مجلة القادسية للعلوم الصرفة المجلد (11) العدد (2) لسنة 2006

Solving Free Boundary Problem in Hydrodynamic Lubrication by Variational Methods

Adnan Yassean Na'ma
Dept. of Mathematics , College of Education , University of Thi-Qar .

Abstract

In this Paper , we will find variational formulation free boundary problem in hydrodynamic lubrication infinite Journal Bearing , after that we solve the direct problem to find pressure function ***p(t)*** numerically by variational methods .

1. Introduction: A Jornal Bearing consists of a rotating cylinder which is separated from a " bearing surface " by a thin film of lubricating fluid (see fig. 1) . The fluid is fed in at **A** and flows out at **B** . The width of the film is smallest at **C** , and we set $t = \theta / \theta_C$ where θ is as show in fig.1 .

Between **C** and **B** , the width of the film increases so that the pressure in the lubricating fluid may be expected to decrease . We assume that for $t=\tau$ the pressure becomes slow that the fluid vapofrizes . The point $t=\tau$, the interfase between the two phases of the fluid , is called the free boundary . Colin W. Cryer [1] used Finite difference method to solve this problem by interest from method of Christopherson . Also Giovawni Cimatti [2] trying to fined variational formulation of lubrication problemand fined solve this problem in tectangular region . We might of studies the problem of lubrication in [3] and solution this problem in rectangular region .In this paper we try to solution this problem in one dimention .

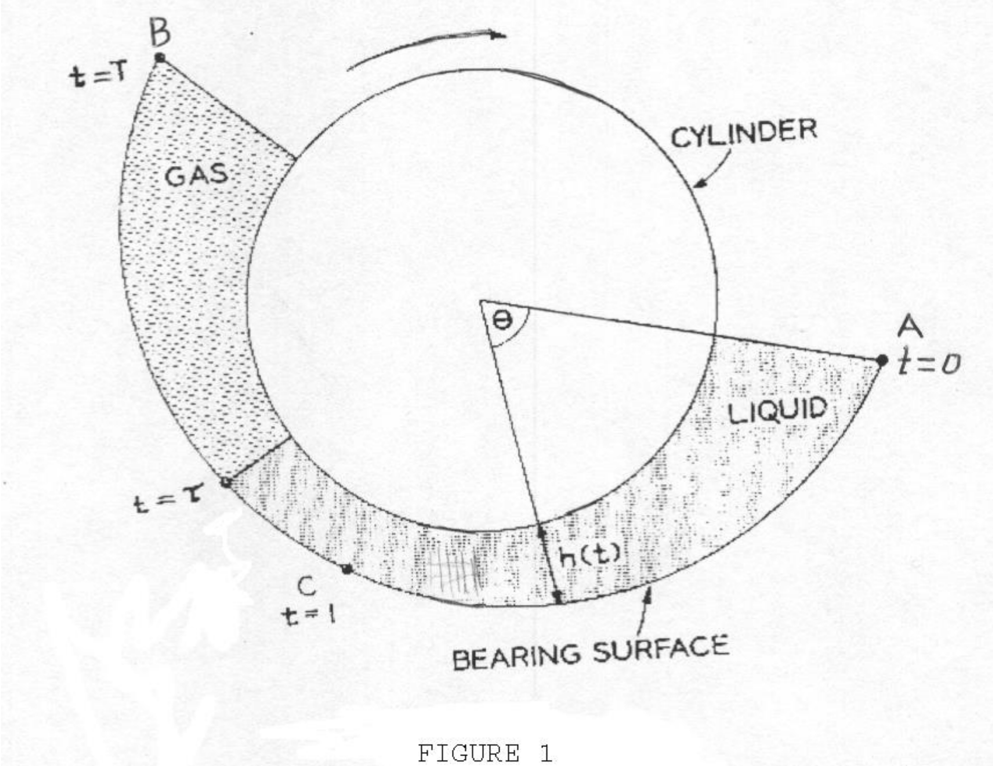

FIGURE 1

2. Mathematical formulation : The mathematical problem can now formulated by Pinkus and Sternlicht (see [1]) :

Problem 1. Find a function ***p(t)*** and a constant τ such that $p \in C[0,T] \cap C^{(2)}(0, \tau)$, and

$$\delta p(t) = \frac{d}{dt}[h^3(t)\frac{dp}{dt}] - \frac{dh}{dt} = 0 \qquad 0 < t < \tau \qquad \text{------(1.1)}$$

$$p(t) = 0 \ , \quad \tau \le t \le T \qquad \text{------(1.2)}$$

$$p(0) = 0 \ , \qquad \text{------(1.3)}$$

$$\frac{d}{dt}p(\tau(= 0 \qquad \text{------(1.4)}$$

where ***p(t)*** is proportional to the fluid pressure , while Eq.(1.1) is Reynold's equation for the pressure in a lubricating film . In order to define the problem 1 , we must to know that the width of the film ***h(t)*** satisfy certain conditions : $h \in C^{(1)}[0,T]$ and that

$$h(t) > 0 \ , \ t \in [0,T] \qquad \text{--------(1.5)}$$

$$\frac{dh}{dt} < 0 \ , \quad t \in (0,1) \ , \qquad \text{--------(1.6)}$$

$$\frac{dh}{dt} > 0 \ , \quad t \in (1,T)$$

$$h(T) \ge h(0) \qquad \text{--------(1.7)}$$

The conditions (1.5)-(1.7) ensure that there exists a unique solution to problem 1[1] .

Now ,we give some of important definitions :

Definition 2.1[4]: A set ***A*** is said to be convex if for any $u,v \in A$, $(1-\alpha)u+\alpha v \in A$, where $\alpha \in (0,1)$.

Definition 2.2 [4]: A functional ***F*** is called convex function if ***F*** defines on the convex set ***A*** .

Definition 2.3 [3]: A bilinear form , defined on a given two linear spaces ***U*** and ***V*** is a functional ***G(u,v)*** , which linear in both ***u*** and ***v*** , where u and v are elements of ***U*** and ***V*** respectively .

Definition 2.4 [3]: Let ***G(.,.)*** be a bilinear form , then ***G(.,.)*** is said to be symmetric if ***G(u,v)=G(v,u)*** for all $u \in U$ and $v \in V$.

Definition 2.5 [5]: The bilinear form $G(.,.): V x V \rightarrow R$ is called ***V***-elliptic if $G(v,v) \ge \alpha \|v\|^2$, $\alpha \ge 0$ and for all $v \in V$.

Definition 2.6 [5]: The bilinear form ***G(.,.)*** is bounded if $G(u,v) \le \mu \|u\| \|v\|$ for all $u \in U$, $v \in V$,where $\mu \in R$.

3.Variational inequalities formulation for boundary value problem

A variational inequality is usually posed with respect to a Hilbert space ***V*** with dual space V^* , a non-empty closed convex set ***K*** in ***V*** , a bilinear form ***a(.,.)*** on $V \times V$ and an element $f \in V^*$ [2] . The basic idea of the subject of variational inequality is to find $u \in K$, such that $a(u,v-u) \geq (f,v-u)$, for all $v \in K$.

Lions and Stampacchia in 1967 [3] introduced one of the foundamental theorems in this field , which is :

Theorem 2.1 [3]: If ***(.,.)*** is ***V***-elliptic , then variational inequality $a(u,v-u) \geq (f,v-u)$ for every $v \in K$ has a unique solution .

Now , we can find the functional ***J*** , which represent the variational formulation the problem ***L(u)=f*** by depending on the next theorem , when ***L*** be a linear operator .

Theorem 2.2 [4]: Assume ***K*** is a non-empty closed subset of the Hilbert sbace ***V*** ,$a(.,.): V \times V \to R$ is a bilinear , symmetric , bounded and ***V***-elliptic , $f \in V^*$, let :

$$J(v) = \frac{1}{2} a(v,v) - (f,v) \, , \, \mathrm{v} \in \mathrm{V}$$

then there exists a unique $u \in K$, such that : $J(v) = \inf_{v \in K} J(v)$

which is also the unique solution of the variational inequalities :

$$u \in K \, , \mathrm{a(u, v - u)} \geq \mathrm{(f, v - u)} \, , \mathrm{for \ \ each \ v} \ \in \mathrm{K} \ \ \mathrm{or} \ \ \mathrm{u} \in \mathrm{K} \, , \mathrm{a(u, v)} = \mathrm{(f, v)} \ \mathrm{for \ \ each \ v} \ \in \mathrm{K}$$

Now , we can use the theorems in above to find a variational formulation of the problem 1 .

4.The variational formulation of the problem 1 .

The solution of the mathematical model , which is defined by eqs (1.1)-(1.4) depending a function $p(t) \in C[0,T] \cap C^{(2)}(0,\tau)$.

The cavitation conditions (1.2)-(1.4) ensure non-negative pressure and the conditions (1.5)-(1.6) on width of the film ***h(t)*** , all these conditions makes ***p(t)*** satisfies the inequalities :

$$\left. \begin{aligned} & -\frac{d}{dt}[h^3(t)\frac{dp}{dt}] + \frac{dh}{dt} \geq 0 \\ & p\{-\frac{d}{dt}[h^3(t)\frac{dp}{dt}] + \frac{dh}{dt}\} = 0 \end{aligned} \right\} \text{-------------------(4.1)}$$

in the domain $\mathbf{C[0,T] \cap C^{(2)}(0,\tau)}$.

To find p(t) such that the equations (1.1) and (4.1) are satisfies . Consider $C^{(2)}[0,T]$ and ***H[0,T]*** with inner product

$$(u,v)=\int_0^T uvdt \quad \text{and} \quad a(u,v)=\int_0^T h^3(u_t v_t)dt$$

Where the associated norms an : $|u|^2=(u,u)$ and $\|u\|^2=a(u,u)$

Denoted by $H^*[0,T]$ the subspace of $H[0,T]$, whose elements v satisfy conditions (1.1) to (1.4) . Let K be a closed convex set of $H^*[0,T]$,defined by : $K=\{v\in H^*[0,T]: v\geq 0$ in $[0,T]\}$

If $p(t)$ is a solution of (1.1) and (4.1) , then $p\in K$ for every $v\in K$, we have :

$$a(p,v-p)=\int_0^T h^3\{p_t(v-p)_t\}dt$$

We can easily prove that ***a(.,.)*** is a bilinear , symmetric , bounded and elliptic form . Also we get :

$$a(p,v-p)\geq(f,v-p), \text{for every } v\in K \text{ where } f=\frac{dh}{dt} \quad \text{-------(4.2)}$$

Thus. , by Lions-Stampacchia theorem , (4.2) has unique solution .

Consider the functional : $J(p)=\frac{1}{2}a(p,p)-(f,p)$ ------(4.3)

The equivalent minimization statement is $J(p)\leq J(v)$, for all $v\in K$, by using theorem (2.2), (2.1) and (4.3) has a unique solution , which is the same solution of (4.2) ,

thus : $J(p)=\frac{1}{2}\int_0^T\{h^3\frac{d}{dt^2p}-\frac{dh}{dt}p\}dt$ ----------(4.4)

Could be regarded as the variational formulation of infinite Journal Bearing problem .

5. Numerical Solution of the Problem

The variational formulation (4.4) with the boundary conditions (1.2)-(1.4) is so difficalt to solve analytically using principle or other analytical methods , therefore direct methods are required to solve this problem . To find the solution p which is the critical point of functional (4.4) , the Rits method [6] will used . The procedure utilized here , can be described in below .

Let the solution $p_a=u$ is approximate by a linear combination of elements of a complete sequence of functions $\{\varphi(t)\}$ defined over the interval ***[0,T]*** and each function satisfying the before conditions . In other words : $u(t)=\sum_{i=1}^n a_i\varphi_i(t)$ --------------(5.1) ,

where a_1 ,a_2 , … ,a_n cofficients to determined .These functions can be defined as follows:

$$u(t)=t(\tau-t)^2[a_0+a_1t+a_2t^2+a_3t^3] \quad \text{---------------(5.2)}$$

Substituting then above function ***u(t)*** in the functional ***J*** defined by Eq. (4.4) , and after carrying out the integration using Gaussian quadrate integration method [4] of degree 7 , the functional ***J*** becomes a function of the real variables a_0 , a_1 , a_2 and a_3 .

Thus , the variational formulation is equivalent to the unconstrainted non-linear minimizing problem of the functional $J(a_0 , a_1 , a_2 , a_3)$.

Hook and Jeeves minimization method [6] have been successfully used to solve the problem , after assuming that the film thickness :

$h(t)=1+0.01\cos(t)$ [1]

where ***0.01*** equals to eccentricity ration . The approximate values of the coefficients a_0 , a_1 , a_2 and a_3 in approximate solution Eq. (5.2) are tabulated below :

$a_0=0.4$, $a_1=0.7$, $a_2=1.69$ and $a_3=0.9$.

6. References

1. Colin W. Cryer , 1971 ,” The Method of Christopherson for Solving Free Boundary Problems for Infinite Journal Bearings by Means Of Finite Differences ,” Mathematics of Computation,Vol.25,No.115,PP.435-443.
2. Adnan Yaseen Na’ma ,”Application of Nonlinear Inverse Problems in the Finite Journal Bearing Problem , 2002 , ” Sc. M. Thesis ,College of Education/Ibn Al-Haitham , University of Baghdad .
3. Lions, J. L., and Stampacchia, G., 1967, “Variational Inequalities”, Communications on Pure and Applied Mathematics, Vol.xx, pp.493-519
4. Atkinson, K. and Han, W., 2001 ,” Theorical Numerical Analysis “,Springer , New York .
5. Boas, M. L., 1983, “Mathematical Methods in the Physical Sciences”, 2nd Edition, John Wiley and Sons, Inc., U.S.A.
6. Bunday, B. D., 1984, “Basic Optimization Methods”, Edward Arnold, London.