On Solving Special Case of System Of Initial Value Problems using Semi-Analytic Technique

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Abstract:-

The aim of this paper is to present a method for solution special case of system of first order nonlinear initial value problems of ordinary differential equation by a semi-analytic technique with constructing polynomial solutions. The original problem is concerned with using two-point osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0 , 1] .

1. Introduction:-

Many problems in engineering and science can be formulated in terms of differential equations [1],[2] .

In this paper we introduce the mathematical model of the damped simple pendulum which introduced in [3] for some of its orbits.

Consider the second-order ODE that models the motion of an undriven pendulum of length L supporting a metal bob of mass m and subject to a damping force :

$$\mathbf{mL\, y_1'' + cLy_1' + mg\sin(y_1) = 0}$$

where c is the damping coefficient ,y_1 is the angle of the pendulum measured counter- clockwise from the vertical ,and y_1' is the angular velocity .Divide by mL ,set c/ m = α and g / L = 10. Using y_1 and $y'_1 = y_2$ as state variable and imposing specific initial conditions ,we have the IVP :

$y'_1 = y_2$, $y_1(0) = 0$

$y'_2 = -10\sin(y_1) - \alpha y_2$, $y_2(0) = 10$

the initial conditions model the pendulum hanging downward at angle 0 radians and with an initial counterclockwise angular velocity of 10 radians / sec .

How changing the Mass Affects the Motion of the Pendulum for α = c / m , and c is assumed to be a fixed positive number ,we see that changing the mass in the pendulum model(1) has an inverse effect on damping force .So the pendulum with a large mass (hence , light "friction") might whirl all the way up and over the pivot several times before frictional forces gradually compel the pendulum .

To settle down to back-and-forth decaying oscillations about a downward vertical position .The big change occurs when α switches from value 0 to a positive number, i.e., When the damping is "turned on" .Of about 3 radians ,rightmost point on the leftmost orbit. The pendulum that stops and falls back into a mode of decaying oscillations about $y_1 = 0$, $y_2 = 0$.

It would be interesting to determine those crucial values of α which divide orbits that approach the equilibrium point (2π ,0) from those approaching the equilibrium point ((2n-2)π , 0). The systems̉ behavior is quite sensitive to small changes in the value of α near these critical values .

2. Description of Method

In this section ,we illustrates the semi-analytic technique which has general application to equations of the following type :

$$y'_1 = y_2 \quad , \quad y_1(0) = 0$$
$$y'_2 = -10\sin(y_1) - \alpha\ y_2 \quad , \quad y_2(0) = 10 \quad \ldots\ldots\ldots \quad (1)$$

Where $\alpha \in (0, 2\pi)$

We are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0 , 1] ,wherein a useful and succinct way of writing osculatory interpolant $P_{2n+1}(x)$ of degree 2n + 1 was given for example by Phillips [4] as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ y^{(j)}(0)\, q_j(x) + (-1)^j\ y^{(j)}(1)\, q_j(1-x) \} \quad \ldots\ldots\ldots\ldots(2)$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j! \quad \ldots\ldots\ldots\ldots(3)$$

So that (2) with (3) satisfies :

$y^{(j)}(0) =(1)$, j = 0, 1, 2,…, n . $P_{2n+1}^{(j)}$ (1) = $^{(j)}(0)$, y $P_{2n+1}^{(j)}$

Implying that $P_{2n+1}(x)$ agrees with the appropriately truncated Taylor series for y(x) about x = 0 and x = 1.

Finally we observe that (2) can be written directly in terms of the Taylor coefficients a_i and b_i about x = 0 and x = 1 respectively, as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ a_j\, Q_j(x) + (-1)^j\, b_j\, Q_j(1-x) \} \quad \ldots\ldots\ldots \quad (4)$$

The simple idea behind the use of two-point polynomials is to replace y(x) in problem by a P_{2n+1} (equation(2) or (4)) which enables any unknown derivatives of y(x) to be computed . The first step therefore is to construct the P_{2n+1} . To do this we need the Taylor coefficients of $y_1(x)$ and $y_2(x)$ respectively about x = 0 :

$$y_1 = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \qquad \ldots\ldots\ldots \quad (5a)$$

$$y_2 = b_0 + b_1 x + \sum_{i=2}^{\infty} b_i x^i \qquad \ldots\ldots\ldots \quad (5b)$$

where $y_1(0) = a_0$, $y_1'(0) = a_1$, … , $y_1^{(i)}(0) / i! = a_i$, i = 2, 3, ……
and $y_2(0) = b_0$, $y_2'(0) = b_1$, … , $y_2^{(i)}(0) / i! = b_i$, i = 2, 3, ……

then insert the series forms (5a) and (5b) respectively into (1) and equate coefficients of powers of x .

Also ,we need Taylor coefficients of $y_1(x)$ and $y_2(x)$ about x = 1, respectively

$$y_1 = c_0 + c_1 (x-1) + \sum_{i=2}^{\infty} c_i (x-1)^i \qquad \ldots\ldots\ldots \quad (6a)$$

$$y_2 = d_0 + d_1 (x-1) + \sum_{i=2}^{\infty} d_i (x-1)^i \qquad \ldots\ldots\ldots \quad (6b)$$

where $y_1(1) = c_0$, $y_1'(1) = c_1$, … , $y_1^{(i)}(1) / i! = c_i$, i = 2, 3, ……
and $y_2(1) = d_0$, $y_2'(1) = d_1$, … , $y_2^{(i)}(1) / i! = d_i$, i = 2, 3, ……
then insert the series forms (6a) and (6b) respectively into (1) and equate coefficients of powers of (x – 1) .

The resulting system of equations can be solved using MATLAB version 7.9 to obtain a_i , b_i , c_i and d_i ,for all $i \geq 2$, we see that c_i's and d_i's coefficients depend on indicated unknowns c_0 and d_0 .

The algebraic manipulations needed for this process .We are now in a position to construct a $P_{2n+1}(x)$ and $\widetilde{P}_{2n+1}(x)$ from (5) and (6) of the form (2) by the following :

$$P_{2n+1}(x) = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i c_i Q_i(1-x) \} \qquad \ldots\ldots\ldots\ldots(7a)$$

and

$$\widetilde{P}_{2n+1}(x) = \sum_{i=0}^{n} \{ b_i Q_i(x) + (-1)^i d_i Q_i(1-x) \} \qquad \ldots\ldots\ldots\ldots(7b)$$

Where $Q_i(x)$ defined in (3) ,

We see that (7) have only two unknowns c_0 and d_0 .

Now, integrate equation (1) to obtain :

$$c_0 - a_0 = \int_0^1 f_1(x, y_1, y_2)\, dx \qquad \ldots\ldots\ldots \quad (8a)$$

$$d_0 - b_0 = \int_0^1 f_2(x, y_1, y_2)\, dx \qquad \ldots\ldots\ldots \quad (8b)$$

use P_{2n+1} and $\tilde{P}_{2n+1}$ as instead of y_1 and y_2 respectively in (8) .

Since we have only the two unknowns c_0 and d_0 to compute for any n, we only need to generate two equations from this procedure the two equations are already supplied by (8) and initial condition. Then solve this system of algebraic equations using MATLAB version 7.9 to obtain c_0 and d_0 ,so insert it into (7) thus (7) represent the solution of (1) .

The difficulty of this problem that is not in general possible to perform the integration involving sin(P_{2n+1} (x)) by using (8) .

Consider the use of alternative strategies .However we choose to continue with the integral form and replace sin(P_{2n+1}(x)) itself by a two-point polynomial $q_{2n+1}(x)$ in (8). Thus by (5) and (6) we can write :

$$\mathbf{\sin(y(x)) = \sum_{j=0}^{M} A_j(a_0, a_1)\, x^j}$$

$$\mathbf{\sin(y(x)) = \sum_{j=0}^{M} B_j(b_0, b_1)\, (x-1)^j}$$

for which we can construct the $q_{2n+1}(x)$. Hence (8) become :

$$\mathbf{c_0 - 0 = \int_0^1 \tilde{P}_{2n+1}(x)\, dx} \qquad \ldots\ldots\ldots \quad \mathbf{(9)}$$

$$\mathbf{d_0 - 10 = -10 \int_0^1 q_{2n+1}(x)\, dx - \alpha \int_0^1 \tilde{P}_{2n+1}(x)\, dx} \qquad \ldots\ldots\ldots \quad \mathbf{(10)}$$

Here (9) and (10) becomes :

$$\mathbf{F(c_0,d_0,w) = -c_0 +} \int_0^1 \tilde{P}_{2n+1}(x)\, dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$\mathbf{G(c_0,d_0,w) = 10 - d_0 - 10} \int_0^1 q_{2n+1}(x)\, dx - w \int_0^1 \tilde{P}_{2n+1}(x)\, dx \quad \ldots\ldots\ldots\ldots \quad (12)$$

Now ,for each value of w there is corresponding for c_o and d_0 .This suggests that there exists w $= w^*$ such that for $w > w^*$. There are no solutions of the IVP, while for $w<w^*$ there are two. This of course is a well- known feature of the problem .What we do now is to compute the threshold value w^* using our two-points method. Essentially this involves finding double root of (11) and (12) for (c_0,d_0). Thus we have to solve (11) and(12) together with :

$$\frac{\partial F}{\partial c_o}\frac{\partial G}{\partial d_o}-\frac{\partial G}{\partial c_o}\frac{\partial F}{\partial d_o}=0$$

For the unknowns c_0,d_0 and w. The results for n=2,3 are shown in table (1). We note that there is no difficulty in taking higher values of n, if we wished to refine this value. Then from equations (2) and (3) we have :

$$P_5 = -2.7107042776\,x^5 + 3.7020658293\,x^4 + 5.442563977570236\,x^3 - 14.2142307069\,x^2 + 10\,x$$

$$P_7 = -0.2860462788x^7 + 9.2735798449x^6 - 9.1060233661x^5 + 30.4059634083x^4 - 4.5572962569x^3 - 13.4774091037\,x^2 + 10x$$

$$\tilde{P}_5 = 51.8376420299x^5 - 143.1476264633x^4 + 118.4835473770126x^3 - 9.59112908226x^2 - 28.4284614138x + 10$$

$$\tilde{P}_7 = -128.83009481802\,x^7 + 448.9030079118\,x^6 - 524.0939476192\,x^5 + 176.5451202145\,x^4 + 57.2088062242\,x^3 - 13.6718887705\,x^2 - 26.9548182073\,x + 10$$

For more details ,table (1) gives the results of different nodes in the domain , for n = 2 , 3 . Table (2) gives a comparison between different methods to illustrate the accuracy of suggested method .

Table 1 : The result of the method for n = 2 , 3 of example

	P_5	P_7	$\tilde{P}_5$	$\tilde{P}_7$
c_o	2.219694822391575359	2.252768247672827718	2.219694822391575359	2.252768247672827718
d_o	-0.8460275517985956	-0.8938150580593765	-0.8460275517985956	-0.8938150580593765
w	2.842846141378345752	2.695481820727516889	2.842846141378345752	2.695481820727516889
X	P_5	P_7	$\tilde{P}_5$	$\tilde{P}_7$
0	0	0	10	10
0.1	0.863643356448571	0.863427393788980	7.16592972894538	7.23785769033205
0.2	1.48002716350200	1.46437072749269	4.66608277606805	4.66167415390903
0.3	1.89106818559481	1.84624472994892	2.77378543340702	2.68326464856785
0.4	2.14106245488569	2.06981049178053	1.54322003036112	1.47239865906943
0.5	2.27343242612440	2.19446012332854	0.871630104124934	0.919478014902600
0.6	2.32747413151871	2.26574588809978	0.561525570124699	0.726634070419731
0.7	2.33510433560132	2.30900964440493	0.382887892454248	0.562313579515600
0.8	2.31760769009681	2.32896842786205	0.135375254310834	0.214422897061533
0.9	2.28238388878854	2.31511200744141	-0.289472271568860	-0.322899860693918
1	2.21969482238558	2.25276824772689	-0.846027552472862	-0.893815064545724

Table 2: A comparison between different methods of example

X_i	Y_1 by using RK method	Y_1 by using AMB method	P_7 by using Osculatory interpolation
0	0	0	0
0.1	0.862623503985462	0.862669856256729	0.863427393788980
0.2	1.45797311117741	1.45819725104960	1.46437072749269
0.3	1.82726417857771	1.82744679245478	1.84624472994892
0.4	2.02488229881121	2.02533707557721	2.06981049178053
0.5	2.09686353714629	2.09725218368877	2.19446012332854
0.6	2.07572364294706	2.07610269179243	2.26574588809978
0.7	1.98257884957417	1.98297735088020	2.30900964440493
0.8	1.83095294173355	1.83138106782495	2.32896842786205
0.9	1.63049143871119	1.63095014139590	2.31511200744141
1	1.39021250706271	1.39070333814617	2.25276824772689

X_i	Y_2 by using RK method	Y_2 by using AMB method	$\tilde{P}_7$ by using Osculatory interpolation
0	10	10	10
0.1	7.24919381452187	7.2491601043333	7.2378576903320
0.2	4.73292943114355	4.7329184366403	4.6616741539090
0.3	2.74704714454101	2.7474732904896	2.6832646485678
0.4	1.28263229675577	1.2821759504236	1.4723986590694
0.5	0.210727222320128	0.2105663865417	0.9194780149026
0.6	0.598981593153335	0.5989529888894	0.7266340704197

0.7	-1.24198646242225	1.2418807869361	0.5623135795156
0.8	-1.77499394396582	1.7748696709399	0.2144228970615
0.9	-2.21964146069200	2.2195531898594	0.3228998606939
1	-2.56791543515899	2.5679723802559	0.8938150645457

3. WELL POSED PROBLEM

Most problems obtained by observing physical phenomena generally thus, we have only approximate the actual situation, it is of interest to know whether small changes in the statement of the problem will introduce correspondingly small changes in the solution. This is also important because of the possibility of rounding errors when numerical methods are used .

To discuss this problem we need the following definition :

Definition 1 [5]

The initial value problem :

$$\frac{dy}{dx} = (f, y), \quad a \le x \le b, \; y(a) = \alpha \qquad \ldots\ldots \quad (13)$$

is said to be a **well-posed problem** if :

i) a unique solution, to the problem exists ;

ii) a perturbed problem : small perturbations of the data (initial conditions!) cause only small perturbations of the solution .

Otherwise the problem is said to be **ill-posed problem .**

For more details , the following theorem give specifies conditions which ensure that an initial value problem is well-posed. The proof of this theorem can be found in the [5] .

Theorem 1

Suppose $D = \{(x, y) \mid a \le x \le b \text{ and } -\infty < y < \infty\}$, the initial value problem :

$$\frac{dy}{dx} = f(x, y), \quad a \le x \le b, \quad y(a) = \alpha. \quad ...... \quad (14)$$

is well-posed provided is continuous and satisfies a Lipschitz condition in the variable y on the set D .

Consider the perturbed problem :

$$\frac{dz}{dx} = -z + x + 1 + \delta, \quad 0 \le x \le 1, \quad z(0) = 1 + \varepsilon_{o,}$$

where δ and ε_o are constants. The solutions of equation can be shown to be $y(x) = e^{-x} + x$ and $z(x) = (1 + \varepsilon_o - \delta)e^{-x} + x + \delta,$

respectively. It is easy to verify that if $|\delta| < \varepsilon$ and $|\varepsilon_0| < \varepsilon$, then :

$$|y(x) - z(x)| = |(\delta - \varepsilon_0)e^{-x} - \delta| \le |\varepsilon_0| + |\delta||1 - e^{-x}| \le 2\varepsilon$$

For all x , which corroborates the result obtained by the use of theorem 1 .

In the same manner to determine whether a particular problem of system of first order initial value problem has the property that small changes or perturbations in the statement of the problem introduce correspondingly small changes in the solution ? As usual , we generalize above workable by the following theorem :

Theorem 2

Suppose $D = \{(x, y_1, ..., y_m): a \le x \le b, -\infty \langle y_i \langle \infty, \textit{ for each } i = 1,2,...,m\}$. the system of IVP is well-posed , if it is continuous and satisfies a Lipschitz condition defined on the set D .

References:-

[1] D. Houcque , (2006), **"Ordinary Differential Equations** ", Northwestern University.

[2] R. Bronson, (2003) **, "Differential Equations" , Second Edition .**

[3] L . B . Robert and S . C . Courtney , (1996) , "Differential Equations A Modeling Perspective" , United States of America.

[4] G .M.Phillips , (1973) , "Explicit forms for certain Hermite approximations", BIT 13 , pp. 177-180 .

[5] L. R. Burden and J. D. Faires , (2001), "Numerical Analysis" , Seventh Edition.

حول حـــل حالة خاصة من مسائـل القيم الابتدائية باستخدام التقنية شـــبه التحـليلية

لمى ناجي محمد توفيق هبه عواد عبد الرزاق
كلية التربية ابن الـهيثم كلية العلوم
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الخلاصة :-

الهدف من البحث هو أيجاد طريقة لحل حالة خاصـــة لمنظومة معادلات تفاضـــلية اعتيادية غير خطية من الرتبة الأولى لمسائل القيم الابتدائية باستخدام التقنية شبه التحليلية وذلك بإيجاد الحل بشكل متعددة حدود .

المســالة الأصـلية تتعلق باســتخدام الاندراج التماســي ذو النقطتين والتي تتفق فيها الصــورة وعدد متســاوي من المشتقات المعرفة عند نقطتي نهاية الفترة [0 , 1] مع البيانات المعطاة .