Obtaining the suitable k for $(3+2k)-$cycles

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Abstract

We show that, if k is odd. Then the $(3+2k)$-cycles form a single ambivalent conjugacy class in the alternating group A_n for all $n \geq 5+2k$. This generalize to the following result, if $n \geq 5$, then 3-cycles form a single conjugacy class in A_n [see, (1)].

Keywords: alternating groups, conjugacy classes, ambivalent group, permutations, type α.

1. Introduction

If $\Omega = \{1,2,...,n\}$, then S_n and A_n denote the symmetric and alternating groups of permutation on Ω, respectively. Product of two permutations will be executed from left to right. A cycle $(i_1, i_2, ..., i_l)$ is said to have length l or to be an l-cycle [see, (2)]. Suppose, first, that $\beta \in S_n$. Then the cycle type α of a permutation β is the list of integers $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_l$ such that $\alpha_1 + \alpha_2 + \quad ... + \alpha_l = n$, where α_i, for all $(1 \leq i \leq l)$ are just the lengths of the cycles in the disjoint cycle decomposition of β, 1-cycles being including. Thus the type of the permutation β = (3 8 9) (7 2 4) (5 1) in S_{11} is $\alpha = (3,3,2,1,1,1)$. The permutation of a given type α form one conjugacy class C^{α} in the symmetric group S_n, and if this class C^{α} splits into two conjugacy classes of A_n, we denote these by $C^{\alpha\pm}$. Also, A_n is ambivalent group iff each $C^{\alpha\pm}$ of A_n are ambivalent and $C^{\alpha}(\beta)$ splits into two A_n-classes of equal order iff $n > 1$, and the non-zero parts of $\alpha(\beta)$ are different and odd [see, (3)], so in every other case $C^{\alpha}(\beta)$ does not split. The conjugacy classes and ambivalence in alternating group A_n were studied by many mathematicians such as [(4)-(9)], that if $n \geq 5$ and X is the set of all 3-cycles $(i,\ j,\ k) \in A_n$, and $n \geq i \neq j \neq \ \ k \geq 1$. Then X form a single conjugacy class in the alternating group A_n. In this paper we introduced in the first some theorems in these theorems we prove that if $n \geq 7$ or $n \geq 9$ or $n \geq 5+2k$, then 5-cycles or 7-cycles or $(3+2k)$- cycles, respectively form a single conjugacy class in the alternating group A_n where $k \geq 0$. Finally we prove that if k is an odd. Then for all $n \geq 5+2k$ the (3+2k)-cycles form a single ambivalent conjugacy class in the alternating group A_n.

2. Preliminaries

The following definitions have been used to obtain the results and properties developed in this paper.

2.1 Definition (10):

A partition α is a sequence of nonnegative integers $(\alpha_1, \alpha_2, ...)$ with $\alpha_1 \geq \alpha_2 \geq ...$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as $l(\alpha) = Max\{i \in N; \alpha_i \neq 0\}$ and $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$.

We set $\alpha \vdash n = \{\alpha \ \ partition ; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of n and α_i are called the parts of α.

* We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1 \gamma_2 .... \gamma_l$. With γ_i disjoint cycles of length α_i. Since disjoint cycles commute, we can assume that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_l$. Therefore $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{c(\beta)})$ is a partition of n.

2.2 Definition (10): We call the partition α the cycle-type of $\beta \in S_n$.

2.3 Definition (10): Let α be a partition of n. We define $C^{\alpha} \subset S_n$ to be the set of all elements with cycle type α.

* The permutation of a given type α form one conjugacy class C^{α} in the symmetric group S_n, and if this class C^{α} splits into two conjugacy classes of A_n, we denote these by $C^{\alpha \pm}$, so every pair of permutations γ and β are conjugate iff they have the same cycle type. However, this is not necessarily true in an alternating group.

2.4 Theorem (3): Let $\beta \in C^{\alpha}$ in S_n and $n > 1$, then C^{α} splits into two A_n- classes of equal order iff all the parts of the cycle-type of β are different and odd

2.5 Theorem (1): If $n \geq 5$, then 3-cycles form a single conjugacy class in the alternating group A_n.

3. Obtaining the suitable k for $(3+2k)-$cycles

In this section, we show that which the suitable k satisfies the $(3+2k)$-cycle form a single conjugacy class, two conjugacy classes, a single ambivalent conjugacy class, and two ambivalent conjugacy classes in the alternating group A_n for some positive integer *n*.

3.1 Theorem: If $n \geq 7$, then 5-cycles form a single conjugacy class in the alternating group A_n.

Proof:

Let β denote the cycle $(1\ 2\ 3\ 4\ 5)$, and $\gamma = (a_1\ a_2\ a_3\ a_4\ a_5)$, let λ denote the transposition $(6\ 7)$, but β and γ are two permutations have the same type α, so each of them belong to the conjugacy class C^{α} of S_n. Then there is a permutation $\pi \in S_n$

such that $\gamma = \pi\beta\pi^{-1}$. If π is odd, then $\lambda\pi$ is even. We note that $\beta = \lambda\beta\lambda^{-1}$. Therefore $\gamma = \pi(\lambda\beta\lambda^{-1})\pi^{-1} = (\pi\lambda)\beta(\pi\lambda)^{-1}$.We replace π by $\lambda\pi$. Thus there always is an even permutation π such that $\gamma = \pi\beta\pi^{-1}$, which means that γ is in the conjugacy class of β in the alternating group.

3.2 Lemma

If $7 > n \geq 5$, then 5-cycles form two conjugacy classes in A_5 and in A_6.

Proof:

Let X be the set of all 5-cycles $(i,\ j, k, l, t) \in A_n$, for $(n = 5,6)$ where $n \geq i, j, k, l, t \geq 1$ and different. Moreover, for any $\beta \in X = C^{\alpha}$ the permutation β has cycle-type $\alpha = (5)$ in S_5 and $\alpha = (5,1)$ in S_6. Thus C^{α} splits into two conjugacy classes $C^{\alpha\pm}$ of A_n, for $(n = 5, 6)$ [by Theorem 2.4]. Then for all $7 > n \geq 5$ the 5-cycles form two conjugacy classes in A_5 and in A_6.

3.3 Theorem

If $n \geq 9$, then 7-cycles form a single conjugacy class in the alternating group A_n.

Proof:

Let β denote the cycle $(1\ 2\ 3\ 4\ 5\ 6\ 7)$, and $\gamma = (a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7)$, let λ denote the transposition $(8\ 9)$, but β and γ are two permutations have the same type α, so each of them belongs to the conjugacy class C^{α} of S_n, then there is a permutation $\pi \in S_n$ such that $\gamma = \pi\beta\pi^{-1}$. If π is odd, then $\lambda\pi$ is even. We note that $\beta = \lambda\beta\lambda^{-1}$. Therefore $\gamma = \pi(\lambda\beta\lambda^{-1})\pi^{-1} = (\pi\lambda)\beta(\pi\lambda)^{-1}$. We replace π by $\lambda\pi$. Thus there always is an even permutation π such that $\gamma = \pi\beta\pi^{-1}$, which means that γ is in the conjugacy class of β in the alternating group.

3.4 Lemma

If $9 > n \geq 7$, then 7-cycles form two conjugacy classes in A_7 and in A_8.

Proof:

Let X be the set of all 7-cycles $(i,\ j, k, l, t, r, d) \in A_n$, for $(n = 7,8)$ where $n \geq i, j, k, l, t, r, d \geq 1$ and different. Moreover, for any $\beta \in X = C^{\alpha}$ the permutation β has cycle-type $\alpha = (7)$ in S_7 and $\alpha = (7,1)$ in S_8. Thus C^{α} splits into two conjugacy classes $C^{\alpha\pm}$ of A_n, for $(n = 7, 8)$ [by Theorem 2.4]. Then for all $9 > n \geq 7$ the 7-cycles form two conjugacy classes in A_7 and in A_8.

3.5 Theorem

If $k \geq 0$, then for all $n \geq 5 + 2k$, the $(3 + 2k)$-cycles form a single conjugacy class in the alternating group A_n.

Proof:

1) If $k=0$, then by (Theorem 2.5) we have 3-cycles form a single conjugacy class in the alternating group A_n.

2) If $k=1$, then by (Theorem 3.1) we have 5-cycles form a single conjugacy class in the alternating group A_n.

3) If $k=2$, then by (Theorem 3.3) we have 7-cycles form a single conjugacy class in the alternating group A_n.

4) If $k>2$, so for any $n\geq 5+2k$, assume $l=3+2k$, then $n\geq 2+l$. Therefore the transposition $\lambda=(l+1,\ l+2)\in A_n$, let β denote the cycle $(1\ 2 \ldots l)$, and $\gamma=(a_1\ a_2 \ldots a_l)$, However, β and γ are two permutations have the same type α, then there is a permutation $\pi\in S_n$ such that $\gamma=\pi\beta\pi^{-1}$. If π is odd, then $\lambda\pi$ is even. We note that $\beta=\lambda\beta\lambda^{-1}$. Therefore $\gamma=\pi(\lambda\beta\lambda^{-1})\pi^{-1}=(\pi\lambda)\beta(\pi\lambda)^{-1}$.We replace π by $\lambda\pi$. Thus there always is an even permutation π such that $\gamma=\pi\beta\pi^{-1}$, which means that γ is in the conjugacy class of β in the alternating group A_n . Then for all $n\geq 5+2k$, the $(3+2k)$-cycles form a single conjugacy class in the alternating group A_n.

3.6 Lemma

If $5+2k>n\geq 3+2k$, then $(3+2k)$-cycles form two conjugacy classes in A_{3+2k} and in A_{4+2k}.

Proof:

Let X be the set of all $(3+2k)-$cycles $(a_1,,a_2,a_3,\ldots,a_{3+2k})\in S_n$, for $(n=3+2k,\ 4+2k)$ where $n\geq a_i\geq 1$, $(\forall\ 1\leq i\leq 3+2k)$ and different, since for any $k\geq 0$ we have $3+2k$ is odd number . Moreover, for any $\beta\in X=C^{\alpha}$ the permutation β has cycle-type $\alpha=(3+2k)$ in S_{3+2k} and $\alpha=(3+2k,1)$ in S_{4+2k}. Thus C^{α} splits into two conjugacy classes $C^{\alpha\pm}$ of A_n, for $(n=3+2k,\ 4+2k)$ [by Theorem 2.4]. Then, for all $5+2k>n\geq 3+2k$ the $(3+2k)$-cycles form two conjugacy classes in A_{3+2k} and in A_{4+2k}.

3.7 Theorem

If k is odd, then for all $n\geq 5+2k$, the $(3+2k)$-cycles form a single ambivalent conjugacy class in the alternating group A_n.

Proof:

From [Theorem 3.5], we have the $(3+2k)$-cycles form a single conjugacy class in the alternating group A_n. Now we have to prove that for each permutation $\beta=(b_1,b_2,\ldots,b_{3+2k})$ has $(3+2k)$-cycle is conjugate to its inverse in A_n, where $n\geq 5+2k$. Since k odd number $\Rightarrow$ $\dfrac{(3+2k)-1}{2}$ is even number for each k. Let $\mu=(b_2,b_{3+2k})(b_3,b_{(3+2k)-1})(b_4,b_{(3+2k)-2})\ldots\ldots.$ Then we have $\mu\beta\mu^{-1}=\beta^{-1}$. Now we want to show that μ is an even permutation (i.e $\mu\in A_n$), since μ is a composite of $\dfrac{(3+2k)-1}{2}$ (an even number) of transpositions $\Rightarrow \mu \in A_n$. So for each permutation β has $(3+2k)$-cycle is conjugate to its inverse in A_n. Then $(3+2k)$-cycles form a single ambivalent conjugacy class in the alternating group A_n, for each $n \geq 5+2k$ and k odd number.

3.8 Lemma

If k is odd, and $5+2k > n \geq 3+2k$, then $(3+2k)$-cycles form two ambivalent conjugacy classes in A_{3+2k} and in A_{4+2k}.

Proof:

From [Lemma 3.6], we have the $(3+2k)$-cycles form two conjugacy classes $C^{\alpha\pm}$ in A_{3+2k} and in A_{4+2k}. Assume $\beta = (b_1, b_2, ..., b_{3+2k}) \in C^{\alpha+}$ and $\gamma = (a_1, a_2, ..., a_{3+2k}) \in C^{\alpha-}$. Since k odd number $\Rightarrow \dfrac{(3+2k)-1}{2}$ is even number for each k. That means there are two even permutations $\mu, t \in A_n$, for $(n = 3+2k,\ 4+2k)$ which are satisfy that $\mu\beta\mu^{-1} = \beta^{-1}$, and $t\gamma t^{-1} = \gamma^{-1}$, where $\mu = (b_2, b_{3+2k})(b_3, b_{(3+2k)-1})(b_4, b_{(3+2k)-2})$......., and $t = (a_2, a_{3+2k})(a_3, a_{(3+2k)-1})(a_4, a_{(3+2k)-2})$....... Then both of β and γ are conjugate to their inverses in A_n for $(n = 3+2k,\ 4+2k)$. Moreover, let $\lambda \in C^{\alpha+} \Rightarrow \lambda \underset{A_n}{\approx} \beta \Rightarrow \lambda^{-1} \underset{A_n}{\approx} \beta^{-1}$. However $\beta \underset{A_n}{\approx} \beta^{-1}$. Thus $\lambda^{-1} \underset{A_n}{\approx} \beta$, but $\lambda \underset{A_n}{\approx} \beta$, then $\lambda^{-1} \underset{A_n}{\approx} \lambda$. That means for any class in any group to show this class is ambivalent we need only to find one element belongs to this class and conjugate to its inverse. Thus the conjugacy class $C^{\alpha+}$ of A_n is ambivalent class, and similarity $C^{\alpha-}$ is ambivalent class. Then for all $5+2k > n \geq 3+2k$, the $(3+2k)$-cycles form two ambivalent conjugacy classes in A_{3+2k} and in A_{4+2k}.

4. Concluding Remarks

Suppose that $\beta \in S_n$ and $\beta = \pi_1\pi_2$, where π_1, π_2 are disjoint cycles in S_n of lengths $(3+2k)$ and l respectively. The results of our research can be summarized as follows:

1) If $l = 1$, then A_{4+2k} has two conjugacy classes corresponding to the partition $(3+2k, 1) \vdash (4+2k)$.

2) If k is odd, and $l = 1$, then A_{4+2k} has two ambivalent conjugacy classes corresponding to the partition $(3+2k, 1) \vdash (4+2k)$.

3) If $l = 3+2k$, then A_{2l} has a single ambivalent conjugacy classes corresponding to the partition $(l, l) \vdash 2l$.

The first question we are concerned with is: what is the possible value of l provided that A_{2l+1} with no conjugacy classes corresponding to the partition $(3+2k, l) \vdash (2l+1)$? The answer to this question is that $l = 4+2k$. In another direction, let $\beta = \pi_1\pi_2...\pi_t$, where $\{\pi_i\}_{i=1}^{t}$

are disjoint cycles in S_n of lengths $\{l_i\}_{i=1}^{t}$ respectively. So the second question we are concerned with is: what are the possible values of $\{l_i\}_{i=2}^{t}$ provided that A_n has a single ambivalent conjugacy classes corresponding to the partition $(l_1, l_2, ..., l_t) \vdash n$, where $l_1 = (3+2k)$ and $n = \sum_{i=1}^{t} l_i$?

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الحصول على k المناسبة للدورات – $(3+2k)$

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الخلاصة

بينا في هذا البحث على انه اذا كان k عدد أولي فأن مجموعة التباديل ذات ال– $(3+2k)$ (cycles) في الزمر المتناوبة A_n تشكل صف متغاير أحادي في A_n لكل $n \geq 5+2k$ ، حيث يعتبر هذا تعميم إلى نظرية سابقة و التي تنص على ان مجموعة التباديل ذات ال– 3 (cycles) في الزمر المتناوبة A_n تشكل صف أحادي في A_n لكل $n \geq 5$ ، كما قدمنا عدة نظريات أخرى.