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The Composition Operator C_{ϕ_β} Induced by The Function ϕ_β

By
Aqeel Mohammed Hussain
Department of Mathematics
College of Education
.
University of Qadisyia

Abstract

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} \left|f^{\wedge}(n)\right|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f .

Let ψ be a holomorphic self-map of U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f \circ \psi \qquad (f \in H^2)$$

We have studied the composition operator induced by the automorphism ϕ_β and discussed the adjoint of the composition of the symbol ϕ_β.We have look also at some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function ψ in U.

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

Introduction

This search consists of two sections . In section one ,we are going to the automorphism ϕ_β and properties of ϕ_β,and also discuss the interior and exterior fixed points of ϕ_β and also discuss ϕ_β is rotation a round the origin and ϕ_β is elliptic and ϕ_β is a linear fractional transformation .

In section two, we are going to the Composition Operator C_{ϕ_β} induced by the symbol ϕ_β and properties of C_{ϕ_β}, and also discuss the adjoint of Composition Operator C_{ϕ_β} induced by the symbol ϕ_β and also discuss C_{ϕ_β} is an invertible operator and C_{ϕ_β} is normal operator and define eigenvalue of C_{ϕ_β} .

Section One

Definition(1.1) : [4]

Let U = $\{z \in C : |z| \prec 1\}$ is called unit ball in complex C and ∂U = $\{z \in C : |z| = 1\}$ is called boundary of U

Example(1.2):

For $\beta \in U$, define $\phi_\beta(z) = \dfrac{z}{2+\bar{\beta}z}$ $(z \in U)$. Since the denominator equal zero only at $z = \dfrac{-2}{\bar{\beta}}$, the function ϕ_β is holomorphic on the ball $\{|z| \prec \dfrac{2}{|\beta|}\}$. Since $\beta \in U$, then this ball contains U. Hence ϕ_β take U into U and holomorphic on U .

Definition(1.3) : [10]

Let $\psi : U \to U$ and holomorphic on U. We say that ψ is called conformal automorphism or automorphism of U if and only if ψ is injective and surjective .

Proposition (1.4) :

for $\beta \in U$, ϕ_β is conformal automorphism or automorphism of U .

Proof:

Since $\phi_\beta(z) = \dfrac{z}{2+\bar{\beta}z}$ $(z \ ,\beta \ \in U)$

Suppose $\phi_\beta(z_1) = \phi_\beta(z_2)$ that is $\dfrac{z_1}{2+\bar{\beta}z_1} = \dfrac{z_2}{2+\bar{\beta}z_2}$, therefore $\bar{\beta}z_1 z_2 + 2z_1 = \bar{\beta}z_1 z_2 + 2z_2$, hence $z_1 = z_2$. Thus ϕ_β is injective .

Let $y = \phi_\beta(z)$, that is $y = \dfrac{z}{2+\bar{\beta}z}$, therefore $\bar{\beta} \ z \ y + 2y = z$, then $z - \bar{\beta} \ z \ y = 2y$, hence

$z=\frac{2y}{1-\bar{\beta}y}$, $\phi_\beta(z)=\phi_\beta\left(\frac{2y}{1-\bar{\beta}y}\right)=\frac{\frac{2y}{1-\bar{\beta}y}}{2+\frac{2\bar{\beta}y}{1-\bar{\beta}y}}=\frac{\frac{2y}{1-\bar{\beta}y}}{\frac{2-2\bar{\beta}y+2\bar{\beta}y}{1-\bar{\beta}y}}=y$, for every $y\in U$ there exists $z\in U$ such that $\phi_\beta(z)=y$ .Thus ϕ_β is surjective . Hence ϕ_β is automorphism .

Definition(1.5) : [10]

A point $p\in C$ is a fixed point for the the function ψ, if $\psi(p)=p$.

Proposition (1.6) :

For $\beta\in U$, then 0 , $\frac{-1}{\bar{\beta}}$ are fixed points for ϕ_β .

Proof :

Let $\phi_\beta(z)=z$ that is $\frac{z}{2+\bar{\beta}z}=z$, therefore $\bar{\beta}\, z^2+z=0$. Hence ϕ_β has two fixed points $z_1=0$, $z_2=\frac{-1}{\bar{\beta}}$

Definition(1.7): [4]

Let $\psi:U\rightarrow U$ and holomorphic on U that fixed point r, then:

1) r is interior fixed point for ψ if $r\in U$
2) r is exterior fixed point ψ if $r\notin U$

Proposition (1.8):

Then 0 is interior fixed point and $\frac{-1}{\bar{\beta}}$ is exterior fixed point for ϕ_β.

Proof :

Since ϕ_β has two fixed points $z_1=0$, $z_2=\frac{-1}{\bar{\beta}}$, $|z_1|=|0|=0\prec 1$. Thus z_1 is interior fixed point

Since $\beta \in U$, then $|\beta| \prec 1$, therefore $\left|\frac{-1}{\beta}\right| = \left|\frac{1}{\beta}\right| = \frac{1}{|\beta|} \succ 1$, hence $|z_2| = \left|\frac{-1}{\beta}\right| \succ 1$. Thus z_2 is exterior fixed point

Remark(1.9) :

For $\beta \in U, \phi_\beta^{-1}(z) = \frac{2z}{1-\bar{\beta}z}$

Remark(1.10) :

for $\beta \in U$, then $\phi'_\beta(0) = \frac{1}{2}$, $\phi'_\beta(\beta) = \frac{2}{\left(2+|\beta|^2\right)^2}$.

Definition(1.11) : [11]

Let $\psi : U \to U$ and holomorphic on U. We say that ψ is a rotation round the origin if there exists $\sigma \in \partial U$ such that $\psi(z) = \sigma z$ $(z \in U)$

Proposition (1.12):

$\phi_\beta(z)$ is not a rotation a round the origin

Proof:

Since $\phi_\beta(z) = \frac{z}{2+\bar{\beta}z} \neq \sigma z$, then $\phi_\beta(z)$ is not a rotation a round the origin .

Theorem (1.13) : [11]

Let $\psi : U \to U$ and holomorphic on U, then ψ is elliptic if and only if ψ is automorphism that has an interior fixed point.

Proposition (1.14) :

For $\beta \in U$, ϕ_β is elliptic

Proof :

From (1- 4) , ϕ_β is automorphism , and from (1-8) ϕ_β has an interior fixed Point , hence ϕ_β is elliptic .

Definition(1.15): [10]

A linear fractional transformation is a mapping of the form $\tau(z)=\dfrac{az+b}{cz+d}$, where a , b , c , and d are complex numbers and $\tau(z)=\dfrac{az+b}{cz+d}$ we sometime denote it by $\tau_A(z)$ where A is the non-singular 2×2 complex matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proposition (1.16) :

ϕ_β is a linear fractional transformation .

Proof :

Since $\phi_\beta(z)=\dfrac{z}{2+\bar{\beta}z}=\dfrac{az+b}{cz+d}$ such that a =1 , b = 0 , c = $\bar{\beta}$, d = 2 and a , b , c , and d are complex numbers and $A=\begin{bmatrix} 1 & 0 \\ \bar{\beta} & 2 \end{bmatrix}$, hence by (1.15) ϕ_β is a linear fractional transformation .

Section Two

Definition(2.1): [4]

Let U denote the unit ball in the complex plane, the Hard space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f\,\hat{}\,(n)z^n$ holomorphic on U such that $\sum_{n=0}^{\infty}\left|f\,\hat{}\,(n)\right|^2 \prec \infty$ with $f\,\hat{}\,(n)$ denotes then the Taylor coefficient of f .

Remark (2.2) : [1]

We can define an inner product of the Hardy space functions as follows:

f(z) = $\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ and $g(z)=\sum_{n=0}^{\infty} g(n)\, z^n$, then inner product of f and g is:

$$\langle f,g\rangle = \sum_{n=0}^{\infty} f^{\wedge}(n)\, \overline{g^{\wedge}(z)}$$

Example (2.3) :[10]

Let $\alpha \in U$ and $k_\alpha(z)=\dfrac{1}{1-\overline{\alpha}z}$ $(z \in U)$. Since $\alpha \in U$ then $|\alpha| \prec 1$,hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $k_\alpha \in H^2$ and $k_\alpha(z)=\overline{\alpha}\ z^n$.

Definition(2.4) : [4]

Let $\psi : U \to U$ and holomorphic on U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation $C_\psi\ f = f \circ \psi$ $\left(f \in H^2\right)$

Definition(2.5) : [2]

Let T be a bounded operator on a Hilbert space H, then the norm of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood′s Subordination principle (2.6) : [11]

Let $\psi : U \to U$ and holomorphic on U with $\psi(0) = 0$, then for each $f \in H^2$, $f \circ \psi \in H^2$ and $\|f \circ \psi\| \le \|f\|$. The goal of this theorem $C_\psi : H^2 \to H^2$.

Definition(2.7) :

The composition operator C_{ϕ_β} induced by ϕ_β is defined on H^2 by the equation

$$C_{\phi_\beta} f = f \circ \phi_\beta \ , \left(f \in H^2\right)$$

Proposition(2.8) :

If $\beta \in U$, then for each $f \in H^2$, $f \circ \phi_\beta \in H^2$ and $\|f \circ \phi_\beta\| \le \|f\|$

Proof :

Since $\phi_\beta : U \to U$ and holomorphic on U with $\phi_\beta(0) = 0$, then by (2- 6)

$f \in H^2$, $f \circ \phi_\beta \in H^2$ and $\|f \circ \phi_\beta\| \leq \|f\|$, hence $C_{\phi_\beta} : H^2 \to H^2$

Remark (2.9) : [4]

1) One can easily show that $C_\kappa C_\psi = C_{\psi \circ \kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi$

$= C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$

2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.

3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Theorem (2. 10) : [11]

Let $\psi : U \to U$ and holomorphic on U . C_ψ is an invertible operator on H^2 if and only if ψ automorphism of U and $C_\psi^{-1} = C_{\psi^{-1}}$

Proposition(2.11) :

If $\beta \in U$, then C_{ϕ_β} is an invertible operator on H^2

Proof :

Since ϕ_β is automorphism of U by (1- 4) , hence C_{ϕ_β} is an invertible operator on H^2 .

Definition(2.12): [3]

Let T be an operator on a Hilbert space H , The operator T^* is the adjoint of T if $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for each $x, y \in H$.

Theorem (2.13) : [5]

$V_{\alpha\in U}\{K_\alpha\}$ forms a dense subset of H^2.

Theorem (2.14) : [10]

Let $\psi : U \to U$ and holomorphic on U, then for all $\alpha \in U$

$$C_\psi^* K_\alpha = K_{\psi(\alpha)}$$

Definition(2.15): [6]

Let $g \in H^\infty$, the Toeplits operator T_g is the operator on H^2 given by :

$$(T_g f)(z) = g(z)\ f(z) \quad (f \in H^2,\ z \in U)$$

Remark (2.16) : [7]

For each $f \in H^2$, it is well- know that $T_h^* f = T_{\bar{h}} f$, such that $h \in H^\infty$.

Proposition(2.17) :

Let $C_{\phi_\beta}^* = T_g\ C_\gamma\ T_h$, where $h(z) = (2 + \bar{\beta}z)$, $g(z) = \frac{1}{2}$, $\gamma(z) = \frac{z-\beta}{2}$

Proof :

By (2-16), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\langle T_h^*\ f, k_\alpha \rangle = \langle T_{\bar{h}} f, k_\alpha \rangle = \langle f, T_{\bar{h}}^*\ k_\alpha \rangle \cdots\cdots (2-1)$$

On the other hand ,

$$\langle T_h^*\ f, k_\alpha \rangle = \langle f, T_h f \rangle = \langle f, h(\alpha) k_\alpha \rangle \cdots\cdots (2-2)$$

From (2-1)and (2-2) one can see that $T_{\bar{h}}^*\ k_\alpha = h(\alpha)\ k_\alpha$. Hence $T_h^*\ k_\alpha = \overline{h(\alpha)}\ k_\alpha$.

Calculation give

$$C_{\phi_\beta}^* k_\alpha(z) = k_{\phi_\beta(\alpha)}(z)$$

$$= \frac{1}{1-\overline{\phi_\beta(\alpha)}\ z} = \frac{1}{1-\dfrac{\overline{\alpha} z}{2+\beta\overline{\alpha}}}$$

$$= \frac{1}{\dfrac{2+\beta\overline{\alpha}-\overline{\alpha}z}{2+\beta\overline{\alpha}}} = \frac{2+\beta\overline{\alpha}}{2-\overline{\alpha}(z-\beta)} = \frac{\overline{(2+\overline{\beta}\alpha)}}{2-\overline{\alpha}(z-\beta)}$$

$$= \overline{(2+\overline{\beta}\alpha)} \cdot \left(\frac{1}{2}\right) \cdot \frac{1}{1-\overline{\alpha}\left(\dfrac{z-\beta}{2}\right)}$$

$$= \overline{h(\alpha)} \cdot T_g\ k_\alpha(\gamma(z)) = T_g\ \overline{h(\alpha)}\ k_\alpha(\gamma(z))$$

$$= T_g\ \overline{h(\alpha)}\ C_\gamma\ k_\alpha(z) = T_g\ C_\gamma\ \overline{h(\alpha)}\ k_\alpha(z)$$

$$= T_g\ C_\gamma\ T_h^*\ k_\alpha(z)$$, therefore

$$C_{\phi_\beta}^* k_\alpha(z) = T_g\ C_\gamma\ T_h^*\ k_\alpha(z) \quad (z \in U).$$

But $\overline{V_{\alpha\in U}\ \{K_\alpha\}} = H^2$, then $C_{\phi_\beta}^* = T_g\ C_\gamma\ T_h^*$

Definition (2.18) : [3]

Let T be an operator on a Hilbert space H , T is called normal operator if $T\ T^* = T^*\ T$

Theorem (2.19) : [9]

Let $\psi : U \to U$ and holomorphic on U, then C_ψ is normal if and only if $\psi(z) = \lambda\ z$ for some $\lambda\ ,\ |\lambda| = 1$

Proposition(2.20) :

C_{ϕ_β} is not normal composition operator .

Proof :

Since $\phi_\beta(z)=\dfrac{z}{2+\bar{\beta}z}\neq\lambda z$,hence by (2.19) C_{ϕ_β} is not normal composition operator.

Definition (2.21) : [12]

Let $\psi:U\to U$ and holomorphic on U, the eigenvalue equation for the composition operator is define by $C_\psi f=\kappa\, f$ or $f\circ\psi=\kappa\, f$.

Theorem (2.22): [11]

Let $\psi:U\to U$ and holomorphic on U, and that fixes the point p∈U and suppose that $C_\psi f=\kappa f$ for some non-constant $f\in H^2$ and some $\kappa\in C$. Then $\kappa=(\psi'(p))^n$ for some n = 0, 1, 2, ...

Proposition(2.23) :

If $\beta\in U$, then $\left(\dfrac{1}{2}\right)^n$ is an eigenvalue of C_{ϕ_β} for some n = 0 , 1 , 2 , …..

Proof :

Since $\phi_\beta(z)=\dfrac{z}{2+\bar{\beta}z}$, $\phi'_\beta(z)=\dfrac{(2+\bar{\beta}z)(1)-(\bar{\beta}z)}{(2+\bar{\beta}z)^2}=\dfrac{2}{(2+\bar{\beta}z)^2}$, and since ϕ_β fixed the point 0∈U, and by (2.22) $\kappa=(\phi'_\beta(0))^n=\left(\dfrac{1}{2}\right)^n$ is an eigenvalue of C_{ϕ_β} for some n = 0 , 1 , 2 , …..

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المؤثر التركيبي C_{ϕ_β} المتولد بالدالة ϕ_β

من قبل

عقيل محمد حسين

قسم الرياضيات

كلية التربية

جامعة القادسية

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المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال $f(z)=\sum_{n=0}^{\infty} f^\wedge(n)\ z^n$ التحليلية على U بحيث أن $\sum_{n=0}^{\infty} \left|f^\wedge(n)\right|^2 < \infty$ ، $f^\wedge(n)$ يرمز إلى معاملات تيلر النونية.

لتكن $\psi: U \to U$ دالة تحليلية على U ، المؤثر التركيبي المحتث من ψ يعرف على فضاء هاردي H^2 بواسطة:

$$C_\psi f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث المؤثر التركيبي المحتث من الدالة ϕ_β حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المحتث من الدالة ϕ_β . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج التي أعطيت بدون برهان.