*On Minimal Actions

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Abstract
In this paper, we introduce and study a new type of actions namely "Minimal Actions". Thought this paper, a new concepts have been illustrated including a Minimal Group, Minimal Proper functions, Minimal Bourbaki MG-Space, Minimal Cartan MG-Space and Minimal palais MG-Space and clarified their properties.

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Introduction

One of very important concepts in geometrical topology is the concept of group actions and there are several types of these actions . In 1950 Maki H., Umehara J . and Noiri T . introduced the notions of minimal structure and minimal spaces . They achieved many important results compatible by the general topological case. In [8] Popa V . and Noiri T., introduced the notion of minimal structure. They also introduced the notion of m_x-open sets and m_x-closed sets and characterized those sets by using m_x-closure and m_x-interior operators, respectively .

The applications of minimal structure are found inNakaoka F . and ONA N., as an application of a theory of minimal open sets, they presented a sufficient conditions for a locally finite space to be a pre-Hausdorff space. Bourbaki in [3] defined the proper map and proper actions. And the other hand , Palais in [6] defined proper G-spaces . In this work , we introduce the definition of Minimal Group which is Considcred as a basis of our main definition to Construed the definition of Minimal Group space "mG-space ". we give the definitions of Certain types of minimal group space and investigate their properties . Finally , we introduce the definitions of Minimal limit sets and use it to characterize Certain types of Minimal G-spaces .

1. Preliminaries

In this section, we introduce some elementary concept which we need in our work.

1.1. Definition[8]:
Let X be a nonempty set and $P(X)$ the power set of X . A subfamily M_X of $P(X)$ is called a minimal structure (briefly m-structure) on X if $\emptyset, X \in M_X$. In this case (X, M_X) is said to be minimal space (briefly m-space).

1.2.Example:
Let $X = \{a, b, c, d\}$and $M_X = \{\emptyset, X, \{a\}, \{c\}\}$. Then M_X is an m-structure on X and (X, M_X) is an m-space

1.3. Remarks:
(i) The m-structure M_X is called indiscrete m-structure if it contains only $\emptyset$, X and is called discrete m-structure if $M_X = P(X) = \{A : A \subseteq X\}$.

(ii) Every topological space is an m-space, but the converse is not true in general as shown in example (1.2).

1.4. Definition [7]:
Let X be a nonempty set and M_X be an m-structure on X . A set $A \in P(X)$ is said to be a minimal open (briefly m-open) set if $A \in M_X$, The complement of m-open is called minimal closed (briefly m-closed) .

1.5. Remark [4]:
Let (X, M_X) be an m-space. If A and B are an m-open sets , Then $A \cap B$ and $A \cup B$ are not necessarily m-open sets as the following example shows .

1.6. Example:
Let$X = \{a, b, c, d\}$, $M_X = \{\emptyset, X, \{a, b\}, \{d\}, \{b, d\}\}$, be an m-structure on X. Then $\{a, b\}, \{b, d\} \in M_X$ but$\{a, b\} \cup \{b, d\} = \{a, b, d\} \notin M_X$ and $\{a, b\} \cap \{b, d\} = \{b\} \notin M_X$.

1.7. Definition[4]:
An m-space (X, M_X) is said

(i) um-space if the union of m-open sets is an m-open set .
(ii) im-space if the finite intersection of m-open sets is an m-open set .

1.8. Remark:
The intersection and union of two m-closed sets is not necessarily m-closed set as the following example shows. Let $X = \{a, b, c, d\}$and $M_X = \{\emptyset, X, \{c, d\}, \{b, d\}, \{a, c, d\}\}$ be an m-

structure on X .Then $\{a,b\},\{a,c\}$ are m-closed setsin X but $\{a,b\}\cup\{a,c\} = \{a,b,c\}$, $\{a,b\}\cap\{a,c\} = \{a\}$ are not m-closed.

1.9. Proposition [8]:

Let (X, M_X) be an um-space . Then the intersection of m-closed sets is an m-closed.

1.10. Definition [5]:
Let X be a nonempty set and M_X be an m-structure on X. For a subset A of X, the minimal closure of A (briefly $\overline{A}^m$) and the minimal interior of A (briefly $A^{\circ m}$) , are defined as follows:

$$\overline{A}^m = \cap\{F : A \subseteq F , F^c \in M_X\}$$
$$A^{\circ m} = \cup\{V : V \subseteq A , V \in M_X\}$$

1.11. Remark [4]:
$\overline{A}^m$ is not necessarily an m-closed set and $A^{\circ m}$ is not necessarily an m-open set . As the following example shows.

1.12. Example:

Let$X = \{a,b,c,d,e\}, M_X = \{\emptyset, X, \{e\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$ be an m-structure on X and let $A = \{c,d,e\}$, $B = \{a,b\}$then $A^{\circ m} = \{c,d,e\}$is not m-open set and $\overline{B}^m = \{a,b\}$ is not m-closed set.

1.13. Proposition [10]:
Let(X, M_X) be an um-space thenA^{om} is m-open set and $\overline{A}^m$ is m-closed set.

1.14. Remark:
Let (X, M_X) be an m-space, and A be a subset of X , if $A^{\circ m} = A$and $\overline{B}^m = B$. Then A and B are not necessarily an m-open and m-closed sets respectively . Asthe following example shows .

1.15. Example:
Let$X = \{1,2,3\}$, $M_X = \{\emptyset, X, \{1\}, \{3\}, \{1,2\}\}$ be m-structure on X and let $A = \{1,3\}$, $B = \{2\}$ then:

(i)$A^{\circ m} = A$, but A is not m-open set in X ;
(ii)$\overline{B}^m = B$, but B is not m-closed set in X.

1.16. Proposition [8]:
Let (X, M_X) be an um-space, and A be a subset of X then:

(i) $A \in M_X$if and only if $A^{\circ m} = A$;
(ii) A is an m-closed if and only if $\overline{A}^m = A$.

1.17. Definition [4]:
Let X be an m-space and B be any subset of X . Anm-neighborhood of B is any subset of X that contains an m-open set containing B . The m-neighborhoods of a subset $\{x\}$ consisting of single points are also called m-neighborhoods of the point x.

The collection of all m-neighborhoods of the subset B of X is denoted by $N_m(B)$. In particular the collection of all neighborhoods of the subset $\{x\}$ is denoted by $N_m(x)$.

1.18. Definition [9]:
Let (X, M_X) be an m-space. A subset $K$of X is said to be m-compact if every cover of K by subsets of M_X has a finite subcover. X is called m-compact space if every m-open cover has a finite subcover.

1.19. Definition:
(i)A subset A of m-space $X$is called m-relative m-compact if $\overline{A}^m$ is an m-compact
(ii) An m-space $X$is called m-locally m-compact if every point in X has m-relative compact m-neighborhood .

1.20. Proposition:
Let X be an m-locally m-compact space and K be an m-compact subset of X . Then$K$ has m-compact m-neighborhood .

Proof: Clear .

1.21. Definition [12]:
A set D is called a directed set if there is relation $\geq$ on D satisfying :
(i) $d \geq d$ for each $d \in D$.
(ii) If $d_1 \geq d_2$ and $d_2 \geq d_3$ then $d_1 \geq d_3$
(iii) If $d_1, d_2 \in D$ then there is some $d_3 \in D$, with $d_3 \geq d_1$ and $d_3 \geq d_2$.

1.22. Definition [12]:
A net in a set X is a function $\chi : D \to X$ where D directed set . The point $\chi(d)$ is usually denoted by χ_d.

1.23. Definition [11]:
A subnet of a net $\chi : D \to X$ is the composition $\chi o \varphi$ where $\varphi : H \to D$ and H directed set such that:
(i) $\varphi(h_1) \geq \varphi(h_2)$ whenever $h_1 \geq h_2$.
(ii) for each $d \in D$ there is some $h \in H$ such that $d \geq \varphi(h)$, the point $(\chi o \varphi)(h)$ is often written χ_{dh}.

1.24. Definition [4]:
Let $(\chi_d)_{d\in D}$ be a net in an m-space X, $x \in X$. Then:
(i) $(\chi_d)_{d\in D}$ is an m-converges to x if $(\chi_d)_{d\in D}$ is eventually in every m-neighborhood of X (written $\chi_d \overset{m}{\to} x$) . The point x is called an m-limit point of $(\chi_d)_{d\in D}$.
(ii) $(\chi_d)_{d\in D}$ said to have x as an m-cluster point if $(\chi_d)_{d\in D}$ is frequently in every m-neighborhood of x (written $\chi_d \overset{m}{\propto} x$) .
(iii) $\chi_d \overset{m}{\to} \infty$ if the net $(\chi_d)_{d\in D}$ has no m-convergent subnet .

1.25. Example:
Let $X = \{-1, 0, 1\}$, $M_X = \{ \emptyset, X, \{-1, 1\}, \{0, 1\}\}$ be an m-structure on X , and let $\{(-1)^n\}$ be a net in X then $\{(-1)^n\}$ is not m-converge to 1 but its m-converge to -1 .

1.26. Remark [1]:
Let $f: X \to Y$ be a function from a set X into a set Y , then :
(i) If $(\chi_d)_{d\in D}$ is a net in X then $\{f(\chi_d)\}_{d\in D}$ is a net in Y.
(ii) If $(\gamma_d)_{d\in D}$ is a net in Y then there is a net $(\chi_d)_{d\in D}$ in X such that $f(\chi_d) = \gamma_d$ for each $d \in D$.

1.27. Proposition [4]:
Let X be an m-space and let $A \subseteq X$, $x \in X$. Then $x \in \overline{A}^m$ if and only if there is a net $(\chi_d)_{d\in D}$ in A and $\chi_d \overset{m}{\to} x$.

1.28. Corollary [4]:
Let X be an m-space , $x \in X$ and $A \subseteq X$. Then $x \in \overline{A}^m$ if and only if there is a net $(\chi_d)_{d\in D}$ in A such that $\chi_d \overset{m}{\propto} x$.

1.29. Proposition:
Let X be an um-space. Then X is an m-compact if and only if every net in X has an m-cluster point in X.

Proof: Clear .

1.30. Remark [4]:
Let $(\chi_d)_{d\in D}$ be a net in an m-space X and $x \in X$, Then:
(i) If $\chi_d \overset{m}{\to} x$, then every subnet of $(\chi_d)_{d\in D}$ is an m-convergence to x .
(ii) If $\chi_d = x$ for all $\in D$, then $\chi_d \overset{m}{\to} x$.

1.31. Proposition [4]:
Let $(\chi_d)_{d\in D}$ be a net in an m-space X . Then $\chi_d \overset{m}{\propto} x$ if and only if there exists a subnet $(\chi_{dh})_{dh\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dh} \overset{m}{\to} x$.

2.32. Proposition [4]:
Let X, Y be two m-spaces and $\{(\chi_d, \gamma_d)\}_{d\in D}$ be a net in $X \times Y$ such that $(\chi_d, \gamma_d) \overset{m}{\to} (x, y)$ then $\chi_d \overset{m}{\to} x$ and $\gamma_d \overset{m}{\to} y$.

1.33. Definition [4]:
Let (X, M_X) be an m-space. Then X is said to be m-T_2-space if for every two distinct points x and y in X , there exist two disjoint m-open sets U and V such that $x \in U$ and $y \in V$.

1.34. Proposition:
A minimal space X is an m-T_2-space if and only if every m-convergent net in X has a unique m-limit point .

Proof:
Let X be m-space and $(\chi_d)_{d\in D}$ is a net in X such that $\chi_d \overset{m}{\rightarrow} x$, $\chi_d \overset{m}{\rightarrow} y$. and $x \neq y$. Since X be an m-T_2-space . There are $U \in N_m(x)$ and $V \in N_m(y)$ such that $U \cap V = \emptyset$. Since $\chi_d \overset{m}{\rightarrow} x$, there is $d_0 \in D$ such that $\chi_d \in U$ for all $d \geq d_0$. since $\chi_d \overset{m}{\rightarrow} y$, there is $d_1 \in D$ such that $\chi_{d1} \in V$ for all $d \geq d_1$. since D is directed set and d_0 , $d_1 \in D$, then there is $d_2 \in D$ such tha $d_2 \geq d_0$ and $d_2 \geq d_1$. Then $\chi_d \in U$ for all $d \geq d_2$ and $\chi_d \in V$ for all $d \geq d_2$, thus $U \cap V \neq \emptyset$, this is a contradiction.So $x = y$.

Conversely:
Suppose that X is not m-T_2-space , there are $x, y \in X$ and $x \neq y$, for all $U \in N_m(x)$, $V \in N_m(y)$ such that $U \cap V \neq \emptyset$. put $N_x^y = \{ U \cap V : U \in N_m(x)$ and $V \in N_m(y)\}$, where N_x^y is directed set . Thus for all $D \in N_x^y$, there is $\chi_D \in D$ then $(\chi_D)_{D\in N_x^y}$ is net in X .
To prove $\chi_D \overset{m}{\rightarrow} x$,$\chi_D \overset{m}{\rightarrow} y$, let $G \in N_m(x)$ then $G \in N_x^y$, $G \cap X \neq \emptyset$. Thus $\chi_D \in G$ for all $D \geq G$, so $\chi_D \overset{m}{\rightarrow} x$. Also , let $H \in N_m(y)$ then $H \in N_x^y$, $H \cap X \neq \emptyset$. Thus $\chi_D \in H$ for all $D \geq G$, so $\chi_D \overset{n}{\rightarrow} y$. this is a contradiction .

1.35. Definition [2]:
Let $f: X \longrightarrow Y$ be a function from m-space (X, M_X) into m-space (Y, M_Y). Then f is called minimal continuous (briefly m-continuous)if $f^{-1}(B) \in M_X$, for every $B \in M_Y$.

1.36. Definition [10]:
A function $f: (X, M_X) \longrightarrow (Y, M_Y)$ is said to be minimal closed (briefly m-closed) if for each m-closed set B of X , $f(B)$ is m-closed set in Y.

1.37. Proposition[4]:
Let $f: X \longrightarrow Y$ be a function, $x \in X$ then f is an m-continuous if and only if whenever a net $(\chi_d)_{d\in D}$ in $X$and $\chi_d \overset{m}{\rightarrow} x$ then $f(\chi_d) \overset{n}{\rightarrow} f(x)$.

2. On Minimal Action:
In this section, we introduce the definition of minimal group, minimal group action and its related concepts such as, minimal orbit, minimal stabilizer and minimal kernel . Also we introduce the definitions of $\Lambda^n(x)$and $J^m(x)$and give some properties of these concepts .

2.1. Definition:
A minimal group is a set G with two structures:
(i)(G, μ) is a group .
(ii)(G, M_G) is a minimal space .

Such that the two structures are compatible , i.e; the multiplication function $\mu: G$x $G \longrightarrow G$ which is defined by $\mu(g_1, g_2) = g_1g_2$, for every $g_1, g_2 \in G$ and the inversion function $\text{Ɲ}: G \longrightarrow G$ which is defined by $\text{Ɲ}(g) = g^{-1}$ for all g$\in G$, are both m-continuous function .

2.2. Definition:
Let G be a minimal group and X be a minimal space . A left minimal action of G on X is a m-continuous map $\varphi: G$x $X \longrightarrow X$ such that:

(i) $\varphi(e, x) = x$, for all $x \in X$ where e is the identity element in G .
(ii) $\varphi(g_1, \varphi(g_2, x) = \varphi(\mu(g_1, g_2), x)$, for all $x \in X$ and $g_1, g_2 \in G$.

The m-space X together with minimal action θ is called minimal group space and denoted by mG-space , more precisely (left mG-space) . In similar way one can define a right mG-space .

Note that the difference between the left and right minimal action is not a trivial one , however there is a one to one correspondence between them as follow : if φ is a left minimal action of G on X , then $\varphi': X$x $G \longrightarrow X$ defined by$\varphi'(x,g) = \varphi(g^{-1}, x)$ is a right minimal action of G on X, and similarly for right minimal action .

Thus for every left minimal action there is a conjugate right minimal action and vice versa , so every theorem that is true of left minimal action has a conjugate theorem for right minimal action . Because of this , we will usually use a left action .

2.3. Example:
Let G be a minimal group, then G is mG-space by multiplication $\varphi = \mu: G \times G \longrightarrow G$, $(g_1, g_2) \longrightarrow g_1g_2$, φ is an m-continuous (because G is minimal group)

2.4. Proposition:
Let G be minimal group and $(g_d)_{d\in D}$ be a net in G. Then:
(i) If $g_d \xrightarrow{m} e$, then $g\, g_d \xrightarrow{m} g$ (or $g_d g \xrightarrow{m} g$) for each $g \in G$.
(ii) If $g_d \xrightarrow{m} \infty$, then $g\, g_d \xrightarrow{m} \infty$(or $g_d g \xrightarrow{m} \infty$) for each $g \in G$.
(iii) If $g_d \xrightarrow{m} \infty$, then $g_d^{-1} \xrightarrow{m} \infty$.

Proof:
(i) Since $R_g: G \longrightarrow G$ is an m-continuous and an m-open ,where R_g is right translation by g . thus by proposition (1.37) $g\, g_d \xrightarrow{m} g$ for each $g \in G$.

(ii) Let $g_d \xrightarrow{m} \infty$ and $g \in G$. suppose that $g g_d \xrightarrow{m} g_1$, for some

$g_1 \in G$. Since g^{-1} is an m-continuous , Then by proposition (1.37)$R_g^{-1}(g g_d) \xrightarrow{m} R_g^{-1}(g_1)$. Then $g_d \xrightarrow{m} g_1 g^{-1}$, a contradiction .Thus g $g_d \xrightarrow{m} \infty$.

(iii) Let $g_d^{-1} \xrightarrow{m} g$. Since $N: G \longrightarrow G$ is an m-continuous ,Then $g_d \xrightarrow{m} g^{-1}$. Thus if $g_d \xrightarrow{m} \infty$, then $g_d^{-1} \xrightarrow{m} \infty$.

2.5. Definition:
Let (X,φ) be an mG-space and $x \in X$ .then :

(i) The minimal orbit of x defined to be the set $G_x^m = \{\varphi(g, x) : g \in G\}$, the set of all minimal orbit denoted by X/G and called it minimal orbit space .
(ii) The minimal stabilizer of $x \in X$ defined to be the set

$$S_x^m = \{g \in G: \varphi(g, x) = x\}.$$

(iii) The minimal kernel of the minimal action is defined to be the set $(ker\varphi)^m = \{g \in G: \varphi(g, x) = x , \forall x \in X\}$.

2.6. Preposition:
Let (X,φ) be mG-space , then:

(i) The minimal stabilizer of $x \in X$ is a subgroup of G .
(ii)$(ker\varphi)^m = \bigcap_{x\in X} S_x^m$.
(iii)$(ker\varphi)^m$ is a normal subgroup of G .

Proof:

(i) Let$g_1, g_2 \in S_x^m$then $\varphi(g_1, x) = \varphi(g_2, x) = x$.
$=\varphi(g_1 g_2, x) = \varphi(\mu(g_1, g_2), x)) = \varphi(g_1, \varphi(g_2, x)) = \varphi(g_1, x) = x$
Hence $g_1 g_2 \in S_x^m$.

Now , let $g \in S_x^m$ then $\varphi(g, x) = x$

$\varphi(g^{-1}, x) = \varphi(g^{-1}, \varphi(g, x)) = \varphi(g^{-1}g, x)$
$= \varphi(e, x) = x$
Hence $\varphi(g^{-1}, x) = x$
Therefore $g^{-1} \in S_x^m$.

So S_x^m is a subgroup of G .

(ii) Let $g \in (ker\varphi)^m \Leftrightarrow \varphi(g, x) = x$ for all $x \in X$.

$\Leftrightarrow$ $g \in S_x^m$ for all $x \in X$.

$\Leftrightarrow$ $g \in \bigcap_{x\in X} S_x^m$.

Then $(ker\varphi)^m = \bigcap_{x\in X} S_x^m$.

(iii) From (ii) $(ker\varphi)^m$ subgroup of G .

Let $g \in (ker\varphi)^m$,then $\varphi(g, x) = x$, $\forall x \in X$ and$\varphi(hgh^{-1}, x) = \varphi(h, \varphi(gh^{-1}, x))$

$$= \varphi(h, \varphi(g, \varphi(h^{-1}, x))$$

$$= \varphi(h, \varphi(h^{-1}, x))$$

$$= \varphi(hh^{-1}, x) = \varphi(e, x) = x.$$

Hence $(\varphi(hgh^{-1}, x)) = x$ for all $x \in X$. Hence$hgh^{-1} \in (ker\varphi)^m$. Therefor $h(ker\varphi)^m h^{-1} \subseteq (ker\varphi)^m$ for all $x \in X$ .Since $(ker\varphi)^m \subseteq h(ker\varphi)^m h^{-1}$.

Thus$h(ker\varphi)^m h^{-1} = (ker\varphi)^m$.

Therefor $(ker\varphi)^m$ is a normal subgroup of G

2.7. Theorem:
Let (X,φ) be a m-Hausdorff mG-space, where X is an um-space and G is an m-compact minimal group . Then the minimal action $\varphi: G \times X \longrightarrow X$ is an m-closed map .

Proof:
Let $A \subseteq G \times X$ be an m-closed set in $G \times X$.
Let$y \in \overline{\varphi(A)}^m$ then there exist a net $(y_\alpha)_{\alpha\in D} \in \varphi(A)$ such that $y_\alpha \xrightarrow{m} y$ (proposition 1.27)

this implies there exist $(g_\alpha, x_\alpha)_{\alpha\in D} \in A$ such that $\varphi(g_\alpha, x_\alpha) = y_\alpha$.

Since G is m-compact then the net $(g_\alpha)_{\alpha\in D}$ has m-convergent subnet $(g_{n\alpha})$ such that $g_{n\alpha} \xrightarrow{m} g$ and $y_{n\alpha} \xrightarrow{m} y$.

Now $x_{n\alpha} = \varphi(e, x_{n\alpha}) = \varphi(g_{n\alpha}^{-1}\, g_{n\alpha}, x_{n\alpha})$

$= \varphi(g_{n\alpha}^{-1}, \varphi(g_{n\alpha}, x_{n\alpha})) = \varphi(g_{n\alpha}^{-1}, y_{n\alpha})$
And then $x_{n\alpha} \xrightarrow{m} \varphi(g^{-1}, y)$
Therefor $(g_{n\alpha}, x_{n\alpha}) \xrightarrow{m} (g, \varphi(g^{-1}, y))$and since A is m-closed then $(g, \varphi(g^{-1}, y)) \in A$. Thus $\varphi(g, \varphi(g^{-1}, y)) = \varphi(g\, g^{-1}, y) = \varphi(e, y) = y \in \varphi(A)$ therefore $\varphi(A) = \overline{\varphi(A)}^m$ and then $\varphi(A)$ is m-closed since X is an um-space (proposition 1.1.19) .

2.8. Corollary:
Let (X, φ) be a m-Hausdorff mG-space, where X is an um-space and G is an m-compact minimal group . Then the natural projection $\pi: X \longrightarrow X/G$ is an m-closed map .

Proof:
Let $A \subseteq X$ be an m-closed set in X . Then $\pi^{-1}(\pi(A)) = G.A = \varphi(G \times A)$ is m-closed set in X (Theorem 2.7) . So $\pi(A)$ is an m-closed set in X .

Therefore π is an m-closed map .

Dydo in [6] , developed the concept of the sets $\Lambda(x)$ and $J(x)$ in any space and used only $J(x)$ as a characterization of Cartan G-space as following:

For any point x in a G-space
$J(x) = \{y \in X$: there is a net $(g_d)_{d\in D}$ in G and there is a net $(\chi_d)_{d\in D}$ in X with $g_d \longrightarrow \infty$ and $\chi_d \longrightarrow x$ such that $g_d\chi_d \longrightarrow y\}$
$\Lambda(x) = \{y \in X$: there is a net $(g_d)_{d\in D}$ in G with $g_d \longrightarrow \infty$ such that $g_d\chi_d \longrightarrow y\}$
Also it is clear that the set $\Lambda(x)$ is a subset of $J(x)$.

Now, we introduce the following definition and prove some results.

2.9. Definition:
Let (X, φ) be an mG-space and $x \in X$. Then:

$\Lambda^m(x) = \{Y$: there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{m} \infty$ such that $g_d\chi_d \xrightarrow{m} y\}$ is called minimal limit set of x
$J^m(x) = \{Y$: there is a net $(g_d)_{d\in D}$ in G and there is a net $(\chi_d)_{d\in D}$ in with $g_d \xrightarrow{m} \infty$ and $\chi_d \xrightarrow{m} x$ such that $g_d\chi_d \xrightarrow{m} y\}$ is called minimal first prolongation limit set.
It is clear that the set $\Lambda^m(x)$ is a subset of $J^m(x)$.

2.10. Proposition:
Let (X, φ) be an mG-space and $x \in X$. Then:
(i) $\Lambda^m(x)$ is an invariant sets under G.
(ii) The m-orbit G_x^m is m-closed if and only if $\Lambda^m(x) \subseteq G_x^m$.
(iii) If $x \notin \Lambda^m(x)$, then the m-stabilizer subgroup S_x^m of G is an m-compact.
(iv) $\overline{G_x^m}^m = G_x^m \cup \Lambda^m(x)$.
(v) $g\Lambda^m(x) = \Lambda^m(gx) = \Lambda^m(x)$ for each $g \in G$.

Proof:
(i) Let $y \in \Lambda^m(x)$ and $g \in G$. Then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{m} \infty$ and $g_d\chi_d \xrightarrow{m} y$ and . It is clear that $(gg_d)_{d\in D}$ is a net in G with $gg_d \xrightarrow{m} \infty$, by proposition (2.4.ii) Since the action is minimal continuous , thus $gg_dx \xrightarrow{m} gy$ which is implies that $gy \in \Lambda^m(x)$ and hence $\Lambda^m(x)$ is invariant . The proof of $J^m(x)$ is similar.

(ii) Let G_x^m be an m-closed and let $y \in \Lambda^m(x)$, then there is a net $(g_d)_{d\in D}$ in G such that $g_d \xrightarrow{m} \infty$ and $g_d\chi_d \xrightarrow{m} y$ Since $g_dx \in G_x^m$ and $(g_d)_{d\in D}$ is a net in G_x^m then by proposition(1.27) $y \in \overline{G_x^m}^m = G_x^m$. Therefore $\Lambda^m(x) \subseteq G_x^m$

Conversely:
Let $y \in \overline{G_x^m}^m$. then by proposition (1.27) there is $(y_d)_{d\in D}$ is a net in G_x^m such that $y_d \xrightarrow{m} y$, then $\forall\ d \in D$ there is $g_d \in G$ such that $y_d = g_dx$. Then $(g_d)_{d\in D}$ is a net in G and $g_dx \xrightarrow{m} y$. Now either $g_d \xrightarrow{m} g$ or $g_d \xrightarrow{m} \infty$. If $g_d \xrightarrow{m} g$, then $g_dx \xrightarrow{m} g_dx = y$, which implies that $y \in G_x^m$. If $g_d \xrightarrow{m} \infty$, then $\in \Lambda^m(x) \subseteq G_x^m$, thus G_x^m is an m-closed .

(iii) Let $x \notin \Lambda^{m}(x)$ and suppose that S_x^m is not m-compact. then there is a net $(g_d)_{d\in D}$ in G such that $g_d \overset{m}{\rightarrow} \infty$. Since $g_d \in S_x^m$, then $g_d x = x$. i.e $g_d x \overset{m}{\rightarrow} xg$. then $x \in \Lambda^{m}(x)$ which is a contradiction, thus S_x^m is an m-compact.

(iv) If G_x^m is an m-closed , then $G_x^m = \overline{G_x^m}^m$ from (ii) , $G_x^m = G_x^m \cup \Lambda^m(x)$. Let $y \notin G_x^m$ and $\in \Lambda^m(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \overset{m}{\rightarrow} \infty$ such that $g_d \chi_d \overset{m}{\rightarrow} y$. Since $(hg_d)_{d\in D}$ is a net in G with $hg_d \overset{m}{\rightarrow} \infty$. So $hg_d x \overset{m}{\rightarrow} hy$ which implies that $hy \in \Lambda^m(x)$, but $hy \in G_x^m$ then $\Lambda^m(x) \subseteq G_x^m$, therefore by (ii) , G_x^m is m-closed , hence $\overline{G_x^m}^m = G_x^m \cup \Lambda^m(x)$.

(v) Clear .

2.11. Proposition:

Let (X,φ) be an mG-space and $x \in X$. Then:

(i) $J^m(x)$ is an invariant sets under G.

(ii) $y \in J^m(x)$ if and only if $x \in J^m(y)$.

(iii) $gJ^m(x) = J^m(gx) = J^m(x)$ for each $g \in G$.

Proof:

Similar to prove of proposition (2.10) .

2.12. Proposition:

Let (X,φ) be an mG-space and $x \in X$. Then:

(i) If $x \in J^m(x)$ then for each $y \in G_x^m$ then $y \in J^m(y)$

(ii) If $y \in \Lambda^m(x)$ for some $y \in X$, then $y \in J^m(y)$.

(iii) If $x \notin J^m(y)$ for each $x \in X$, then $\Lambda^m(x) = \emptyset$.

Proof:

(i) Let $x \in J^m(x)$ and $y \in G_x^m$. Since $J^m(x)$ is an invariant , then $x \in J^m(x)$ for each $y \in G_x^m$, by proposition(2.11.ii) , $x \in J^m(x)$. But $J^m(x)$ is an invariant , thus $y \in J^m(x)$

(ii) Let $y \in \Lambda^m(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \overset{m}{\rightarrow} \infty$, such that $g_d x \overset{m}{\rightarrow} y$. Put $y_d = g_d x \overset{m}{\rightarrow} y$. Then it is clear that $g_d^{-1} \overset{m}{\rightarrow} \infty$, proposition(2.3.19-iii) and $g_d^{-1}(y_d) = g_d^{-1} g_d x = x \overset{m}{\rightarrow} x$ thus $x \in J^m(y)$ which is closed and invariant , then we have $g_d x \in J^m(y)$. since $g_d x \overset{m}{\rightarrow} y$, then $y \in J^m(y)$.

(iii) Let $x \notin J^m(x)$ for each $x \in X$. To prove $\Lambda^m(x) = \emptyset$. If $y \in \Lambda^m(x)$, then from (ii) $y \in J^m(y)$ it is a contradiction, thus $\Lambda^m(x) = \emptyset$ for each $x \in X$.

2.13. Definition:

Let $f: X \longrightarrow Y$ be a function from m-space X into m-space Y . then f is said to be a minimal proper (briefly m-proper) if :

(i) f is an m-continuous function ;

(ii) f is an m-closed function ;

(iii) $f^{-1}\{y\}$ is an m-compact set in X , for every $y \in Y$.

2.14. Theorem [5]:

Let $f: X \longrightarrow Y$ be a continuous function . Then the following statements are equivalent:

(i) f is proper function .

(ii) If $(\chi_d)_{d\in D}$ is a net in X and $y \in Y$ is a cluster point of the net $f(\chi_d)$ then there is a cluster point $x \in X$ of $(\chi_d)_{d\in D}$ such that $f(x) = y$.

Simple verification shows that this result remain valid when X and Y are minimal spaces as following:

Let $f: X \longrightarrow Y$ be an m-continuous function from um-compact m-T_1-space X into m-T_1-space . Then the following statements are equivalent:

(i) f is an m-proper function.

(ii) If $(\chi_d)_{d\in D}$ is a net in X and $y \in Y$ is an m-cluster point of $f(\chi_d)$ then there is an m-cluster point $x \in X$ of $(\chi_d)_{d\in D}$ such that $f(x) = y$

Bourbaki in [3] defines a proper G-space as follows :

2.15. Definition [3]:

Let X be G-space . Then X is said to be proper G-space if the function $\varphi : G\text{x}X \longrightarrow X\text{x}X$, defined by $\varphi(g,x) = (x, g.x)$, $\forall (g,x) \in G\text{x}X$, is proper .

Now, we introduce the following definition.

2.16. Definition:
An mG-space (X,φ) is called an minimal proper mG-space (m-Bourbakiproper mG-space) if the function $\theta: G\text{x}X \to X\text{x}X$, $\theta(g,x) = (x, \varphi(g,x))$, $\forall(g,x) \in G\text{x}X$, is an m-proper function .

2.17. Proposition:
Let (X,φ) be an mG-space , where X and G are um-compact T_2-space , then X is an m-proper mG-space if and only if $J^m(x) = \emptyset$ for each $x \in X$.

Proof:
Suppose that $\in J^m(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{m} \infty$ and there is a net $(x_d)_{d\in D}$ in X with $x_d \xrightarrow{m} x$ such that $g_d x_d \xrightarrow{m} y$. since $\theta((g_d, x_d)) = (x_d, g_d x_d)$, but X m-Bourbakiproper mG-space , then there is $(h, x_1) \in G \times X$ such that $(g_d, x_d) \overset{m}{\propto} (h, x_1)$. Thus by proposition (1.31) then$(g_d)_{d\in D}$ has m-convergent subnet , say itself . this is a contradiction , thus $J^m(x) = \emptyset$ for each $x \in X$.

Conversely:
Let $(g_d, x_d)_{d\in D}$be a net in $G \times X$and $(x,y) \in X \times X$. since $((g_d, x_d))$, $= (x_d, g_d x_d)$, thus by proposition (1.31) $(g_d, g_d x_d)_{d\in D}$ has a subnet , say itself , such that $(g_d, g_d x_d) \xrightarrow{m} (x,y)$, by proposition(1.32) $x_d \xrightarrow{m} x$ and $g_d x_d \xrightarrow{m} y$ suppose that $g_d \xrightarrow{m} \infty$ then $y \in J^m(x)$ this is a contradiction . Then there is $h \in G$ such that $g_d \xrightarrow{m} h$, therefore $(g_d, x_d) \xrightarrow{m} (h,y)$and $(h,x) = (x,y)$. Thus X m-Bourbakiproper mG-space .

3. On Minimal Small Sets:

In this section, we introduce the definitions of m-small, m-thin relative and m-cartanmG-spaceand introduce a new types of minimal G-spaces . Also we study the proper mG-spaces in senses ofboth Plasis and Bourbaki .

3.1. Definition [6]:
Let (X,φ) be an G-space. A subset A of X is said to be thin relative to a subset B of X if the set$((A,B)) = \{g \in G \mid gA \cap B \neq \emptyset\}$ has neighborhood whose closure is compact in G . If A is thin relative to itself then it is called thin.

Now, we introduce the following definitions.

3.2. Definition:
Let(X,φ) be an mG-space, A subset A of X is said to be minimal thin relative to a subset B of$X$ iftheset$((A,B)) = \{g \in G \mid gA \cap B \neq \emptyset\}$ has an m-neighborhoodwhose m-closure is an m-compact in G . If A is an m-thin relative to itself then it is called m-thin.

3.3. Proposition:
Let (X,φ) be a mG-space and K_1 , K_2 be an m-compact subset of X. Then:

(i)$((K_1,K_2))$ is an m-closed subset of G .
(ii)$((K_1,K_2))$ is an m-compact when K_1andK_2 relatively m-thin

Proof:
(i) Let $g \in \overline{((K_1,K_2))}^m$. Then by theorem (1.27) there is a net $(g_d)_{d\in D}$ in $((K_1,K_2))$ such that$g_d \xrightarrow{m} g$, since $g_d \in ((K_1,K_2))$, then there is a net $(k_d^1)_{d\in D}$ in K_1 which is an m-compact , such that$g_d k_d^1 \in K_2$, since K_2 is an m-compact , then there exists a subnet $(g_{d_m} k_{d_m}^1)$ of$(g_d k_d^1)$ such that $g_{d_m} k_{d_m}^1 \xrightarrow{m} k_0^2$, where$k_0^2 \in K_2$, but$(k_{d_m}^1)$ in K_1 and K_1 is m-compact , thus there is a point$k_0^1 \in K_1$ and a subnet o f $k_{d_m}^1$say itself such that$k_{d_m}^1 \xrightarrow{m} k_0^1$, then by proposition (1.37) $g_{d_m} k_{d_m}^1 \xrightarrow{m} g k_0^1 = k_0^2$, which mean that $g \in ((K_1,K_2))$, therefore $((K_1,K_2))$ is an m-closed .

(ii) Let K_1 and K_2 are m-compact subset of mG-space X such that K_1 and $K_2$are m-relatively thin , then $((K_1,K_2))$ has an$m$-neighborhood whose m-closure is an m-compact , sinceK_1 and K_2 are m-compact by (i), $((K_1,K_2))$ is an m-closed , thus $((K_1,K_2))$ is an m-compact .

3.4. Definition [11]:
AG-space X is called Cartan G-space if every point in X has a thin neighborhood.

Now, we introduce the following definitions.

3.5. Definition:
A mG-space (X,φ) is said to be an m-cartan mG-space if every point in X has an m-thin m-neighborhood.

3.6. Remark:
It is clear that a mG-space (X,φ) is an m-cartan mG-space if G is an m-compact and um-space.

3.7. Definition [6]:
A subset S of an G-space X is a small subset of X if each point of X has a neighborhood which is thin relative to S.

3.8. Definition [6]:
A G-space X is said to be a palais proper G-space if every point x in X has a small neighborhood .

Now, we introduce the following definitions.

3.9.Definition:
A subset S of an mG-space X is an minimal small (briefly m-small) subset of X if each point of X has an m-neighborhood which is m-thin relative to S .

3.10. Definition:
An mG-space X is said to be an minimal palais proper mG-space (m-palais proper mG-space) if every point x in mG has an m-neighborhood which is an$m$-small set .

3.11.Theorem:
Let X be an mG-space. Then:

(i) If X be im-space then m-small neighborhood of a point x contains an m-thin neighborhood of x.
(ii)A subset of an m-small set is an m-small.
(iii) If X be um-space then a finite union of an m-small sets is an m-small.

(iv) If X be um-space then S is an m-small subset of X and K is an m-compact subset of X then K is an m-thin relative to S.

Proof:
(i) Let S be an m-small neighborhood of x . Then there is an m-neighborhood U of x which is m-thin relative to$S$. Then $((U,S))$ has m-neighborhood whose closure is m-compact. Let $V = U\cap S$. since$X$is an m-space, then V is an m-neighborhood of x and $((V,V)) \subseteq ((U,S))$, therefore V is m-thin neighborhood of x .

(ii) Let S be an m-small set and $K \subseteq S$. let $x \in X$, then there exists an m-neighborhood U of x , which is m-thin relative to S. Then $((U,K)) \subseteq ((U,S))$, thus $((U,K))$ has m-neighborhood whose closure is an m-compact . Then K is an m-small.

(iii) Let$\{S_i\}_{i=1}^{n}$be a finite collection of m-small sets and $y \in X$. Then for$i$there is m-neighborhood K_i of y such that the set $((S_i,K_i))$ has m-neighborhood whose closure is an$m$-compact. Then $\bigcup_{i=1}^{n}((S_i,K_i))$ has m-neighborhood whose closure is an m-compact. But $((\bigcup_{i=1}^{n} S_i , \bigcup_{i=1}^{n} K_i)) \subseteq \bigcup_{i=1}^{n}((S_i,K_i))$, thus $\bigcup_{i=1}^{n} S_i$ is an m-small set.

(iv) Let S be an m-small set and K be m-compact . Then there is an m-neighborhood U_k of K , $U_k \in K$, such that U_k is an m-thin relative to S. Since $K \subseteq \bigcup_{k\in K} U_k$. i.e., $\{U_k\}_{k\in K}$ {is an m-open cover of k , which is an m-compact , so there is a finite sub cover $\{U_{ki}\}_{i=1}^{n}$ of $\{U_k\}_{k\in K}$, since $((U_{ki},S))$ has m-neighborhood whose closure is an m-compact , thus$((\bigcup_{i=1}^{n} U_{ki},S))$ so is . But $((K,S)) \subseteq ((\bigcup_{i=1}^{n} U_{ki},S))$ therefore K is an m-thin relative to S.

3.12.Proposition:
Let X be an um-spaceand(X,φ) be an mG-space. Then:

(i) If $X$is an m-palais proper mG-space, then every m-compact subset of $X$is an m-small set
(ii) If X is an m-palais proper mG-space and K is an m-compact subset of X , then $((K,K))$ is an m-compact subset of G .

Proof:

(i) Let A be a subset of X such that A is an m-compact . let $x \in X$, since X is an m-

proper mG-space then there is an m-neighborhood U which is m-small of x . Then for every $a \in A$ there exist an neighborhood U_a which is an m-small , then $A \subseteq \bigcup_{a \in A} U_a$, since A is an m-compact , then there exists $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$, thus by theorem(3.11.ii-iii) A is an m-small set in X .

(ii) Let X m-palais proper mG-space and K is an m-compact , then by (i) K is an m-small subset of X , and bytheorem (3.11.iv) K is an m-thin , so $((K, K))$ has m-neighborhood whose m-closure is an m-compact . Then by proposition (3.3.i) $((K, K))$ is an m-closed in G . Thus $((K, K))$ is an m-compact.

3.13.Proposition:

Minimal palais proper mG-space is an m-cartan mG-space.

Proof:

Let$X$ be an m-palais proper mG-space and let $x \in X$. Then there is an m-small neighborhood S of x. since S is an m-small neighborhood of x. Then there is an m-neighborhood U of x which is an m-thin relative to S. Thus $((U, S))$ has m-neighborhood whose m-closure is an m-compact, therefore U is an m-thin neighborhood of x. Then X has an m-thin neighborhood of . Therefore X is an m-cartan mG-space .

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الخلاصة:

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الكلمات المفتاحية::

الفضاء الاصغري , الافعال الاصغرية , المجوعات الواهية الاصغرية

<u>*البحث مستل من رسالة ماجستير للباحث الثاني</u>