \in [0, 1)$, $\lambda^n \to 0$ as $n \to \infty$, thus $\lim_{n\to\infty} \rho(u,Tu) = 0$. Hence $u=Tu$.

ii- assume that $T^t x = u$, $T^t x = T^n x$ for some t

$$\rho(u, Tu) = \rho(T^t x, T^{t+1} x) \leq \lambda^t \rho(x,Tx) \to 0 \text{ as } t \to \infty. \text{Hence } u=Tu.$$

To prove the uniqueness of the fixed point, let $v \in X$, $u \neq v$ and $Tv = v$.

Since X is $\epsilon/2$ –chainable, we can find a finite chain $x_0, x_1, x_2, \ldots, x_{r-1}, x_r$ and

by the same sense of (2.10)

$$\rho(u, Tu) < r\, \epsilon/2.$$

$$\rho(T^m u, T^m v) < \lambda^m \rho(u, Tu), \ \forall m \in N.$$
$$< \lambda^m r\epsilon/2$$

Hence $\rho(u, v) = \rho(T^m u, T^m v) < \lambda^m r\epsilon/2 \to 0$ as $m \to \infty$, this implies u must be equal to

This completes the proof of (III).

IV. If $M = \rho(y,Ty)$

$$\rho(Tx, Ty) \leq \lambda\, \rho(y,Ty)$$

Let $y = Tx$, $\rho(Tx, T^2 x) < \lambda\, \rho(Tx, T^2 x)$, but $\lambda \in [0, 1)$,

$\rho(Tx, T^2 x) = 0$. Hence $Ty = y$.

To show that $\lim_{n\to\infty} T^n x = y$.

$$\rho(T^m x, y) = \rho(T^m x, T^m y) \leq \lambda^m \rho(y,Ty) = \lambda^m \rho(y,y) = 0 .$$

To prove the uniqueness, let $v \in X$, $u \neq v$ and $T v = v$.

$$\rho(u, v) = \rho(T u, T v) \leq \lambda\, \rho(v, T v) = 0, \text{ this implies } u=v.$$

This completes the proof of (IV).

V. If $M=\rho(y, Tx)$

$$\rho(Tx, Ty) \le \lambda\, \rho(y, Tx)$$

Let $y= Tx$ we obtain,

$$\rho(Tx, T^2x) \le \lambda\, \rho(Tx, Tx)=0$$

Hence $Ty=y$.

To show that $\lim_{n\to\infty} T^n x = y$.

$$\rho(T^m x,y)= \rho(T^m x, T^m y) \le \lambda^m \rho(y, T^m x),$$

but $\lambda \in [0,1)$,

$$\rho(y, T^m x)=0.$$

To prove the uniqueness, let $v \in X$, $u \ne v$ and $Tv = v$.

$$\rho(u, v) = \rho(Tu, Tv) \le \lambda\, \rho(v, Tu)= \lambda\, \rho(v,u),$$

but $\lambda \in (0,1) \Rightarrow \rho(u, v)=0$. This implies $u=v$. This completes the proof of (V). This completes the proof of theorem▪

Corollary2.10: *Theorem (1) in [1].*

Corollary2.11: *Let X as in Theorem 2.9, indeed ,condition(2)in Theorem2.9satisfied .If we replace condition (1) in Theorem2 by one of the following*

1- *If $\rho(x,y) < \delta \Rightarrow \rho(Tx,Ty) \le q \max\{\rho(x,Tx), \rho(x,y)\}$*
2- *If $\rho(x,y) < \delta \Rightarrow \rho(Tx,Ty) \le q \max\{(\rho(x,Tx) + \rho(y,Ty))/2, \rho(x,y)\}$*
3- *If $\rho(x,y) < \delta \Rightarrow \rho(Tx,Ty) \le \alpha\{\rho(x,Ty)+ \rho(y,Tx)\}$*
4- *If $\rho(x,y) < \delta \Rightarrow$ condition (2.1) satisfied*

Then T has a unique fixed point.

Finally, if we replace T in above by T^k(for positive integer k) then T also has unique fixed point .

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الخلاصة

مبرهنتين حول النقطة الصامدة في الفضاءات المترية المعممة الكاملة مساريا

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ان الهدف لهذا البحث هو برهنة نتيجتين حول وجود نقطة صامدة وحيدة لتطبيق معرف على الفضاء المتري المعمم الكامل مساريا(او الفضاء المتري المعمم الكامل مساريا ذو السلسلة).هذه النتائج تعتمد على تطبيق انكماشي معمم وتطبيق انكماشي محلي معمم والتي تتضمن بعض النتائج المعروفة كحالات خاصة،وعليه ،فأن نتائجنا توحد وتوسع نتائج كل من داس وداس - دي وسيرك.