On Projective Plane Over A Finite Field of Order Seventeen and its Application to Error-Correcting Codes

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Abstract

The aim of the paper is to classify certain geometric structures, called arcs. The main computing tool is the mathematical programming language GAP. In the plane $PG(2,17)$, the important arcs are called complete and are those that cannot be increased to a larger arc. So far, all arcs up to and including size eight have been classified, as have complete 10-arcs, 11-arcs, 12-arcs, 13-arcs and 14-arcs. In the plane of order seventeen, the maximum size is eighteen. Each of these arcs gives rise to an error-correcting code that corrects the maximum possible number of errors for its length.

Key words: Arcs, Projective plane, codes, orbits, stabilizer groups.

1. INTRODUCTION:

A projective plane is an incidence structure of points and lines with the following properties.

- Every two points are incident with a unique line.
- Every two lines are incident with a unique point.
- There are four points, no three collinear.

A *Desarguesian* projective plane $PG(2,q)$ has as points one-dimensional subspaces and as lines two-dimensional subspaces of a three-dimensional vector space over the finite field $\mathbb{F}_q$ of q elements. A k-arc in $PG(2,q)$ is a set of k points no three of which are collinear. A k-arc is *complete* if it is not contained in a $(k+1)$-arc. A $(k;3)$-arc in $PG(2,q)$ is a set of k points in which no four points but some three points are collinear.

The main aims of this paper is to classify arcs of all sizes in projective plane $PG(2,q)$, and classify those arcs which are contained in a *conic*, each of these arcs gives rise to an error-correcting code that corrects the maximum possible number of error for its length.

Arcs in $PG(2, q)$ for $q = 2, 3, 4, 5, 7, 8, 9, 11, 13$ have been classified; (1).

We are looking at the plane of order seventeen, as it is the next in the sequence. A brief history, associated to any topic in mathematics is its history. Arcs were first introduced by (2) in connection with designs in statistics. Further development began with (3); he showed that every $(q + 1)$-arc in $PG(2, q)$ is a conic. An important result is that of Ball, Blokhuis and Mazzocca showing that maximal arcs cannot exist in a plane of odd order. (4) found important applications of curves over finite fields to coding theory. As geometry over a finite field, it has been thoroughly studied (5).

An $(n, M, d)_q$ code C is a set of M words, each with n symbols taken from an alphabet of size q, such that any two words differ in at least d places. A code $(n, M, d)_q$ has the following desirable properties:

- Small n: fast transmission;
- Large M: many messages;
- Large d: correct many errors.

If the code is linear, it can more easily be used for encoding and decoding; in this case, $M = q^k$ for some positive integer k, the dimension of the code, and C is called an $[n, k, d]_q$ code. The main Coding Theory problem is to find codes optimizing one parameter with the other two fixed. Mathematically, such a code can also be viewed as a set of n points in $PG(k - 1, q)$ with at most $n - d$ points in a subspace of dimension $k - 2$ for more details see (1), (6), (7), (8), (9).

2. PREVIOUS RESULTS

Definition(2.1): The set denoted by $\mathbb{F}_P$, with P prime, consists of the residue classes of the integers modulo P under the natural addition and multiplication.

Definition(2.2): Given a homogenous polynomial F in three variables x_0, x_1, x_2 over $\mathbb{F}_q$, a *curve* $\mathcal{F}$ is the set $\mathcal{F} = v(F) = \{P(X): F(X) = 0\}$

Where $P(X)$ is the point of $PG(2, q)$ represented by $X = (x_0, x_1, x_2)$.

If F has degree three, that is,

$$F = a_0 {x_0}^2 + a_1 {x_1}^2 + a_2 {x_2}^2 + b_2 x_0 x_1 + b_1 x_0 x_2 + b_0 x_1 x_2,$$

Then $\mathcal{F}$ is called a *quadric.* For q odd, the discriminant of a quadric $\mathcal{F}$ is the determinant $D = \begin{vmatrix} 2a_0 & b_2 & b_1 \\ b_2 & 2a_1 & b_0 \\ b_1 & b_0 & 2a_2 \end{vmatrix}$ A quadric $\mathcal{F}$ is non-singular if its discriminant D is non-zero.

Definition(2.3): A *conic* $\boldsymbol{C}$ is a non-singular quadric $\mathcal{F}$.

Definition(2.4): An $[n, k, d]_q$ code C is a subspace of $V(n, q) = (\mathbb{F}_q)^n$, where the dimension of C is dim $C = k$, and the minimum distance is $d(C) = d =$ min $d(x, y)$.

Definition(2.5): For any $[n, k, d]_q$ code we have $d \leq n - k + 1$.

Definition(2.6): Let λ be a root of f which irreducible polynomial . Then f is *primitive* if the smallest power s of λ such that $\lambda^s = 1$ is $s = q^n - 1$. It is *subprimitive* if the smallest power s of λ such that $\lambda^s \in \mathbb{F}_q$ is $s = q^{n-1} + \cdots + q + 1$.

Definition(2.7): Denote by S and S^* two subspaces of $PG(n, K)$, A *projectivity* $\beta: S \rightarrow S^*$ is a bijection given by a matrix T, necessarily non-singular, where $P(X^*) = P(X)\beta$ if $tX^* = XT$, with $t \in K$. Write $\beta = M(T)$; then $\beta = M(\lambda T)$ for any λ in K. The group of projectivities of $PG(n, K)$ is denoted by $PGL(n + 1, K)$.

Definition(2.8): A group G acts on a set Λ if there is a map $\Lambda \times G \rightarrow \Lambda$ such that given g, g′ elements in G and 1 its identity, then

a. $x1 = x$,
b. $(xg)g' = x(gg')$ for any x in Λ.

Definition(2.9): The orbit of x in Λ under the action of G is the set $x\mathrm{G} = \{x\mathrm{g} | \mathrm{g} \in \mathrm{G}\}$.

Definition(2.10): The stabilizer of x in Λ under the action of G is the group

$$\mathrm{G}_x = \{g \in G | xg = x\}.$$

Definition(2.11): Let K be a k -arc and P a point of $PG(2, q)\backslash K$. Then if exactly i bisecants of K pass through P, then P is said to be a point of index i. The number of these points is denoted by c_i.

Lemma(2.12): The constants c_i of a k – arc K in $PG(2, q)$ satisfy the following equations with the summation taken 0 to n for which $c_i \neq 0$:

$$\sum c_i = q^2 + q + 1 - k, \quad \ldots \quad 1$$

$$\sum i c_i = k(k - 1)(q - 1)/2, \quad \ldots \quad 2$$

$$\sum i(i - 1)c_i/2 = k(k - 1)(k - 2)(k - 3)/8. \quad \ldots \quad 3$$

3. Results and Applications

3.1 The algorithm to calculate k-arcs, $k \leq 18$

The strategy to compute the complete k-arcs is the following.

- The way of calculating c_0 for $(k - 1)$-arcs is by listing the points not on the bisecants of the $(k - 1)$-arcs.
- The points represented by the number c_0 are separated into orbits.
- The k-arcs are constructed by adding one point from each orbit.
- For a given k-arc K, the set S of points not on the bisecants of K is found.

- If S is empty, then K is complete. Otherwise K is incomplete.
- All possible k-arcs from a given $(k-1)$-arcs are listed.
- The next step is to select the non-identical complete k-arcs among the total number constructed.
- Calculate the transformations between them. By use of The Fundamental Theorem of Projective Geometry, there is a unique projectivity of $PG(2,q)$ transforming four points no three on a line to any other four points no three on a line. Two k-arcs K_1 and K_2 are equivalent if $K_1\beta = K_2$ and β is given by a matrix T and $\beta = M(T)$ with $M(\lambda T) = M(T)$, $\lambda \in \mathbb{F}_{17}\{0\}$.

3.2 Preliminary to $PG(2,17)$

In $PG(2,q)$, the projective plane of order 17, $\theta_1 = 18$, $\theta_2 = 307$, where

$$\theta_n = |PG(n,q)| = (q^{n+1}-1)/(q-1);$$

Hence we have 307 points, 307 lines, 18 points on each line and 18 lines passing through each point.

Let $P_0 = (1,0,0)$, and $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 14 & 1 & 0 \end{pmatrix}$ be a non-singular matrix such that the points of $PG(2,17)$ are generated as following. $P_i = P_0T^i$, $i = 0,\dots,306$, such $P_0 = (1,0,0)$, $P_1 = (0,1,0)$, $P_2 = (0,0,1), \dots, P_{306} = (16,0,0)$. We will write the points of $PG(2,17)$ in numeral forms as follows: $P_i = i, i = 0,\dots,306$, $P_{307} \sim P_0$.The lines of $PG(2,17)$ are as follows:

$\ell_1 = \{0,1,3,45,58,62,73,96,110,122,142,149,178,196,267,277,286,302\}$,

$\ell_2 = \{1,2,4,46,59,63,74,97,111,123,143,150,179,197,268,278,287,303\}$,

$$\vdots$$

$\ell_{307} = \{306,0,2,44,57,61,72,95,109,121,141,148,177,195,266,276,285,301\}$.

3.3 Stabilizer of the frame

The stabilizer of any 4-arc is the group of 24 projectivities found by shifting the 4-arc to its 24 permutations. The frame points in $PG(2,17)$ are 0,1,2,253. The two projectivities

$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 0 & 16 & 0 \\ 0 & 0 & 16 \\ 1 & 1 & 1 \end{pmatrix}$ which generate $\boldsymbol{S_4}$, the stabilizer of the frame, partitions the points in $PG(2,17)$ into 21 disjoint orbits.

3.4 The 5-arcs

Let K be a k-arc in $PG(2,q)$. For $k = 4$, the equations in Lemma (2.12) become

$$c_0 = (q-2)(q-3),$$
$$c_1 = 6(q-2),$$
$$c_2 = 3;$$

Another way to calculate c_0 is by listing the points not on the bisecants of the 4-arc. The points represented by the number c_0 are separated into orbits. Then 5-arcs are constructed by adding one point from each orbit. This gives the following result.

Theorem 1: In $PG(2,17)$ there are precisely four projectively distinct 5-arcs, given in Table 1.

Symbol	5-arc	Stabilizer
A_1	{0,1,2,253,6}	$\boldsymbol{Z_4}$
A_2	{0,1,2,253,7}	$\boldsymbol{Z_2}$
A_3	{0,1,2,253,9}	$\boldsymbol{Z_2}$
A_4	{0,1,2,253,11}	$\boldsymbol{Z_2}$

Table 1: The distinct 5-arcs

3.5 The 6-arcs

The number of the points on the bisecant of any 5-arc is $L(5,q) = 10q - 20$; that is, 150 for $q = 17$. Hence there are 307-150=157 points of the plane not on the bisecant of any of the four 5-arcs. Let K be a k-arc in $PG(2,q)$. For $k = 5$, the equations in Lemma (2.12) become

$$c_0 = (q-4)(q-5)+1,$$
$$c_1 = 10(q-4),$$
$$c_2 = 15;$$

Another way to calculate c_0 is by listing the points not on the bisecants of the 5-arc. The points represented by the number c_0 are separated into orbits. Then 6-arcs are constructed by adding one point from each orbit. For a specific 5-arc, points of index zero are divided into orbits by the stabilizer of that 5-arc. The points of index zero for every 5-arc as a number of orbits with the size of the orbits in brackets are given in Table 2.

5-arc	c_0	Orbits
A_1	157	36(4),6(2),1(1)
A_2	157	72(2),13(1)
A_3	157	72(2),13(1)
A_4	157	72(2),13(1)

Table 2: The orbits

So far, the number of 6-arcs constructed is 295. The method to compute the transformations between the 6-arcs is by use of The Fundamental Theorem of Projective Geometry. This gives the following result.

Theorem 2: In $PG(2,17)$ there are precisely 74 projectively distinct 6-arcs, the numbers of 6-arcs and their stabilizers are given in Table 3.

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{Z_3}$	$\boldsymbol{Z_4}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{S_3}$	$\boldsymbol{S_4}$	$\boldsymbol{D_6}$	$\boldsymbol{A_4}$
Number	32	16	9	3	2	7	1	1	3

Table 3: The stabilizers of 6-arcs

3.6 The 6-arcs on a conic

The ten distinct hexads (An unordered set of six points) on $PG(1,17)$ can be mapped to ten distinct 6-arcs on a conic. If the points $U_0 = (1,0,0), U_1 = (0,1,0), U_2 = (0,0,1)$ are on the conic, then the general equation of the conic reduces to the following:

$$x_0x_1 + a_0x_0x_2 + a_1x_1x_2 = 0$$

Therefore, $(a_0, a_0) = (-7,6), (-3,2), (-2,1), (-5,4)$ are the coefficients of the equations of the conic containing the respective four 5-arcs

$$\{U_0, U_1, U_2, U_3, U_4\}, \{U_0, U_1, U_2, U_3, U_5\}, \{U_0, U_1, U_2, U_3, U_6\}, \{U_0, U_1, U_2, U_3, U_7\},$$

Where

$$U_3 = (1,1,1), U_4 = (-8,-6,1), U_5 = (-8,4,1), U_6 = (-8,-5,1), U_7 = (-7,6,1).$$

Substituting the point of each 6-arc in the corresponding conic shows the ten 6-arcs on a conic as given in Table 4.

Symbol	Conic	6-arc	Stabilizer
B_1	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	{0,1,2,253,6,13}	$\boldsymbol{Z_2}$
B_2	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	{0,1,2,253,6,41}	$\boldsymbol{I}$
B_3	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	{0,1,2,253,6,84}	$\boldsymbol{Z_2}$
B_4	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	{0,1,2,253,6,269}	$\boldsymbol{S_4}$
B_5	$x_0x_1 - 3x_0x_2 + 2x_1x_2$	{0,1,2,253,7,23}	$\boldsymbol{Z_2}$
B_6	$x_0x_1 - 3x_0x_2 + 2x_1x_2$	{0,1,2,253,7,33}	$\boldsymbol{Z_2 \times Z_2}$
B_7	$x_0x_1 - 3x_0x_2 + 2x_1x_2$	{0,1,2,253,7,98}	$\boldsymbol{Z_2}$
B_8	$x_0x_1 - 3x_0x_2 + 2x_1x_2$	{0,1,2,253,7,153}	$\boldsymbol{Z_2 \times Z_2}$
B_9	$x_0x_1 - 2x_0x_2 + x_1x_2$	{0,1,2,253,9,235}	$\boldsymbol{D_6}$
B_{10}	$x_0x_1 - 5x_0x_2 + 4x_1x_2$	{0,1,2,253,11,182}	$\boldsymbol{S_3}$

Table 4: The distinct 6-arcs on a conic

3.7 The 7-arcs

Let K be a k-arc in $PG(2,q)$. For $k = 6$, the equations in Lemma (2.12) become

$$c_0 = (q-7)^2 + 6 - c_3,$$

$$c_1 = 3\{5(q-7) + c_3\},$$

$$c_2 = 3\{15 - c_3\};$$

The constant c_3 and hence c_0, c_1 and c_2 are calculated. Another way of calculating c_0 is by listing the points not on the bisecants of the 6-arc. The points represented by the number c_0 are separated into orbits. Then 7-arcs are constructed by adding one point from each orbit. This gives the following result.

Theorem 3: In $PG(2,17)$ there are precisely 733 projectively distinct 7-arcs, the numbers of 7-arcs and their stabilizers are given in Table 5.

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{S_3}$	$\boldsymbol{Z_3}$
Number	644	75	2	12

Table 5: The stabilizers of 7-arcs

3.8 The 7-arcs on a conic

The ten distinct heptads (An unordered set of seven points) on $PG(1,17)$ can be mapped to ten distinct 7-arcs on a conic.

Substituting the points of each 7-arc in the corresponding conic shows the ten 7-arcs on a conic as given in Table 6.

Symbol	Conic	7-arc	Stabilizer
C_1	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_1 \cup \{41\}$	$\boldsymbol{I}$
C_2	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_1 \cup \{84\}$	$\boldsymbol{Z_2}$
C_3	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_1 \cup \{152\}$	$\boldsymbol{Z_2}$
C_4	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_1 \cup \{167\}$	$\boldsymbol{Z_2}$
C_5	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_1 \cup \{175\}$	$\boldsymbol{I}$
C_6	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_2 \cup \{84\}$	$\boldsymbol{I}$
C_7	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_2 \cup \{167\}$	$\boldsymbol{Z_2}$
C_8	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_2 \cup \{175\}$	$\boldsymbol{Z_2}$
C_9	$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$B_2 \cup \{205\}$	$\boldsymbol{Z_2}$
C_{10}	$x_0x_1 - 3x_0x_2 + 2x_1x_2$	$B_5 \cup \{33\}$	$\boldsymbol{Z_2}$

Table 6: The distinct 7-arcs on a conic

3.9 The 8-arcs

Let K be a k-arc in $PG(2,q)$. For $k = 7$, the equations in Lemma (2.12) become

$$c_0 = (q-10)^2 + 20 - c_3,$$
$$c_1 = 3\{7(q-11) + c_3\},$$
$$c_2 = 3\{35 - c_3\};$$

The constant $c_3$and hence c_0, c_1 and $c_2$are calculated. Another way of calculating c_0 is by listing the points not on the bisecants of the 7-arc. The points represented by the number c_0 are separated into orbits. Then 8-arcs are constructed by adding one point from each orbit. This gives the following result.

Theorem 4: In $PG(2,17)$ there are precisely 5441 projectively distinct 8-arcs, the numbers of 8-arcs and their stabilizers are given in Table 7.

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{Z_4}$	$\boldsymbol{D_4}$	$\boldsymbol{D_8}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{Z_8 \rtimes Z_2}$
Number	5027	389	4	3	1	16	1

Table 7: The stabilizers of 8-arcs

3.10 The 8-arcs on a conic

The seventeen distinct octads (An unordered set of eight points) on $PG(1,17)$ can be mapped to seventeen distinct 8-arcs on a conic. The 8-arcs in $PG(2,17)$ on a conic are given in Table 8.

Conic	8-arc	Stabilizer
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{84\}$	$\boldsymbol{I}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{135\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{152\}$	$\boldsymbol{I}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{175\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{185\}$	$\boldsymbol{I}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{205\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_1 \cup \{269\}$	$\boldsymbol{Z_2 \times Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_2 \cup \{167\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_2 \cup \{175\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_3 \cup \{167\}$	$\boldsymbol{Z_2 \times Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_3 \cup \{175\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_4 \cup \{175\}$	$\boldsymbol{I}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_4 \cup \{185\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_5 \cup \{298\}$	$\boldsymbol{D_4}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_6 \cup \{135\}$	$\boldsymbol{Z_2}$
$x_0x_1 - 7x_0x_2 + 6x_1x_2$	$C_6 \cup \{175\}$	$\boldsymbol{Z_2 \times Z_2}$
$x_0x_1 - 3x_0x_2 + 2x_1x_2$	$C_{10} \cup \{240\}$	$\boldsymbol{D_8}$

Table 8: The distinct 8-arcs on a conic

3.11 The 9-arcs on a conic

The seventeen distinct nonads on $PG(1,17)$ can be mapped to seventeen distinct 9-arcs on a conic as given in Table 9. The 9-arcs all lie on the conic

$$\nu(x_0x_1 - 7x_0x_2 + 6x_1x_2).$$

9-arc	Stabilizer	9-arc	Stabilizer	9-arc	Stabilizer
$E_1 \cup \{135\}$	$\boldsymbol{Z_2}$	$E_1 \cup \{205\}$	$\boldsymbol{I}$	$E_3 \cup \{269\}$	$\boldsymbol{S_3}$
$E_1 \cup \{152\}$	$\boldsymbol{Z_3}$	$E_1 \cup \{269\}$	$\boldsymbol{I}$	$E_4 \cup \{187\}$	$\boldsymbol{Z_4}$
$E_1 \cup \{167\}$	$\boldsymbol{I}$	$E_1 \cup \{300\}$	$\boldsymbol{I}$	$E_5 \cup \{269\}$	$\boldsymbol{D_9}$
$E_1 \cup \{175\}$	$\boldsymbol{Z_2}$	$E_2 \cup \{187\}$	$\boldsymbol{I}$	$E_6 \cup \{175\}$	$\boldsymbol{Z_2}$
$E_1 \cup \{185\}$	$\boldsymbol{Z_2}$	$E_3 \cup \{175\}$	$\boldsymbol{I}$	$E_7 \cup \{300\}$	$\boldsymbol{Z_8}$
$E_1 \cup \{187\}$	$\boldsymbol{Z_2}$	$E_3 \cup \{187\}$	$\boldsymbol{Z_2}$		

Table 9: The distinct 9-arcs on a conic

Where,

$E_1 = C_1 \cup \{84\}$, $E_2 = C_1 \cup \{135\}$,
$E_3 = C_1 \cup \{152\}, E_4 = C_1 \cup \{185\}$,
$E_5 = C_2 \cup \{167\}, E_6 = C_3 \cup \{185\}$, $E_7 = C_4 \cup \{185\}$.

3.12 The complete k-arcs, $k \geq 10$

From section (3.1), we have the following results.

Theorem 5: The numbers of projectively distinct complete k-arcs in $PG(2,17)$ for $k \geq 10$ are given in Table 10.

k	10	11	12	13	14	15	16	17	18
Number	560	2644	553	8	1	–	–	–	1

Table 10: The numbers of the complete k-arcs

The numbers of the complete k-arcs, $k = 10,11,12,13,14$ and their stabilizers are given in Table 11,12,13,14 and 15.

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{A_4}$	$\boldsymbol{D_9}$	$\boldsymbol{Z_3}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{S_3}$	$\boldsymbol{Z_4}$	$\boldsymbol{Z_8 \rtimes Z_2}$	$\boldsymbol{Q_4}$	$\boldsymbol{S_4}$
Number	343	178	2	1	9	8	9	7	1	1	1

Table 11: The stabilizers of the complete 10-arcs

Let K_1 be the complete 10-arc with group isomorphic to $\boldsymbol{D_9}$ in Table 9. Then $G(K_1)$ is generated by g_1, g_2 where

$$g_1 = \begin{pmatrix} 0 & 0 & 16 \\ 0 & 11 & 0 \\ 15 & 0 & 0 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 13 & 13 & 13 \\ 6 & 13 & 12 \\ 6 & 15 & 13 \end{pmatrix}.$$

Then $G(K_1)$ has the following orbits on K_1: one orbit M_1 of size 9 and one orbit $M_2 = \{P\}$ of size 1. Then K_1 consists of M_1 on conic $\boldsymbol{C}$ and P not on $\boldsymbol{C}$. The number of the points on no bisecant of M_1 is $c_0 = 19$. So P is not unique and we can select it from any of these ten points not on $\boldsymbol{C}$.

Let K_2 be the complete 10-arc with group isomorphic to $\boldsymbol{Z_8} \rtimes \boldsymbol{Z_2}$ in Table 9. Then $G(K_2)$ is generated by g_1, g_2 where

$$g_1 = \begin{pmatrix} 0 & 0 & 1 \\ 15 & 0 & 0 \\ 13 & 3 & 4 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 8 & 0 & 0 \\ 14 & 5 & 16 \\ 12 & 12 & 12 \end{pmatrix}.$$

Then $G(K_2)$ has the following orbits on K_2: one orbit of size 8 and one orbit of size 2. The group $G(K_2)$ stabilizes a line containing the orbit of size two, and partitions the line into one orbit of size 8, two of size 4, and one orbit of size 2.

Let K_3 be the complete 10-arc with group isomorphic to $\boldsymbol{S_4}$ in Table 9. Then $G(K_3)$ is a generated by g_1, g_2 where

$$g_1 = \begin{pmatrix} 0 & 12 & 0 \\ 16 & 2 & 8 \\ 8 & 8 & 8 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 10 & 12 & 1 \end{pmatrix}.$$

Then $G(K_3)$ has the following orbits on K_3: one orbit of size 6 and one orbit of size 4.

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$
Number	2569	75

Table 12: The stabilizers of the complete 11-arcs

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{Z_3}$	$\boldsymbol{Z_2 \times Z_2}$	$\boldsymbol{Z_4}$	$\boldsymbol{S_3}$	$\boldsymbol{D_4}$	$\boldsymbol{D_6}$	$\boldsymbol{S_4}$
Number	337	152	17	18	1	20	2	3	3

Table 13: The stabilizers of the complete 12-arcs

Stabilizer	$\boldsymbol{I}$	$\boldsymbol{Z_2}$	$\boldsymbol{Z_3}$	$\boldsymbol{Z_4}$	$\boldsymbol{S_3}$
Number	1	4	1	1	1

Table 14: The stabilizers of the complete 13-arcs

Stabilizer	$\boldsymbol{D_4}$
Number	1

Table 15: The stabilizers of the complete 14-arcs

Let K_4 be the complete 14-arc with group isomorphic to $\boldsymbol{D_4}$ in Table 12. Then $G(K_4)$ is generated by g_1, g_2 where

$$g_1 = \begin{pmatrix} 0 & 6 & 0 \\ 3 & 0 & 0 \\ 2 & 12 & 16 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 11 & 1 & 15 \\ 12 & 12 & 12 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $G(K_4)$ has the following orbits on K_4: one orbit O_4 of size 8, one orbit O_5 of size 4 and one orbit O_1 of size 2. The group $G(K_4)$

stabilizes a line ℓ containing O_1on a conic $\boldsymbol{C}$, and partitions the line ℓ into three orbits of size 4 and three orbits O_1, O_2, O_3 of size 2. Then K_4 consists of ten points on $\boldsymbol{C}$, two of them on ℓ, and eight points in $O_4 = \{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$ on $\boldsymbol{C}$. Let Q_1, Q_2, Q_3, Q_4be the four points in O_5 not on $\boldsymbol{C}$. Let $O_1 = \{P_1, P_1{}'\}$, $O_2 = \{P_2, P_2{}'\}$,$O_3 = \{P_3, P_3{}'\}$ on ℓ, where

$$P_2 = Q_1Q_3 \cap \ell = Q_2Q_4 \cap \ell, P_2{}' = Q_1Q_4 \cap \ell = Q_2Q_3 \cap \ell, P_3 = Q_1Q_2 \cap \ell, P_3{}' = Q_3Q_4 \cap \ell.$$

The tetrad $O_1 \cup O_2$ is a harmonic (if $P_1, P_2, P_3, P_4$are distinct points, then $P_1$and P_2 separate $P_3$and P_4 harmonically if $\lambda = \frac{(t_1-t_3)(t_2-t_4)}{(t_1-t_4)(t_2-t_3)} = -1$, with t_1, t_2, t_3, t_4 are the coordinates of P_1, P_2, P_3, P_4) and the tetrads $O_1 \cup O_3$, $O_2 \cup O_3$ are neither harmonic nor equianharmonic (if $\lambda = \frac{1}{1-\lambda}$).

The tangents at P_1 and $P_1{}'$ to $\boldsymbol{C}$ meet at R. The lines

$$R_1R,\ R_2R,\ R_3R,\ R_4R,\ R_5R,\ R_6R,\ R_7R,\ R_8R,\ P_1R,\ P_1{}'R;$$

are part of a pencil . However $O_4{}' = \boldsymbol{C} - \{O_2 \cup O_4\}$ is inequivalent to O_4. The other eight lines of the pencil meet $\boldsymbol{C}$ in O_4.

3.13 **Links with Coding Theory**

From Definition (2.5) and (1), there is a natural one-to-one correspondence between linear $[n, k, n-k+1]$ MDS code and n-arcs in $PG(k-1, q)$. In the case that $k = 3$ and $d = n-2$ of an $[n, k, d]$ code, the code C converts to a set K of n points on the projective plane $PG(2, q)$.

The parameters n, k and d for k-arcs in $PG(2, q)$ up to 18 and the number e of errors that can be corrected are given in Table 16.

$(k;2)$-arc	n	k	d	e	$(k;2)$-arc	n	k	d	e
$(4;2)$-arc	4	3	2	1	$(10;2)$-arc	10	3	8	3
$(5;2)$-arc	5	3	3	1	$(11;2)$-arc	11	3	9	4
$(6;2)$-arc	6	3	4	1	$(12;2)$-arc	12	3	10	4
$(7;2)$-arc	7	3	5	2	$(13;2)$-arc	13	3	11	5
$(8;2)$-arc	8	3	6	2	$(14;2)$-arc	14	3	12	5
$(9;2)$-arc	9	3	7	3	$(18;2)$-arc	18	3	16	7

Table 16: The parameters for $(k;2)$-arcs

If C has minimum distance d, then it can detect $d-1$ errors and correct $e = \lfloor (d-1)/2 \rfloor$ errors, where $\lfloor m \rfloor$ denotes the integer part of m:

d	1	2	3	4	5	6	7	8
e	0	0	1	1	2	2	3	3

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حول مستوي الاسقاط لحقل منتهي من الرتبة السابعة عشرة و تطبيقاته لتصحيح أخطاء الرموز

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الخلاصة

هدف البحث هو تصنيف أشكال هندسية تدعى بالأقواس. أن أدوات الحسابات الأساسية هي برمجة الرياضيات بلغة كاب. في المستوي من الرتبة السابعة عشر الأقواس المهمة تدعى بالكاملة والتي لا يمكن أن تكون متزايدة لأكبر قوس. كل الأقواس التي تحتوي الحجم الثامن تم تصنيفها. مثل الأقواس الكاملة من الحجم 10و11و12و13و14. في المستوي من الرتبة السابعة عشر اكبر حجم هو ثمانية عشر. كل هذه الأقواس أعطت تصحيح اكبر عدد ممكن من الأخطاء للرموز من نفس الطول.