Fuzzy Strongly Locally Connected Space

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Abstract

In this paper , we have dealt with the concept of fuzzy strongly connected set in fuzzy topological spaces .Throughout its definition ,we have defined the fuzzy regular strongly connected set and found that the every fuzzy strongly connected set is a fuzzy regular strongly connected set . We have also defined the concept of fuzzy regular locally connected by means of which we have defined fuzzy regular strongly locally connected where we have proved that every fuzzy regular strongly locally connected space is fuzzy regular locally connected .

Introduction

Throughout the present paper X and Y always denote fuzzy topological spaces on which no separation axioms are assumed unless explicitly stated .For a fuzzy set A in X , $\overset{0}{A}$ and $\overline{A}$ denote the fuzzy interior and fuzzy closure of A respectively . By 0_X and 1_X we will mean the fuzzy sets with constant function 0 (Zero function) and 1 (Unit function) respectively. This paper includes three sections. In the first section we have dealt with the concepts "fuzzy regular connected set ", and " fuzzy regular connected space" .In the second section we have discussed the concepts "fuzzy strongly set" and "fuzzy regular strongly connected" and their relation with each other ,we have also dealt with the concepts " fuzzy RT_0 -space" and "fuzzy RT_1-space" and shown that fuzzy RT_1-space is fuzzy RT_0 -space . Finally ,in the third section ,we have dealt with " fuzzy regular strongly locally connected" and some theorems related to it .

1. Prilimeries

1.1 Definition [1]

A fuzzy subset A of a fuzzy topological space X is said to be **fuzzy regular open** if and only if $A = \overset{0}{\overline{A}}$,and its complement is said to be **fuzzy regular closed** set .

1.2 Remarks

1) A fuzzy subset A is fuzzy regular closed if and only if $A = \overline{A^0}$.

2) If A is any fuzzy subset of a fuzzy topological space ,then ($\overset{0}{\overline{A}}$) is fuzzy regular open .

1.3 Remark [3]

Every fuzzy regular open set is a fuzzy open set , and every fuzzy regular closed set is a fuzzy closed set.
The converse of remark (1.3) is not true in general as the following example .

1.4 Example

Let $X = \{x, y\}$ be a set and $T = \{0_X, \{x_{0.7}, y_{0.5}\}, \{x_{0.5}, y_{0.5}\}, \{x_{0.7}, y_{0.3}\}, \{x_{0.5}, y_{0.3}\}, 1_X\}$ topology on X .
Then $A = \{x_{0.7}, y_{0.5}\}$ is a fuzzy open set in X , but it is not fuzzy regular open and $B = \{x_{0.3}, y_{0.5}\}$ is a fuzzy closed set in X but it is not fuzzy regular closed .

1.5 Remark [4]

Let A and B are two fuzzy sets in the fuzzy topological space X . Then

1) If A and B are fuzzy regular open sets , then (their intersection) $A \wedge B$ is a fuzzy regular open set .
2) If A and B are fuzzy regular closed sets , then (their union) $A \vee B$ is a fuzzy regular closed set .

1.6 Definition

A fuzzy topological space X is said to be **fuzzy regular disconnected** if and only if it is the union of two non-empty disjoint fuzzy regular open sets, otherwise is said to be **fuzzy regular connected** .

1.7 Lemma

A fuzzy subset A in a fuzzy topological space X which is both fuzzy open and fuzzy closed is fuzzy regular open and fuzzy regular closed in the same time .

Proof :

Let A is both fuzzy open and fuzzy closed set in X .

$A = \overline{A}$ because A is fuzzy closed, and since A is fuzzy open ,then $A = \overset{0}{A} = \overset{0}{\overline{A}}$.

Then A is fuzzy regular open .

Similarly

Since $A = \overset{0}{A}$,because A is fuzzy open ,and since A is fuzzy closed ,then $A = \overline{A} = \overline{A^0}$,then A is fuzzy regular closed .

1.8 Theorem

The following statements are equivalent.
1) X is fuzzy regular disconnected space .
2) There are two non-empty fuzzy regular closed A, B such that $A \wedge B = 0_X$,and $A \vee B = 1_X$.
3) There exist fuzzy regular open and fuzzy regular closed A in the same time such that $A \neq 0_X$, $A \neq 1_X$.
4) There exist fuzzy subset A in a fuzzy topological space X such that $A \neq 0_X$, $A \neq 1_X$ and $b(A) = 0_X$.

Proof :-

1 $\Longrightarrow$ 2

Since X is fuzzy regular disconnected ,then there are two non-empty fuzzy regular open sets A, B such that $A \wedge B = 0_X$,and $A \vee B = 1_X$.

Then $A^c \vee B^c = 1_X$,and $A^c \wedge B^c = 0_X$.

Since A and B are fuzzy regular open, then A^c and B^c are fuzzy regular closed .

2 $\Longrightarrow$ 3

Since there are two non-empty fuzzy regular closed A and B such that $A \wedge B = 0_X$,and $A \vee B = 1_X$.

Since $A \wedge B = 0_X$,then $A \leq B^c$,and since $A \vee B = 1_X$,then $B^c \leq A$.

Then $A = B^c$.

Since B^c is fuzzy regular open, then A is fuzzy regular open and fuzzy regular closed in the same time.

If $A = 1_X$,since $A \wedge B = 0_X$,then $B = 0_X$,this contradicting.

Then $A \neq 0_X$, $A \neq 1_X$.

3 $\Longrightarrow$ 4

Let A be fuzzy regular open and fuzzy regular closed in the same time such that $A \neq 0_X$, $A \neq 1_X$.

Since A is fuzzy regular closed ,then A is fuzzy closed, so $b(A) \wedge A = 0_X$.

Since A is fuzzy regular open, then A^c is fuzzy closed ,so $b(A^c) \wedge A^c = 0_X$.

Then $b(A) = b(A^c)$,thus $b(A) = 0_X$.

4 $\Longrightarrow$ 1

Since $b(A) = 0_X$, then A is fuzzy open and fuzzy closed in the same time, so A is fuzzy regular open and fuzzy regular closed in the same time by (1.7) .

Since $A \neq 1_X$,then $A^c \neq 0_X$, and since $A \wedge A^c = 0_X$,and $A \vee A^c = 1_X$ .Then X is fuzzy regular disconnected .

1.9 Theorem

A fuzzy topological space X is fuzzy regular connected if and only if it is fuzzy connected .

Proof :-

Suppose X is fuzzy disconnected ,then there are fuzzy open sets A and B such that $A \wedge B = 0_X$,and $A \vee B = 1_X$.

Then $A^c \wedge B^c = 0_X$,thus $B^c \leq A$.

Since $A \wedge B = 0_X$,then $A \leq B^c$.

Then $A = B^c$,that is A is both fuzzy open and fuzzy closed .

Then A is fuzzy regular open from (1.7)

Similarly ,we can prove that B is fuzzy regular open .

Then X is fuzzy regular disconnected .

The converse it is clear ,since every fuzzy regular open set is fuzzy open .

Now ,we introduce the following definition

A fuzzy subset A of a fuzzy topological space (X,T) is said to be **fuzzy regular connected** if the fuzzy subspace (A,T_A) is fuzzy regular connected ,otherwise A is **regular disconnected** .Then from (1.9), we have the following results.

1.10 Proposition

A fuzzy subset A in a fuzzy topological space X is fuzzy regular connected if and only if it is fuzzy connected .

1.11 Proposition

If $\{A_i \,/\, i \in I\}$ is a collection of fuzzy regular connected sets from a fuzzy topological space (X,T) such that $\bigwedge_{i \in I} A_i \neq 0_X$, $\bigvee_{i \in I} A_i = 1_X$ .Then X is fuzzy regular connected .

2. Fuzzy Regular Strongly Connected Sets in Fuzzy Topology

2.1 Definition

A fuzzy subset A of a fuzzy topological space X is said to be **fuzzy strongly connected** if and only if for each fuzzy open sets B and C such that $A \leq B \vee C$,then $A \leq B$ or $A \leq C$.

2.2 Theorem

If A is fuzzy strongly connected set ,then A is fuzzy connected.

Proof :-

Suppose A is fuzzy disconnected set, then there are two fuzzy open sets B and C , such that $A \leq B \vee C$, $B \wedge C = 0_X$.

Since $A \leq B \vee C$ and $B \wedge C = 0_X$, then A not contained in B and A not contained in C .

Then A is fuzzy strongly disconnected , this contradiction , thus A is fuzzy connected .

2.3 Definition
A fuzzy set A of a fuzzy topological space X is said to be **fuzzy regular strongly connected** if and only if for each fuzzy regular open sets B and C such that $A \leq B \vee C$,then $A \leq B$ or $A \leq C$.

2.4 Example
Singelation set is fuzzy regular strongly connected in any fuzzy topological space .

2.5 Theorem
Every fuzzy strongly connected set is fuzzy regular strongly connected .

Proof:-

Let A be fuzzy strongly connected set and let B, C be two fuzzy regular open sets such that $A \leq B \vee C$. Since B and C are fuzzy regular open sets ,then B and C are fuzzy open sets from (1.3) , and since A is fuzzy strongly connected ,then $A \leq B$ or $A \leq C$,So A is fuzzy regular strongly connected.

2.6 Remark
It is easy to see that if A is fuzzy regular strongly connected ,then A is fuzzy regular connected .

2.7 Theorem
If A is fuzzy regular strongly connected set ,then A is fuzzy connected .

The proof is clear from (2.6) and (1.9) .

2.8 Definition
A fuzzy topological space X is said to be **regular strongly connected** if and only if the only non-empty fuzzy subset of X which is both fuzzy regular open and fuzzy regular closed in X is 1_X .

2.9 Theorem
A fuzzy topological space X is regular strongly connected if and only if is fuzzy connected space

Proof :-

Suppose X is fuzzy disconnected ,then there exists a fuzzy subset A such that $A \neq 0_X , A \neq 1_X$,and A is both fuzzy open and fuzzy closed in the same time.

Then by (1.7), A is both fuzzy regular open and fuzzy regular closed in the same time, which contradicts being X is fuzzy regular strongly connected .

The converse proof is trivial .

2.10 Definition
A fuzzy subset A of a fuzzy topological space X is said to be **fuzzy regular weakly disconnected** if and only if it is not fuzzy regular strongly connected .

2.11 Definition [2]
Let f be a mapping from fuzzy topological space (X,T) into a fuzzy topological space (Y,T') ,then f is said to be **fuzzy regular continuous** if the inverse image of any fuzzy regular open (fuzzy regular closed) in Y is fuzzy regular open (fuzzy regular closed) in X .

2.12 Theorem
If $f:(X,T) \to (Y,T')$ is fuzzy regular continuous and if A is fuzzy regular strongly connected in X ,then $f[A]$ is fuzzy regular strongly connected in Y .

Proof :-

Let $f[A]$ be a fuzzy regular weakly disconnected set in Y ,then there exist two fuzzy regular open sets B, C in Y such that $f[A] \leq B \vee C$, $f[A]$ not contained in B and $f[A]$ not contained in C .

Thus $A \leq f^{-1}(f[A]) \leq f^{-1}(B \vee C) = f^{-1}(B) \vee f^{-1}(C)$.

Then $A \leq f^{-1}(B) \vee f^{-1}(C)$, A not contained in $f^{-1}(B)$ and A not contained in $f^{-1}(C)$.

Since f is fuzzy regular continuous and since B,C are fuzzy regular open ,then $f^{-1}(B), f^{-1}(C)$ are fuzzy regular open in X .Then A is fuzzy regular weakly disconnected .

2.13 Definition

A fuzzy topological space (X,T) is said to be **fuzzy** RT_0 if and only if for each pair of fuzzy points x_α, y_β such that $x \neq y$ in X ,there exists a fuzzy regular open set in X which contains one of them and not the other.

2.14 Example

Let $X = \{a,b\}$ and $T = \{0_X, \{a_{0.5}\}, \{a_{0.5}, b_{0.5}\}, 1_X\}$ be fuzzy topology on X , then (X,T) is fuzzy RT_0 .

2.15 Definition

A fuzzy topological space (X,T) is said to be **fuzzy** RT_1 if and only if for each pair of fuzzy points x_α, y_β such that $x \neq y$ in X ,there exists A,B be two fuzzy regular open sets such that $x_\alpha \in A, y_\beta \notin A, x_\alpha \notin B, y_\beta \in B$.

2.16 Example

Every discrete fuzzy topological space is fuzzy RT_1-space .

2.17 Theorem

Every fuzzy RT_1-Space is fuzzy RT_0 -Space .

Proof :-

Let (X,T) be fuzzy RT_1-Space and let x_α, y_β are two fuzzy points such that $x \neq y$ in X .

Since X is fuzzy RT_1 -Space ,then there are fuzzy regular open sets A,B such that $x_\alpha \in A, y_\beta \notin A, x_\alpha \notin B, y_\beta \in B$.

Then there exists fuzzy regular open set in X which contains one of them and not the other .

Then (X,T) is fuzzy RT_0 -space .

2.18 Remark

The converse of the above theorem is not true in general (see example 2.14) .

2.19 Definition

A fuzzy topological space (X,T) is said to be **regular totally weakly disconnected** if and only if fuzzy singelation sets are the only fuzzy regular strongly connected sets .

2.20 Theorem

A fuzzy topological space (X,T) is fuzzy RT_1 if and only if it is fuzzy regular totally weakly disconnected .

Proof :-

Fuzzy singelation sets are clearly fuzzy regular strongly connected .

Now

Suppose A is a fuzzy subset of X with two or more points .

let x_α, y_β are fuzzy points such that $x \neq y$ in A ,then $\{x_\alpha\}$ and $\{y_\beta\}$ are non-empty disjoint fuzzy regular closed subsets of A ,then A is regular weakly disconnected .Then X is fuzzy regular totally weakly disconnected .

Conversely

Let x_α, y_β are two fuzzy points ,such that $x \neq y$ in X .

Then $A = \{x_\alpha, y_\beta\}$ is not fuzzy regular strongly connected .Thus there are two fuzzy regular open sets B,C such that $A \leq B \vee C$, A not contained in B and A not contained in C .

Since $A \leq B \vee C$,then $x_\alpha \in B \vee C$.

If $x_\alpha \in B$,then $y_\beta \notin B$ and $y_\beta \in C, x_\alpha \notin C$.

Then X is fuzzy RT_1-Space .

3. Fuzzy Strongly Locally Connectivity

3.1 Definition

A fuzzy topological space (X,T) is said to be **fuzzy locally connected** if and only if for each fuzzy point $a_\alpha \in X$ and for each fuzzy open set A in X such that $a_\alpha \in A$,there exists a fuzzy connected open set B such that $a_\alpha \in B \le A$.

3.2 Definition

A fuzzy topological space (X,T) is said to be **fuzzy regular locally connected** if and only if for each fuzzy point $a_\alpha \in X$ and for each fuzzy open set A in X such that $a_\alpha \in A$,there exists a fuzzy regular open connected set B such that $a_\alpha \in B \le A$.

3.3Theorem

Every fuzzy regular locally connected space is fuzzy locally connected .

Proof :-

Let (X,T) be fuzzy regular locally connected topological space, and let a_α be fuzzy point in X , A is fuzzy open set such that $a_\alpha \in A$.

Since (X,T) is regular locally connected ,then there exists fuzzy regular open connected set B such that $a_\alpha \in B \le A$.

Then X is locally connected (because every fuzzy regular open connected set is fuzzy open connected).

3.4 Definition

If $x_\alpha \in X$ be a fuzzy point ,the largest fuzzy regular connected subset $C_{x\alpha}$ of a fuzzy space X containing x_α is said to be the **fuzzy regular component of** x_α .It exists being just the union of all fuzzy regular connected subsets of X containing x_α .

3.5 Proposition

The fuzzy regular component of a fuzzy regular connected space are fuzzy regular open (fuzzy regular closed).

Proof :-

Let (X,T) be fuzzy regular locally connected space.

Then X is locally connected space by theorem (3.3) .

Then the proof is complete by theorem (1.9) .

3.6 Theorem

Let (X,T) be a fuzzy topological space, then the following statements are equivalent .

1) X is fuzzy regular locally connected

2) If C is fuzzy regular component of a fuzzy regular subspace Y in X ,then $b(C) \le b(Y)$.

3) Every fuzzy regular component of a fuzzy regular open subspace in X be fuzzy regular open .

Proof :-

$1 \Longrightarrow 2$

Let $Y \le X$,and C be fuzzy regular component of a fuzzy subspace Y in X .

Let $a_\alpha \in b(C)$,then $a_\alpha \in \overline{C} = \mathrm{b(C)} \vee \mathrm{C}$.

Since $C \le Y$,then $\overline{C} \le \overline{Y}$ and $a_\alpha \in \overline{Y}$,thus $a_\alpha \in b(Y) \vee Y^0$.

Suppose $a_\alpha \notin b(Y)$,then $a_\alpha \in Y^0$.

Since $Y^0 \in T$ and X is fuzzy regular locally connected ,then there is a fuzzy regular open connected set A in X such that $a_\alpha \in A \le Y^0$.

Since $a_\alpha \in b(C)$,then $A \wedge C \ne 0_X$.

Since A and C are fuzzy connected (because every fuzzy regular connected set is fuzzy connected).
Then $A \vee C$ is fuzzy connected ,but C is fuzzy regular component ,then $C = A \vee C$,thus $A \le C$.
So $A \in C^0$,this contradicts, then $a_\alpha \in b(Y)$ and $b(C) \le b(Y)$.

$2 \Longrightarrow 3$

Let Y be fuzzy regular open subspace in X ,and let C be fuzzy regular component of Y.
Then we have $b(C) \le b(Y)$ from 2.

Since $C \le Y$,then $b(C) \wedge C \le b(Y) \wedge Y \le b(Y) \wedge Y^0 = 0_X$.

Then $b(C) \wedge C = 0_X$ .Thus $C = C^0$,hence C is fuzzy regular open .

$3 \Longrightarrow 1$

Let x_α be a fuzzy point in X ,and A is a fuzzy open set in X such that $x_\alpha \in A$.

Let $C_{x\alpha}$ be a fuzzy regular component of x_α in a fuzzy subspace (A, T_A).

From (3) C_x is fuzzy regular open connected set and $x_\alpha \in C_x \le A$.

Then X is fuzzy regular locally connected.

3.7 Corollary

Every fuzzy regular component in a fuzzy locally connected space is fuzzy regular open .

3.8 Definition

A fuzzy topological space (X, T) is said to be **fuzzy regular strongly locally connected** if and only if for each fuzzy point $a_\alpha \in X$ and for each fuzzy open set A in X such that $a_\alpha \in A$,there exists a fuzzy regular strongly connected open set B such that $a_\alpha \in B \le A$.

3.9 Theorem

If X is fuzzy regular strongly locally connected space , then X is fuzzy regular locally connected .

Proof :-

Let a_α be a fuzzy point in X and A be a fuzzy open set such that $a_\alpha \in A$.

Since X is fuzzy regular strongly locally connected space, then there is a fuzzy regular strongly connected open set B such that $a_\alpha \in B \le A$.

Since every fuzzy regular strongly connected is regular connected ,then (X, T) is fuzzy regular locally connected .

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الفضاء الضبابي المتصل محلياً بقوة

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الخلاصة

في هذا البحث تناولت مفهوم الاتصـال المنتظم في الفضـاءات التبولوجية المضـبضـبة ومن خلاله قمت بتعريف الاتصـال الأقوى المنتظم ووجدت أن كل مجموعة متصلة بقوة تكون منتظمة متصلة بقوة والعكس غير صحيح دائماً. أيضاً تناولت مفهوم الاتصال المحلي المنتظم ومن خلاله عرفت مفهوم الاتصـال المحلي الأقوى المنتظم وبرهنت أن كل فضـاء يحقق مفهوم الاتصـال المحلي الأقوى المنتظم فانه يحقق مفهوم الاتصال المحلي المنتظم والعكس غير صحيح دائماً .