The complete Solution of the Linear second Order partial Differential Equations with constant coefficients

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المستخلص:-

هدفنا في هذا البحث هو الحصول على الحل الكامل للمعادلات التفاضلية الخطية الجزئية من الرتبة الثانية ذات المعاملات الثابتة والتي صيغتها العامة $A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0$ حيث ان z هو المتغير المعتمد و x, y هما متغيرين مستقلين وان المعاملات $A_i = (i = 1,2,...,6)$ هي ثوابت.

Abstract

Our aim in this paper is to get the complete solution of the second order partial differential equations with constant coefficients which is general formula $A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0$ where z be the dependent variable, x and y the independent variables, A_i $(i = 1,2,...,6)$ are constants.

1- Introduction

Let us consider the general form of the linear second order partial differential equation with constant coefficients

$$A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0 \quad \dots (1)$$

to find the complete solution of the equation (1) we will search a new functions u(x) and v(y) such that the assumption

$$z = e^{\int u(x)dx \int v(y)dy} \quad \dots (2)$$

gives the complete solution of equation (1) , this assumption will transform the above equation (1) to linear first order ordinary equation and contains two independent functions u(x) and v(y).

2- Basis defentions and concepts

Def 1 [5] : we call to the ordinary differential equation

$$\frac{dy}{dx} + p(x)\,y = Q(x) \quad \dots (3)$$

the Linear differential equation.

Def 2 [2] : we call to the ordinary differential equation

$$\frac{dy}{dx} + p(x)y = Q(x)\,y^n \;;n \neq 1 \quad \dots (4)$$

the Bernoulli equation .

Def 3 [8] : A partial Differential equation is an equation that contains partial derivatives, in contrast to ordinary differential equation, where unknown function depends only on one variable, in partial differential equations, the unknown function depends on several variables.

Def 4 [7] Equation that their derivatives of the first degree and they are not multiplied together are called linear differential equation .

Def 5[4] : Any relation between dependent variable and independent variables which satisfies partial differential equation and free from partial derivatives is said to be a solution of the partial differential equation .

Def 6 [4] : the solution of the partial differential equation which contains only arbitrary constants is called complete solution.

3- classification of partial Differential Equations [1]

second- order partial differential equations and systems can usually be classified as :-

1) parabolic Equations: parabolic equations describe heat flow and diffusion processes and satisfy the property $A_2^2 - 4A_1A_3 = 0$, atypical example is the heat equation $z_y = c^2 z_{xx}$, where c is an arbitrary constant .

2) Hyperbolic Equations: Hyperbolic equations describe vibrating systems and wave motion and satisfy the property $A^2 - 4A_1A_3 > 0$,a typical example is the wave equation $z_{yy} = c^2 z_{xx}$, where c is an arbitrary constant .

3) Elliptic Equations: Elliptic equations describe steady – state phenomena and satisfy the property $A^2 - 4A_1A_3 < 0$,a typical example is the Laplace equation $z_{xx} + c^2 z_{yy} = 0$, where c is an arbitrary constant.

Note 1: consider that the $z_1(x)$ be a function only of x and $z_2(y)$ be a function only of y and suppose that

$$f_1(x, z_1(x) + z_1'(x), \ldots) + f_2(y, z_2'(y), z_2''(y), \ldots) = 0 \quad \ldots (5)$$

Inasmuch as x and y independent of each other , each side of equation (5) must be affixed constant, hence, we can write

$f_1(x, z_1', z_1'', \ldots) = a$ and $f_2(y, z_2', z_2'', \ldots) = b$

$\Rightarrow a + b = 0 \quad \Rightarrow a = -b$

so, now we can solve each of this ordinary differential equation .

Now , we make an important observation, namely , that we want the separation constant (b) to be negative , write this in mind, it is general practice to rename $(b = -\lambda^2)$.

4- the complet solution

Now we want to find the complete solution to the partial differential equation , which have the form

$$A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0 \quad \dots (6)$$

so, for this purpose we present a new method to find the complete solution by a simple assumption and to do this we classify the equation (6) to the following kinds:-

1-

a) $A_1 z_{xx} = 0 \quad \dots (7)$

i.e. $A_2 = A_3 = \dots = A_6 = 0 \ni A_1$ is not identically zero .

b) $A_2 z_{xy} = 0 \quad \dots (8)$

i.e $A_1 = A_3 = \dots = A_6 = 0 \ni A_2$ is not identically zero .

c) $A_3 z_{yy} = 0$ ... (9)

i.e $A_1 = A_2 = A_4 = \dots = A_6 = 0 \ni A_3$ is not identically zero.

2-

a) $A_1 z_{xx} + A_2 z_{xy} = 0$ ... (10)

i.e $A_3 = A_4 = \dots = A_6 = 0 \ni A_1$ *and* A_2 are not identically zero .

b) $A_1 z_{xx} + A_3 z_{yy} = 0$ ... (11)

i.e $(A_1 = A_2 = A_4 = \dots = A_6 = 0) \ni A_1$ and A_3 are not identically zero .

c) $A_2 z_{xy} + A_3 z_{yy} = 0$... (12)

i.e $A_1 = A_4 = \dots = A_6 = 0 \ni A_2$ and A_3 are not identically zero.

d) $A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} = 0$... (13)

i.e $A_4 = \dots = A_6 = 0 \ni A_1, A_2$ and A_3 are not identically zero.

3-

a) $A_1 z_{xx} + A_4 z_x + A_5 z_y + A_6 z = 0$... (14)

i.e $(A_2 = A_3 = 0) \ni A_1, A_4, A_5$ and A_6 are not identically zero .

b) $A_2 z_{xy} + A_4 z_x + A_5 z_y + A_6 z = 0$ ... (15)

i.e $(A_1 = A_3 = 0) \ni A_2, A_4, A_5$ and A_6 are not identically zero .

c) $A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0$... (16)

i.e $(A_1 = A_2 = 0) \ni A_2, A_4, A_5$ and A_6 are not identically zero .

4-

a) $A_1 z_{xx} + A_2 z_{xy} + A_4 z_x + A_5 z_y + A_6 z = 0 \quad \dots (17)$

i.e $A_3 = 0 \ni A_1, A_2, A_4, A_5$ and A_6 are not identically zero .

b) $A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0 \ldots(18)$

i.e $A_1 = 0 \ni A_2, A_3, A_4, A_5$ and A_6 are not identically zero .

c) $A_1 z_{xx} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0 \ \ldots (19)$

i.e $A_2 = 0 \ni A_1, A_3, A_4, A_5$ and A_6 are not identically zero .

5) $A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0 \ \ldots (20)$

$\ni A_1 ..., A_6$ are not identically zero .

Now , to find the complete solution to the all above kinds of partial differential equations, we search a new functions $u(x)$ and $v(y)$ such that the assumption

$$z(x, y) = e^{\int u(x)\,dx + \int v(y)\,dy} \ \ldots (21)$$

Represents the complete solutions to the partial differential equations (6), (7), …, and (20) , by finding z_x, z_{xx}, z_{xy}, z_y *and* z_{yy} and after substituting all these in equation (6) we get

$$A_1(u'(x) + u^2(x)) + A_2 u(x) v(y) + A_3 (v'(y) + v^2(y)) + A_4 u(x) + A_5 v(y) + A_6 = 0 \ \ \ldots (22)$$

Now, we will give the general form of the complete solutions to each kinds of the above differential equations and how we can prove their.

1) the complete solutions to any kind of the partial differential equations as in kind (1) are given by :-

a) $z(x, y) = B\Psi(y)(x - c)$

where B and c are arbitrary constants and $\Psi(y)$ is an arbitrary function of y .

b) Either i) $z(x, y) = B_1 \Psi_1(y)$

or ii) $z(x, y) = B_2 \Psi_2(x)$

or iii) $z(x, y) = B_3$

where $B_i \ (i = 1,2,3)$ are arbitrary constants and $\Psi_1(y)$ and $\Psi_2(x)$ are arbitrary functions of y and x respectively .

c) $z(x, y) = B_4 \Psi_4(x)(y - c)$

where B_4 and c are arbitrary constants and $\Psi_4(x)$ is an arbitrary function of x

proof a) since $A_2 = A_3 = ... = A_6 = 0$,

so equation (22) becomes

$$A_1(u' + u^2(x)) = 0$$

$$\Rightarrow \; -\frac{1}{u} = c - x \quad \Rightarrow$$

$$u = \frac{1}{x - c} \quad ,$$

so the equation (21) becomes

$$z(x,y) = e^{\int \frac{dx}{x-c} + \int v(y)\,dy}$$

$$= e^{\ln(x-c) + \phi(y) + b}$$

$$= B\psi(y)(x - c)$$

where B and c are arbitrary constants and $\Psi(y)$ is an arbitrary function of y.

proof b : since $A_1 = A_3 = \ldots = A_6 = 0$

so equation (22) becomes

$$A_2\, u(x)\, v(y) = 0$$

$$\Rightarrow \; either \quad i)\; u(x) = 0 \;\; and \;\; v(y) \neq 0$$

$$\Rightarrow \; equation \; (21) \; becomes$$

$$z(x,y) = e^{\int 0\,dx + \int v(y)\,dy}$$

$$= e^{a + \phi_1(y)}$$

$$= B_1\, \Psi_1(y)$$

where B_1 is an arbitrary constant and $\Psi_1(y)$ is an arbitrary function of y.

ii) $u(x) \neq 0 \;\; and \;\; v(y) = 0$

so equation (21) becomes

$$z(x,y) = e^{\int u(x)\,dx + \int 0\,dy}$$

$$= e^{\phi_2(x) + d}$$

$$= B_2\, \Psi_2(x)$$

where B_2 is an arbitrary constant and $\Psi_2(x)$ is an arbitrary function of x.

or iii) $u(x) = 0 = v(y)$

so equation (21) becomes

$$z(x,y) = e^{\int 0\,dx + \int 0\,dy} = e^{b_3} = B_3$$

where B_3 is an arbitrary constant

proof c) : by the same method as in case (a) we can prove

$$z(x,y) = B_4\, \psi_4(x)(y - c)$$

2) the complete solutions to each kind of the partial differential equations as in kind (2) are given by

a) $z(x,y) = e^{\frac{A_1}{A_2}\lambda^2 y}\,(B_1 e^{-\lambda^2 x} + B_2)$

where B_1 *and* B_2 are arbitrary constants

b) $z(x,y)=(d_1\cos\sqrt{\frac{1}{A_1}}\lambda x+d_2\sin\sqrt{\frac{1}{A_1}}\lambda x)\,(b_1\cosh\sqrt{\frac{1}{A_3}}\lambda y+b_2\sinh\sqrt{\frac{1}{A_3}}\lambda y)$

where d_1, d_2, b_1, b_2 *and* λ are arbitrary constants

c) $z(x,y)=e^{\frac{A_3}{A_2}\lambda^2 x}(B_1 e^{-\lambda^2 y}+B_2)$

where B_1 *and* B_2 are arbitrary constants

d) either i)

$$z(x,y)=e^{-\frac{A}{2}x+\lambda y}\,(d_1\cos\sqrt{B-\frac{A^2}{4}}\,x+d_2\sin\sqrt{B-\frac{A_2}{4}}\,x)$$

If $B\neq\frac{A^2}{4}$; $A=\frac{A_2}{A_1}\lambda$ *and* $B=\frac{A_3}{A_1}\lambda^2$

Where as d_1, d_2 and λ are arbitrary constants

Or ii) $z(x,y)=D\,e^{-\frac{A}{2}x+\lambda y}\,(x-c)$

If $B=\frac{A^2}{4}$; $A=\frac{A_2}{A_1}\lambda$ *and* $B=\frac{A_3}{A_1}\lambda$

Where as D, c and λ are arbitrary constants

Proof a): since $A_3=A_4=...=A_6=0$

So equation (22) becomes

$$A_1(u'(x)+u^2(x))+A_2u(x)v(y)=0$$

$$\Rightarrow\quad \frac{u'(x)+u^2(x)}{u(x)}=-\frac{A_2}{A_1}v(y)=-\lambda^2 ,$$

$$so\quad v(y)=\frac{A_1}{A_2}\lambda^2$$

$$and\quad u'(x)+u^2(x)+\lambda^2u(x)=0 ,$$

the last equation is similar to Bernoulli equation and it's solution is given by:-

$$u(x)=\frac{e^{-\int\lambda^2dx}}{\int e^{-\int\lambda^2dx}dx},$$

so, equation (21) becomes

$$z(x,y)=e^{\int\frac{e^{-\int\lambda^2dx}}{\int e^{-\int\lambda^2dx}dx}dx+\int\frac{A_1}{A_2}\lambda^2dy}$$

$$=e^{\ln(\int e^{-\int\lambda^2dx}dx)+\frac{A_1}{A_2}\lambda^2y+b}$$

$$= B(\int e^{-\int \lambda^2 dx}\ dx\ )e^{\frac{A_1}{A_2}\lambda^2 y}\quad ; B = e^b$$

$$= Be^{\frac{A_1}{A_2}\lambda^2 y}\ (-\frac{1}{\lambda^2}e^{-\lambda^2 x} - \frac{g}{\lambda^2})\ ,$$

so the complete solution is given by

$$z(x,y) = e^{\frac{A_1}{A_2}\lambda^2 y}\ (B_1\, e^{-\lambda^2 x} + B_2)\ ,$$

where $B_1 = \frac{-B}{\lambda^2}$, $B_2 = -\frac{Bg}{\lambda^2}$ *and* λ are arbitrary constants.

b) since $A_2 = A_4 = ... = A_6 = 0$, so equation (22) becomes

$$A_1(u'(x) + u^2(x)) + A_3(v'(y) + v^2(y)) = 0$$

$$\Rightarrow\ A_1(u'(x) + u^2(x)) = -A_3(v'(y) + v^2(y)) = -\lambda^2$$

$$\Rightarrow u'(x) + u^2(x) + \frac{\lambda^2}{A_1} = 0 \quad \ldots (23)$$

and $v'(y) + v^2(y) - \frac{\lambda^2}{A_3} = 0$ … (24)

we get from (23)

$$u(x) = b_1 \tan(f_1 - b_1 x)\ ;\ b_1^2 = \frac{\lambda^2}{A_1}$$

where b_1 and f_1 are arbitrary constants .

Also, we get from equation (24)

$$v(y) = b_2 \tanh(f_2 + b_2 y)\ \ ; b_2^2 = \frac{\lambda^2}{A_3}$$

where b_1 and f_1 are arbitrary constants.

Therefore equation (21) becomes

$$z(x,y) = e^{\int b_1 \tan(f_1 - b_1 x) dx + \int b_2 \tan(f_2 + b_2 y) dy}$$

so, the complete solution is given by

$$z(x,y) = (d_1 \cos\sqrt{\frac{1}{A_1}}\lambda x + d_2 \sin\sqrt{\frac{1}{A_1}}\lambda x)\ (b_1 \cosh\sqrt{\frac{1}{A_3}}\ \lambda y + b_2 \sinh\sqrt{\frac{1}{A_3}}\lambda y)$$

c) since $A_1 = A_4 = ... = A_6 = 0$, so equation (22) becomes

$$A_2 u(x)\ v(y) + A_3\ (v'(y) + v^2(y)) = 0$$

And by the same way as in case (a) we can prove that the complete solution is given by

$$z(x,y) = e^{\frac{A_3}{A2}\lambda^2 x}\ (B_1\, e^{-\lambda^2 y} + B_2)\ ,$$

where $B_1 = \frac{-B}{\lambda^2}$, $B_2 = -\frac{Bg}{\lambda^2}$ and λ are arbitrary constants.

d) since $A_4 = ... = A_6 = 0$, so equation (22) becomes

$$A_1(u'(x)+u^2(x))+A_2u(x)v(y)+A_3(v'(y)+v^2(y))=0,$$

here we can't separate the variables, so we suppose that $v(y)=\lambda$, where λ is an arbitrary constant, then the last equation becomes

$$u'(x)+u^2(x)+\frac{A_2}{A_1}\lambda u(x)+\frac{A_3}{A_1}\lambda^2=0$$

let $A=\frac{A_2}{A_1}\lambda$ and $B=\frac{A_3}{A_1}\lambda^2$, then the last equation becomes

$u'(x)+u^2(x)+Au(x)+B=0$,

we can solve this equation by using variable separable method, hence:-

i) If $B\neq\frac{A^2}{4}$, we get

$u(x)=b\tan(f-bx)-\frac{A}{2}$; b and f are constants .

there fore by using equation (21) we get

$$z(x,y)=e^{\int(b\tan(f-bx)-\frac{A}{2})dx+\int\lambda dy}$$

$=De^{-\frac{A}{2}x+\lambda y}\cos(f-bx)$; D is an arbitrary constant .

So, the complete solution is given by

$$z(x,y)=e^{-\frac{A}{2}x+\lambda y}\left(d_1\cos\sqrt{B-\frac{A^2}{4}}\,x+b_1\sin\sqrt{B-\frac{A^2}{4}}\,x\right)$$

where d_1, b_1 *and* λ are arbitrary constants.

ii) if $B=\frac{A^2}{4}$, we get

$u(x)=\frac{1}{x-c}-\frac{A}{2}$; $A=\frac{A_2}{A_1}\lambda$, $B=\frac{A_3}{A_1}\lambda^2$

therefore by substituting in equation (21) we get

$z(x,y)=e^{\int(\frac{1}{x-c}-\frac{A}{2})dx+\int\lambda dy}$

$=e^{\ln(x-c)-\frac{A}{2}x+\lambda y+g}$,

so the complete solution is given by

$z(x,y)=De^{-\frac{A}{2}x+\lambda y}(x-c)$; D, c *and* λ are arbitrary constants.

Example 1:- to solve the partial differential equation $z_{xx}+\frac{1}{4}z_{yy}=0$ (Laplace equation , $c^2=\frac{1}{4}$) , we note that $A_2=A_4=...=A_6=0$, then by using the formula that is given in kind (2-b) we get the complete solution to the above partial equation , which is form

$$z(x,y)=(d_1\cos\lambda x+d_2\sin\lambda x)\,(b_1\cosh 2\lambda y+b_2\sinh 2\lambda y)\ ,$$

where d_1, d_2, b_1, b_2 *and* λ and arbitrary constants .
3)
a) since $A_2 = A_3 = 0$, so the equation (22) becomes
$A_1(u'(x)+u^2(x))+A_4u(x)+A_5v\ (y))+A_6=0$
and the complete solution of equation (14) is given by

either i) $z(x,y)=e^{-\frac{A}{2}x+\frac{\lambda^2-A_6}{A_5}y}\ (d_1\cos\sqrt{B-\frac{A^2}{4}}\ x+b_1\sin\sqrt{B-\frac{A^2}{4}}\ x)$

if $B\neq\frac{A^2}{4}$; $A=\frac{A_4}{A_1}$ *and* $B=\frac{\lambda^2}{A_1}$,
where d_1, b_1 *and* λ are arbitrary constants .

or ii) $z(x,y)\ =De^{-\frac{A}{2}x+\frac{\lambda^2-A_6}{A_5}y}\ (x-c)$

if $B=\frac{A^2}{4}$; $A=\frac{A_4}{A_1}$ *and* $B=\frac{\lambda^2}{A_1}$,
where D,c and λ are arbitrary constants.
b) since $A_1 = A_3 = 0$, so the equation (22) becomes

$$A_2u(x)v(y)+A_4u(x)+A_5v\ (y)+A_6=0$$

and the complete solution of equation (15) by using assumption that is given in equation (21) is given by

$z(x,y)=Ae^{\lambda x-(\frac{A_4\lambda+A_6}{A_5+A_2\lambda})\,y}$

where A and λ are arbitrary constants.
c) since $A_1 = A_2 = 0$, so the equation (22) becomes

$$A_3z_{yy}+A_4z_x+A_5z_y+A_6z=0,$$

and the complete solution of equation (16) by using the assumption that is given in equation (21) is given by

either i) $z(x,y)=e^{-\frac{A}{2}y+\frac{\lambda^2-A_6}{A_4}x}\ (d_1\cos\sqrt{B-\frac{A^2}{4}}\ y+b_1\sin\sqrt{B-\frac{A^2}{4}}\ y)$

if $B\neq\frac{A^2}{4}$; $A=\frac{A_5}{A_3}$ *and* $B=\frac{\lambda^2}{A_3}$
where d_1, b_1 *and* λ are arbitrary constants .

or ii) $z(x,y)\ =De^{\frac{\lambda^2-A_6}{A_4}x-\frac{A}{2}y}\ (y-c)$

if $B=\frac{A^2}{4}$; $A=\frac{A_5}{A_3}$ *and* $B=\frac{\lambda^2}{A_3}$,
where D ,c and λ are arbitrary constants.

Proof (a) :- since $A_2 = A_3 = 0$, so equation (22) becomes

$$A_1(u'(x)+u^2(x))+A_4u(x)+A_5v\ (y)+A_6 = 0$$

$$\Rightarrow\ u'(x)+u^2(x)+\frac{A_4}{A_1}u(x)=-\frac{A_5}{A_1}v(y)-\frac{A_6}{A_1}=\frac{-\lambda^2}{A_1}$$

so $v(y)=\dfrac{\lambda^2-A_6}{A_5}$ and $u'(x)+u^2(x)+\dfrac{A_4}{A_1}u(x)+\dfrac{\lambda^2}{A_1}=0$

let $A=\dfrac{A_4}{A_1}$ *and* $B=\dfrac{\lambda^2}{A_1}$ therefore the last equation becomes

$$u'(x)+u^2(x)+A\,u(x)+B=0\,,$$

this equation we can solve it by variable separable method and its solution is given by :-

either i) $u(x)=b\ \tan(f-bx)-\dfrac{A}{2}$ *if* $B\neq\dfrac{A^2}{4}$

where $b^2=B-\dfrac{A^2}{4}$

Hence by using equation (21) we get

$$z(x,y)=e^{\int(b\tan(f-bx)-\frac{A}{2})dx+\int\frac{\lambda^2-A_6}{A_5}dy}$$

$$=c_1\ e^{-\frac{A}{2}x+\frac{\lambda^2-A_6}{A_5}y}\ \cos(f-bx)\ ,$$

so, the complete solution is given by:-

$$z(x,y)=e^{-\frac{A}{2}x+\frac{\lambda^2-A_6}{A_4}y}\ (d_1\cos\sqrt{B-\frac{A_2}{4}}\,x+b_1\sin\sqrt{B-\frac{A^2}{4}}\,x)$$

where d_1, b_1 *and* λ are arbitrary constants .

or ii) If $B=\dfrac{A^2}{4}$, we get

$$u(x)=\frac{1}{x-c}-\frac{A}{2}$$

$$\Rightarrow z(x,y)=e^{\int(\frac{1}{x-c}-\frac{A}{2})dx+\int\frac{\lambda^2-A_6}{A_5}dy}$$

$$=De^{-\frac{A}{2}x+\frac{\lambda^2A_6}{A_5}y}\ (x-c),$$

where c, D and λ are arbitrary constants.

Note (2) :- If we write

$$A_1(u'(x)+u^2(x))+A_4u(x)+A_6=-A_5v(y)=-\lambda^2$$

then the complete solution is given by:-

either i) $z(x,y)=e^{-\frac{A}{2}x+\frac{\lambda^2}{A_5}y}\ (d_1\cos\sqrt{B-\dfrac{A^2}{4}}\,x+b_1\sin\sqrt{B-\dfrac{A^2}{4}}\,x)$

If $B\neq\dfrac{A^2}{4}$; $A=\dfrac{A_4}{A_1}$ *and* $B=\dfrac{A_6+\lambda^2}{A_1}$

where d_1, b_1 *and* λ are arbitrary constants .

or ii) $z(x,y) = De^{-\frac{A}{2}x+\frac{\lambda^2}{A_5}y}\ (x-c)$

If $B = \frac{A^2}{4}$; $A = \frac{A_4}{A_1}$ *and* $B = \frac{A_6+\lambda^2}{A_1}$

where c, D and λ are arbitrary constants.

b) since $A_1 = A_3 = 0$, so equation (22) becomes

$$A_2 u(x)v(y) + A_4 u(x) + A_5 v(y) + A_6 = 0$$

$$\Rightarrow \frac{A_5 v(y) + A_6}{A_2 v(y) + A_4} = -u(x) = -\lambda$$

$$\Rightarrow u(x) = \lambda \quad and\ v(y) = -\frac{A_4\lambda + A_6}{A_5 + A_2\lambda} \ ,$$

there for by using equation (21) we get

$$z(x,y) = Ae^{\lambda x - \frac{A_4\lambda + A_6}{A_5 + A_2\lambda}y} \ ,$$

where A and λ are arbitrary constants.

c) By the same way as in (a) we can get the proof

Note (3) :- If we write

$$A_3(v'(y) + v^2(y)) + A_5 v(y) + A_6 = -A_4 u(x) = -\lambda^2 ,$$

then by the same method as in case (c) , we can prove that the complete solution is given by:-

either i) $z(x,y) = e^{-\frac{A}{2}y+\frac{\lambda^2}{A_4}x}\ (d_1 \cos\sqrt{B - \frac{A^2}{4}}\ y + b_1 \sin\sqrt{B - \frac{A^2}{4}}\ y)$

if $B \neq \frac{A^2}{4}$; $A = \frac{A_5}{A_3}$ *and* $B = \frac{A_6+\lambda^2}{A_3}$

where d_1, b_1 *and* λ are arbitrary constants .

or ii) $z(x,y) = De^{-\frac{\lambda^2}{A_4}x-\frac{A}{2}y}\ (y-c)$

if $B = \frac{A^2}{4}$; $A = \frac{A_5}{A_3}$ *and* $B = \frac{A_6+\lambda^2}{A_3}$

where D, c and λ are arbitrary constants.

Example 2 :- To solve the partial defferential equation

$$z_{yy} + z_x + 2z_y + z = 0 \ ,$$

we see that $A_1 = A_2 = 0$, so by using the formula which appears in case (c) we get the complete solution to the above equation which is form

$$z(x,y) = e^{-y+(\lambda^2-1)x}\ (d_1 \cos\sqrt{\lambda^2-1}\,y + b_1 \sin\sqrt{\lambda^2-1}\ y)$$

where λ, d_1 *and* b_1 are arbitrary constants.

4-

(a) since $A_3 = 0$, so the equation (6) becomes

$$A_1 z_{xx} + A_2 z_{xy} + A_4 z_x + A_5 z_y + A_6 z = 0$$

and the complete solution of the partial differential equation (17) is given by

either i) $z(x,y) = e^{-\frac{A}{2}x+\lambda^2 y}\,(d_1 \cos\sqrt{B-\frac{A_2}{4}}\,x + b_1 \sin\sqrt{B-\frac{A^2}{4}}\,x)$

if $B \neq \frac{A^2}{4}$; $A = \frac{A_4 + A_2\lambda^2}{A_1}$ *and* $B = \frac{A_6 + A_5\lambda^2}{A_1}$,

where d_1, b_1 *and* λ are arbitrary constants .

or ii) $z(x,y) = De^{\lambda^2 y - \frac{A}{2}x}\,(x-c)$

if $B = \frac{A^2}{4}$; $A = \frac{A_4 + A_2\lambda^2}{A_1}$ *and* $B = \frac{A_6 + A_5\lambda^2}{A_1}$,

where D, c and λ^2 are arbitrary constants.

b) since $A_2 = 0$, so the equation (6) becomes

$$A_1 z_{xx} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0,$$

and the complete solution of the partial differential equation (18) is given by :-

either i)

$$z(x,y) = e^{-\frac{A}{2}x-\frac{c}{2}y}\,(d_1 \cos\sqrt{B-\frac{A^2}{4}}\,x + a_1 \sin\sqrt{B-\frac{A^2}{4}}\,x)(d_2 \cos\sqrt{D-\frac{c^2}{4}}\,y + a_2 \sin\sqrt{D-\frac{c^2}{4}}\,y)\ ,$$

if $B \neq \frac{A^2}{4}$ *and* $D \neq \frac{c^2}{4}; A = \frac{A_4}{A_1}, B = \frac{\lambda^2}{A_1}, c = \frac{A_5}{A_3}$ *and* $D = \frac{A_6 - \lambda^2}{A_3}$

where d_1, d_2, a_1, a_2 *and* λ are arbitrary constants .

or ii) $z(x,y) = e^{-\frac{A}{2}x-\frac{c}{2}y}\,(y-c)(d_1 \cos\sqrt{B-\frac{A^2}{4}}\,x + b_1 \sin\sqrt{B-\frac{A^2}{4}}\,x)$

if $B \neq \frac{A^2}{4}$ *and* $D = \frac{c^2}{4};\ A = \frac{A_4}{A_1}, B = \frac{\lambda^2}{A_1}, c = \frac{A_5}{A_3}$ *and* $D = \frac{A_6 - \lambda^2}{A_3}$

where d_1, b_1 *and* c are arbitrary constants .

or iii) $z(x,y) = e^{-\frac{A}{2}x-\frac{c}{2}y}\,(x-c)(d_1 \cos\sqrt{D-\frac{c^2}{4}}\,y + b_1 \sin\sqrt{D-\frac{c^2}{4}}\,y)$

if $B = \frac{A^2}{4}$ *and* $D \neq \frac{c^2}{4}; A = \frac{A_4}{A_1}, B = \frac{\lambda^2}{A_1}, c = \frac{A_5}{A_3}$ *and* $D = \frac{A_6 - \lambda^2}{A_3}$

where d_1, b_1 *and* c are arbitrary constants .

or iv) $z(x,y) = ke^{-\frac{A}{2}x-\frac{c}{2}y}\,(x-c_1)(y-c_2)$

if $B = \frac{A^2}{4}$ *and* $D = \frac{c^2}{4}; A = \frac{A_4}{A_1}, B = \frac{\lambda^2}{A_1}, c = \frac{A_5}{A_3}$ *and* $D = \frac{A_6 - \lambda^2}{A_3}$

where c_1, c_2 *and* k are arbitrary constants .

c) since $A_1 = 0$ so, the equation (6) becomes

$$A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0$$

and the complete solution to the equation (19) by using equation (21) is given by

either i) $z(x,y) = e^{-\frac{A}{2}y+\lambda^2 x}[d_1 \cos\sqrt{B-\frac{A^2}{4}}\, y + b_1 \sin\sqrt{B-\frac{A^2}{4}}\, y]$

if $B \neq \frac{A^2}{4}$; $A = \frac{A_5 + A_2\lambda^2}{A_3}$ *and* $B = \frac{A_4\lambda^2 + A_6}{A_3}$

where d_1, b_1 *and* λ are arbitrary constants .

or ii) $z(x,y) = De^{\lambda^2 x - \frac{A}{2}y}\,(y-c)$

if $B = \frac{A^2}{4}$; $A = \frac{A_5 + A_2\lambda^2}{A_3}$ *and* $B = \frac{A_4\lambda^2 + A_6}{A_3}$,

where D ,c are arbitrary constants .

proof (a) :- since $A_3 = 0$, so equation (22) becomes

$$A_1(u'(x)) + u^2(x)) + A_2 u(x)v(y) + A_4 u(x) + A_5 v(y) + A_6 = 0$$

$$\Rightarrow \frac{A_1(u'(x) + u^2(x))}{A_2 u(x) + A_5} + \frac{A_4 u(x) + A_6}{A_2 u(x) + A_5} = -v(y) = -\lambda^2 ,$$

so $v(y) = \lambda^2$,

and $\quad A_1(u'(x) + u^2(x)) + A_4 u(x) + A_6 = -\lambda^2 (A_2 u(x) + A_5)$

$$\Rightarrow u'(x) + u^2(x) + Au(x) + B = 0$$

where $A = \frac{A_4 + A_2\lambda^2}{A_1}$ *and* $B = \frac{A_6 + A_5\lambda^2}{A_1}$,

so, the last equation we can solve it by variable separable method and its solution is given by

either i) $u(x) = b\tan(f - bx) - \frac{A}{2}$; $b = B - \frac{A^2}{4}$ and f is an arbitrary constant and $B \neq \frac{A^2}{4}$ $(i.e \quad b \neq 0)$ therefore by substituting in equation (21) we get

$$z(x,y) = De^{-\frac{A}{2}x + \lambda^2 y}\cos(f - bx) ,$$

Hence , the complete solution of equation (17) is given by

$$z(x,y) = e^{-\frac{A}{2}x + \lambda^2 y}\,(d_1 \cos\sqrt{B-\frac{A^2}{4}}\, x + b_1 \sin\sqrt{B-\frac{A^2}{4}}\, x) ,$$

where d_1, b_1 *and* λ are arbitrary constants .

or ii) If $B = \frac{A^2}{4}$, we get

$$u(x) = \frac{1}{x-c} - \frac{A}{2} ,$$

there for by substituting u(x) and v(y) in equation (21) we get

$$z(x,y) = e^{\int(\frac{1}{x-c}-\frac{A}{2})dx+\int \lambda^2 dy},$$

so the complete solution of equation (17) is given by

$$z(x,y) = De^{\lambda^2 y-\frac{A}{2}x}\ (x-c)$$

where as D, c, λ are arbitrary constants .

b)since $A_2 = 0$, so equation (22) becomes

$$A_1(u'(x)+u^2(x))+A_3(v'(y)+v^2(y))+A_4u(x)+A_5v(y)+A_6 = 0$$

$$\Rightarrow A_1(u'(x)+u^2(x))+A_4u(x) = -A_3(v'(y)+v^2(y))-A_5v(y)-A_6 = -\lambda^2$$

$$\Rightarrow u'(x)+u^2(x)+Au(x)+B = 0\ \ ;\ A = \frac{A_4}{A_1}\ \ and\ \ B = \frac{\lambda^2}{A},$$

also $v'(y)+v^2(y)+cv(y)+D = 0\ ;\ c = \frac{A_5}{A_3}\ \ and\ D = \frac{A_6-\lambda^2}{A_3}$

We can solve the last two equations by variable separable mothod and we get

either i) If $B \neq \frac{A^2}{4}\ and\ D \neq \frac{c^2}{4} \Rightarrow$

$$u(x) = b_1\tan(f_1-b_1x)-\frac{A}{2}\ ;\ b_1^2 = B-\frac{A^2}{4},$$

and f_1 is an arbitrary constant .

Also $v(y) = b_2\tan(f_2-b_2y)-\frac{c}{2}\ ;\ b_2^2 = D-\frac{c^2}{4}$,

and f_2 is an arbitrary constant.

Therefor by substituting in equation (21) we get

$$z(x,y) = e^{\int(b_1\tan(f_1-b_1x)-\frac{A}{2})dx+\int(b_2\tan(f_2-b_2y)-\frac{c}{2})dy},$$

so , the complete solution of equation (18) is given by

$$z(x,y) = e^{-\frac{A}{2}x-\frac{c}{2}y}\left[d_1\cos\sqrt{B-\frac{A^2}{4}}\ x+a_1\sin\sqrt{B-\frac{A^2}{4}}\ x\right]\left[d_2\cos\sqrt{D-\frac{c^2}{4}}y+a_2\sin\sqrt{D-\frac{c^2}{4}}\ y\right],$$

where $d_1, d_2, a_1\ and\ a_2$ are arbitrary constant.

Or ii) If $B \neq \frac{A^2}{4}\ and\ D = \frac{c^2}{4}$, we get

$u(x) = b\tan(f-bx)-\frac{A}{2}\ ;b^2 = B-\frac{A^2}{4}$ and f is an arbitrary constant.

Also $v(y) = \frac{1}{y-c_1}-\frac{c}{2}\ ;\ c_1$ is an arbitrary constant.

Therefore by substituting in equation (21) we get

$$z(x,y) = e^{\int(b\tan(f-bx)-\frac{A}{2})dx+\int(\frac{1}{y-c_1}-\frac{c}{2})dy}$$

So, the complete solution of equation (18) is given by

$$z(x,y)=e^{-\frac{1}{2}(Ax+cy)}\,(y-c_1)(d_1\cos\sqrt{B-\frac{A^2}{4}}x+b_1\sin\sqrt{B-\frac{A^2}{4}}x)$$

Where d_1, b_1 *and* c_1 are arbitrary constant.

Or iii) if $B=\frac{A^2}{4}$ *and* $D\neq\frac{c^2}{4}$, we get

$u(x)=\frac{1}{x-c_2}-\frac{A}{2}$

and $v(y)=b\tan(f-by)-\frac{c}{2}$; $b^2=D-\frac{c^2}{4}$,

and f is an arbitrary constant.

Therefore by substituting u(x) and v(y) in equation (21) we get

$$z(x,y)=e^{\int(\frac{1}{x-c_2}-\frac{A}{2})dx+\int(b\tan(f-by)-\frac{c}{2})dy},$$

hence the complete solution of equation (18) is given by

$$z(x,y)=e^{-\frac{1}{2}(Ax+cy)}\,(x-c_2)(d_1\cos\sqrt{D-\frac{c^2}{4}}y+b_1\sin\sqrt{D-\frac{c^2}{4}}y)$$

Where d_1, b_1 *and* c_2 are arbitrary constant.

Or iv) if $B=\frac{A^2}{4}$ *and* $D=\frac{c^2}{4}$, we get

$u(x)=\frac{1}{x-c_1}-\frac{A}{2}$ and $v(y)=\frac{1}{y-c_2}-\frac{c}{2}$

therefore by substituting u(x) and v(y) in equation (21) we get

$$z(x,y)=e^{\int(\frac{1}{x-c_1}-\frac{A}{2})dx+\int(\frac{1}{y-c_2}-\frac{c}{2})dy},$$

so, the complete solution of equation (18) is given by

$$z(x,y)=K\,e^{-\frac{1}{2}(Ax+cy)}\,(x-c_1)(y-c_2)$$

where K, c_1 *and* c_2 are arbitrary constants.

Note (4) :- if we write

$$A_1(u'(x)+u^2(x))+A_4u(x)+A_6=-A_3(v'(y)+v^2(y))-A_5v(y)=-\lambda^2,$$

then the complete solution can established by the same way as in above cases , but

$A=\frac{A_4}{A_1}, B=\frac{A_6+\lambda^2}{A_1}, c=\frac{A_5}{A_3}$ *and* $D=\frac{-\lambda^2}{A_3}$

c) since $A_1=0$, so equation (22) becomes

$$A_2u(x)v(y)+A_3(v'(y)+v^2(y))+A_4u(x)+A_5v(y)+A_6=0,$$

Then by the same way as in case (a), we get

$u(x)=\lambda^2$ *and* $v'(y)+v^2(y)+Av(y)+B=0$,

where $A=\frac{A_5+A_2\lambda^2}{A_3}$ *and* $B=\frac{A_6+A_4\lambda^2}{A_3}$. Now

i) if $B \neq \frac{A^2}{4}$, we get

$$v(y) = b\tan(f - by) - \frac{A}{2}\ ;\ b = B - \frac{A^2}{4}$$

and f is an arbitrary constant.
Hence by substituting u(x) and v(y) in equation (21) we get

$$z(x,y) = e^{\int \lambda^2 dx + \int (b\tan(f-by) - \frac{A}{2})dy},$$

so the complete solution of equation (19) is given by:-

$$z(x,y) = e^{\lambda^2 x - \frac{A}{2}y}\,(d_1\cos\sqrt{B - \frac{A^2}{4}}\,y + b_1\sin\sqrt{B - \frac{A^2}{4}}\,y).,$$

Where $d_1, b_1\ and\ \lambda$ are arbitrary constants.
ii) if $B = \frac{A^2}{4}$, then

$$v(y) = \frac{1}{y-c} - \frac{A}{2}$$

Hence by substituting u(x) and v(y) in equation (21) we get

$$z(x,y) = e^{\int \lambda^2 x\,dx + \int (\frac{1}{y-c} - \frac{A}{2})dy},$$

so, the complete solution of equation (19) is given by

$$z(x,y) = De^{\lambda^2 x - \frac{A}{2}y}\,(y-c),$$

where D ,c and λ are arbitrary constants .
Example 3 :- To solve the partial differential equation

$$z_{xx} - z_{yy} + 3z_x + 2z_y + z = 0,$$

we note that $A_2 = 0$, so by using the formula which appears in case (b-i) (because $B \neq \frac{A^2}{4}\ \ and\ D \neq \frac{c^2}{4}$) we get the complete solution which is form

$$z(x,y) = e^{-\frac{3}{2}x + y}\,(d_1\cos\sqrt{\lambda^2 - \frac{9}{4}}\ x + a_1\sin\sqrt{\lambda^2 - \frac{9}{4}}\ x)(d_2\cos\sqrt{\lambda^2 - 2}\,y + a_2\sin\sqrt{\lambda^2 - 2}\ y),$$

where as $d_1, d_2, a_1, a_2\ and\ \lambda$ and arbitrary constants.
5) If no any one of the coefficients equal to zero, then the partial differential equation is given by

$$A_1 z_{xx} + A_2 z_{xy} + A_3 z_{yy} + A_4 z_x + A_5 z_y + A_6 z = 0,$$

and equation (22) stay at the same formula which is

$$A_1(u(x) + u^2(x)) + A_2 u(x)v(y) + A_3(v'(y) + v^2(y)) + A_4 u(x) + A_5 v(y) + A_6 = 0,$$

so, the complete solution of equation (20) is given by
either i) $z(x,y) = e^{-\frac{A}{2}x + \lambda y}\,(d_1\cos\sqrt{B - \frac{A^2}{4}}x + b_1\sin\sqrt{B - \frac{A^2}{4}}x)$

if $B \neq \frac{A^2}{4}$; $A = \frac{A_2\lambda + A_4}{A_1}$ *and* $B = \frac{A_3\lambda^2 + A_5\lambda + A_6}{A_1}$

Where d_1, b_1 *and* λ are arbitrary constants.

Or ii) $z(x,y) = De^{-\frac{A}{2}x+\lambda y}\ (x-c)$

If $B = \frac{A^2}{4}$; $A = \frac{A_2\lambda + A_4}{A_1}$ *and* $B = \frac{A_3\lambda^2 + A_5\lambda + A_6}{A_1}$

where D,c and λ are arbitrary constants .

proof: since no any one of coefficients are equal to zero so ,

$$A_1(u'(x) + u^2(x)) + A_2 u(x)v(y) + A_3(v'(y) + v^2(y)) + A_4 u(x) + A_5 v(y) + A_6 = 0$$

Here the last equation, we can't separate the variables, so, to solve this problem we consider that $v(y) = \lambda$ therefore

$$u'(x) + u^2(x) + Au(x) + B = 0$$

where as $A = \frac{A_2\lambda + A_4}{A_1}$ *and* $B = \frac{A_3\lambda^2 + A_5\lambda + A_6}{A_1}$,

the last equation can be solved by using variable separable method and we get :-

i) if $B \neq \frac{A^2}{4}$, then

$$u(x) = b\ \tan(f - bx) - \frac{A}{2}\ ;\ b^2 = B - \frac{A^2}{4}\ ,$$

and f is an arbitrary constant, therefore by using equation (21) we get

$$z(x,y) = e^{\int (b\tan(f-bx) - \frac{A}{2}\ ;\ B - \frac{A^2}{4}}$$

So, the complete solution of equation (20) is given by

$$z(x,y) = e^{\lambda y - \frac{A}{2}x}\ (d_1 \cos\sqrt{B - \frac{A^2}{4}}x + b_1 \sin\sqrt{B - \frac{A^2}{4}}\ x)$$

Where d_1, b_1 *and* λ are arbitrary constants.

ii) if $B = \frac{A^2}{4}$, then

$$u(x) = \frac{1}{x-c} - \frac{A}{2}$$

therefore by substituting u(x) and v(y) in equation (21) we note that

$$z(x,y) = e^{\int (\frac{1}{x-c} - \frac{A}{2})dx + \int \lambda\, dy},$$

so, the complete solution to equation (20) is given by

$$z(x,y) = De^{-\frac{A}{2}x + \lambda y}\ (x-c)$$

where D,c and λ are arbitrary constants .

Example 4 :- To solve the partial differential equation

$$z_{xx}+z_{xy}-\frac{1}{2}z_{yy}+z_x+\frac{1}{2}z_y+\frac{1}{4}z=0,$$

we note that $B\neq\frac{A^2}{4}$, so by using the formula in case (5-i) we can get the complete solution , which is form is given by :-

$$z(x,y)=e^{-\frac{(1+\lambda)}{2}x+\lambda y}\ (d_1\cos\frac{\sqrt{3}}{2}i\,\lambda x+b_1\sin\frac{\sqrt{3}}{2}i\,\lambda x)$$

$$=e^{-\frac{(1+\lambda)}{2}x+\lambda y}\ (c_1\ e^{-\frac{\sqrt{3}}{2}\lambda x}+c_2\ e^{\frac{\sqrt{3}}{2}\lambda x})\ ,$$

where $c_1=\frac{1}{2}(d_1-ib_1),c_2=\frac{1}{2}(d_1+ib_1)\ \textit{and}\ \lambda$ are arbitrary constants.

Example 5 :- To solve the partial differential equation

$$z_{xx}+z_{xy}+\frac{1}{4}z_{yy}+2z_x+z_y+z=0,$$

we note that $B=\frac{A^2}{4}$, so by using the formula in case (5-ii) we can get the complete solution , which is form

$$z(x,y)=De^{\lambda y-(1+\frac{1}{2}\lambda)x}\ (x-c),$$

where D ,c and λ are arbitrary constants .

Example 6: to solve the partial differential equation $4z_{xx}=z_y$ (parabolic equation) which describe heat flow and diffusion processes under the boundary conditions $z(0,y)=z(\pi,y)=0$ and the initial condition $z(x,0)=3\sin 2x$ we will apply the formula which is given in kind (3-a) and hence

$$z(x,y)=e^{-\lambda^2 y}\ (d_1\ \cos\frac{1}{2}\lambda x+b_1\sin\frac{1}{2}\lambda x)$$

Where $\lambda,d_1\ \textit{and}\ b_1$ are constants.

Since $z(0,y)=0\ \Rightarrow e^{-\lambda^2 y}\ d_1=0\Rightarrow d_1=0$

So $z(x,y)=e^{-\lambda^2 y}\ b_1\ \sin\frac{1}{2}\lambda x$

Since $z(\pi,y)=0\Rightarrow b_1\ e^{-\lambda^2 y}\ \sin\frac{\pi}{2}\lambda=0$

$\Rightarrow\ \frac{\pi}{2}\lambda=n\pi\ ;\ n=0,1,2,...$

$\Rightarrow\lambda=2n\Rightarrow z(x,y)=b_1\,e^{-4n^2 y}\ \sin nx\quad ,and\sin ce\qquad z(x,0)=3\sin 2x$

$\Rightarrow b_1=3,n=2$

Therefore $z(x,y)=3e^{-16y}\ \sin 2x$

Example 7 : To solve the partial differential equation $4z_{xx}-z_{yy}=0$ (hyperbolic equation) which describe vibrating systems and wave motion under the boundary conditions $z(0,y)=z(5,y)=0$ and the initial conditions $z(x,0)=0$, $z_y(x,0)=3\sin 2\pi x$

We will apply the formula which is given in kind (2-b) and hence

$$z(x,y)=(d_1\cos\frac{1}{2}\lambda x+d_2\ \sin\frac{1}{2}\lambda x)(b_1\cosh i\lambda y+b_2\sinh i\lambda y)$$

$$\Rightarrow z(x,y)=(d_1\cos\frac{1}{2}\lambda x+d_2\sin\frac{1}{2}\lambda x)(b_1\cos\lambda y+a_1\sin\lambda y)$$

; $a_1=ib_2$

Since $z(0,y)=0\Rightarrow d_1(b_1\cos\lambda y+a_1\sin\lambda y)=0$

$$\Rightarrow d_1=0\ ,\ so\ z(x,y)=\sin\frac{1}{2}\lambda x(c_1\cos\lambda y+c_2\sin\lambda y)$$

Where $c_1=d_2b_1$ and $c_2=d_2\,a_1$

Since $z(5,y)=0\Rightarrow\sin\frac{5}{2}\lambda\ (c_1\cos\lambda y+c_2\sin\lambda y)=0$

$$\Rightarrow\frac{5}{2}\lambda=n\pi\,;\ n=0,1,2,...$$

$$\Rightarrow\lambda=\frac{2n\pi}{5}\Rightarrow$$

$$z(x,y)=\sin\frac{n\pi}{5}x(c_1\cos\frac{2n\pi}{5}y+c_2\ \sin\frac{2n\pi}{5}y)$$

Since $z(x,0)=0\Rightarrow c_1\sin\frac{n\pi}{5}x=0$

$$\Rightarrow c_1=0\Rightarrow$$

$$z(x,y)=c_2\sin\frac{4\pi}{5}x\ \sin\frac{2n\pi}{5}y\Rightarrow$$

$$z_y=\frac{2n\pi c_2}{5}\ \sin\ \frac{n\pi}{5}\ x os\frac{2n\pi}{5}y$$

Since $z_y(x,0)=3\sin 2\pi x$

$$\Rightarrow\frac{2n\pi}{5}c_2\ \sin\frac{n\pi}{5}x=3\sin 2\pi x$$

$$\Rightarrow\frac{n\pi}{5}=2\pi\Rightarrow n=10$$

$$and\ \frac{2n\pi}{5}c_2=3\Rightarrow c_2=\frac{3}{4\pi}$$

$$\Rightarrow z(x,y)=\frac{3}{4\pi}\sin 2\pi x\sin 4\pi y$$

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