Using The Steady-State Approximation to Find The Axial Compression Of Tall Solid Cylinders With a Small Friction

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Abstract

In this paper the steady-state approximation (SSA) method has been applied to determine an effect of small friction on deformation of tall solid cylinders in the axial compression. Where a first order solution is represent compression of a solid cylinder. The zero order solution (with no friction) is trivial and can be obtained from a general law with no difficulty.

Keyword: Steady-State Approximation method and mathematical model.

1. Introduction

The steady-state approximation (SSA) has been applied in axisymmetric compression of a solid cylinder , typically argue that some of the intermediates are highly reactive, so that they are removed as fast as they are made. Then set the corresponding rates of change to zero. What we are saying is not that these rates are identically zero, of course, but that they are much smaller than the other rates of reaction. The steady-state approximation is often surprisingly accurate, but the classical arguments lack rigor and don't lead to any satisfying mathematical generalization which can be used to improve on the approximation. There is mathematical model which build on the (SSA) and provide the required generalization .Slow manifold theory: The curves or surfaces which arise from the (SSA) can in fact be understood as approximations to slow invariant manifolds of the differential equations .To understand the (SSA), we will find that we are led to problems in which the small parameter multiplies a derivative. [1]

Axial and plane-strain compression tests are widely used for the determination of stress-strain data, ductile fracture conditions and friction laws. In the case of the determination of stress-strain curves, one of the principal limitations of the test is caused by friction . Also, one of the ideal basic tests for determining workability diagram assumes the axial compression with no friction [2]. In such conditions, the friction stress is very small. For example, assuming Coulomb's frictional law the coefficient of friction has been evaluated to less than 0.01 or even 0.001. [3]

2. Formulating the Mathematical Model

Consider a solid cylinder of initial radius R_0 and the height $2H_0$ subject to compression between two parallel, well-lubricated plates, and introduce a cylindrical coordinate system (Fig.1a). The side surface of the cylinder is stress free. The radial and axial components of the velocity will be denoted by u_r and u_z, respectively, and the non-zero components of the stress tensor in the cylindrical coordinates by σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, and σ_{rz}. The plates move with a velocity of the magnitude u_0 which may depend on the

current half-height of the cylinder, H. Because of symmetry, it is sufficient to find the solution at $z \geq 0$.(4)
Obviously,

$$\frac{dH}{dt} = -u_0 \qquad (1)$$

where t is the time, and

$$u_z = -u_0 \qquad (2)$$

at $z = H$. At the axis of symmetry, r = 0,

$$u_r = 0 \quad \text{and} \quad \sigma_{rz} = 0 \qquad (3)$$

Also, at the plane of symmetry, z = 0,

$$u_z = 0 \quad \text{and} \quad \sigma_{rz} = 0 \qquad (4)$$

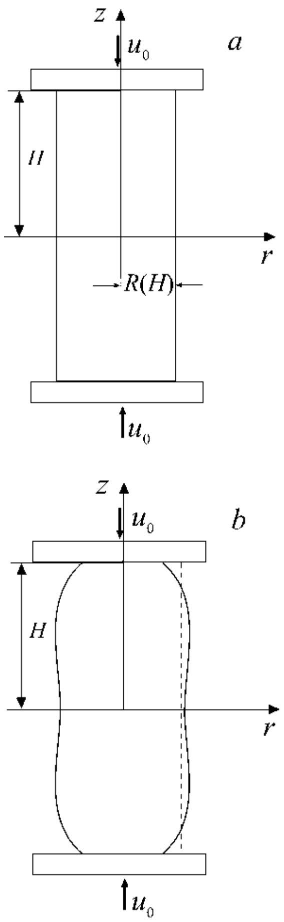

Figure 1: shapes of cylinders compressed without (a) and with (b) friction

The boundary conditions Eqns. (2) through (4) are valid independently of barreling of the free surface. Barreling develops because of friction. It is supposed in the present paper that the friction stress, τ_f, is small and is given by

$$\sigma_{rz} = -\tau_f = \delta\varphi(r,H), \qquad \delta << 1 \qquad (5)$$

at z = H. In general, the function φ depends on physical quantities such as the tangent velocity at the friction surface or the normal stress acting on this surface. However, it will be seen later that for the case under consideration it can be represented in the form of (5). If $\delta \neq 0$ barreling develops (Fig.1b) and the conditions at the free boundary are:

$$\sigma_n = 0 \quad \text{and} \quad \tau_n = 0 \qquad (6)$$

where σ_n and τ_n are the components of the traction vector. In the case of deformation with no barreling, Eqn. (6) reduces to

$$\sigma_{rr} = 0 \quad \text{and} \quad \sigma_{rz} = 0 \qquad (7)$$

at $r = R$, where $R = R(H)$ is the current radius of the cylinder. The shape of the free surface can be described as

$$r = R + \delta\psi\left(z, H\right) \qquad (8)$$

A direct problem consists of prescribing the function φ involved in Eqn. (5). Then, the function ψ involved in (8) should be found from the solution. A great difficulty here is that the function φ is in fact unknown since the direct measurement of the friction stress is very difficult and many methods of experimental determination of the friction stress are based of measurements of geometric parameters . On the other hand, the current shape of the free surface can be found experimentally with a high accuracy (4). a general solid constitutive law has the following form :

$$\xi_{ij} = \lambda s_{ij}\,, \qquad (9)$$

$$s_{ij}s_{ij} = \frac{2}{3}\sigma_Y^2 \qquad (10)$$

where ξ_{ij} are the components of the strain rate tensor, s_{ij} are the components of the stress deviator tensor, λ is non-negative multiplier, and σ_Y is the tensile yield stress. σ_Y may depend on the equivalent strain rate ξ_{eq}, the equivalent strain ε_{eq} defined respectively by

$$\xi_{eq}=\sqrt{\frac{2}{3}\xi_{ij}\xi_{ij}}\,,\qquad \frac{d\varepsilon_{eq}}{dt}=\xi_{eq}\,, \tag{11}$$

and other internal variables. Equation (9) includes the incompressibility equation. Equations (9) and (10) should be complemented with the equilibrium equations. In the cylindrical coordinates, the non-trivial equations have the following form

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{zr}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)=0\,,$$

$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{r}\sigma_{rz}=0\,. \tag{12}$$

3. Zero Order Solution

The zero order solution, $\delta=0$, is trivial for the constitutive equations in their general form. The stress distribution is given by

$$\bar{\sigma}_{rz}=\bar{\sigma}_{rr}=\bar{\sigma}_{\theta\theta}=0 \quad \text{and} \quad \bar{\sigma}_{zz}=-\sigma_Y \tag{13}$$

This representation satisfies the stress boundary conditions of Eqns. (3) and (4), the condition (5) at $\delta=0$, and the boundary conditions (7). Equation (10) is also satisfied. The distribution (13) is compatible with the equilibrium equations (12) if σ_Y is independent of *z*. Assume the velocity field in the form:

$$\bar{u}_z=-u_0\frac{z}{H} \quad \text{and} \quad \bar{u}_r=\frac{ru_0}{2H} \tag{14}$$

Then, the components of the strain rate tensor are :

$$\bar{\xi}_{rr}=\bar{\xi}_{\theta\theta}=\frac{u_0}{2H}\,,\quad \bar{\xi}_{zz}=-\frac{u_0}{H}\,,\quad \bar{\xi}_{rz}=0\,. \tag{15}$$

It follows from Eqn. (15) the incompressibility equation is satisfied and from Eqns. (15) and (11) that $\bar{\xi}_{eq}$ and $\bar{\varepsilon}_{eq}$ are independent of the space coordinates. The latter means that σ_Y is independent of the space coordinates and thus the equilibrium equations are satisfied. It is possible to verify by substitution of Eqns. (13) and (15) into (9) that the latter is also satisfied. Using (10) and the solution (15) the multiplier in the flow rule (9) is determined in the form:

$$\bar{\lambda}=\frac{3u_0}{2H\sigma_Y(\bar{\xi}_{eq},\bar{\varepsilon}_{eq})}\,. \tag{16}$$

4. First Order Approximation

The first order approximation is restricted by:

$$\sigma_Y(\xi_{eq},\varepsilon_{eq})=\sigma_0=const\,. \tag{17}$$

All values of the order $O(\delta)$ will be noticed by upper tilde. Thus, the flow rule and incompressibility equation take now the form:

$$\tilde{\xi}_{rz}=\bar{\lambda}\tilde{s}_{rz}\,,\quad \tilde{\xi}_r-\tilde{\xi}_\theta=\bar{\lambda}(\tilde{s}_r-\tilde{s}_\theta)\,,$$
$$\tilde{\xi}_z+2\tilde{\xi}_r=\bar{\lambda}(\tilde{s}_z+2\tilde{s}_r)\,,$$
$$\tilde{\xi}_r+\tilde{\xi}_z+\tilde{\xi}_\theta=0\,. \tag{18}$$

The yield criterion (10) in the first approximation gives:

$$\tilde{s}_r+\tilde{s}_\theta-2\tilde{s}_z=0\,. \tag{19}$$

As a result, one can obtain the following linear relationships between the components of the stress deviator and strain rate tensor:

$$2\bar{\lambda}\tilde{s}_r=-2\bar{\lambda}\tilde{s}_\theta=\tilde{\xi}_r-\tilde{\xi}_\theta\,,\ \ \tilde{\mathbf{s}}_z=\mathbf{0}\,. \tag{20}$$

Substituting the aforementioned formulae in the equilibrium conditions one can obtain, after some algebra, the equation in terms of strain rate components:

$$\left(\frac{1}{2}\frac{\partial^2}{\partial r\partial z}+\frac{1}{r}\frac{\partial}{\partial z}\right)\left(\tilde{\xi}_r-\tilde{\xi}_\theta\right)+\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial r^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)\tilde{\xi}_{zr}=0. \tag{21}$$

At the top of the specimen, the boundary conditions follow from (2), (14) and (5):

$$\tilde{u}_z(r,z)\big|_{z=H}=0,$$
$$\tilde{s}_{rz}(r,z)\big|_{z=H}=\varphi(r,H), \tag{22}$$

At the symmetry plane, $z=0$, and the symmetry axis, $r=0$, the respective boundary conditions (3) and (4) transform to:

$$\tilde{u}_z(r,z)\big|_{z=0}=0, \qquad \tilde{s}_{rz}(r,z)\big|_{z=0}=0.$$

and:

$$\tilde{u}_r(r,z)\big|_{r=0}=0, \qquad \tilde{s}_{rz}(r,z)\big|_{r=0}=0.$$

Finally, the free boundary conditions (6) can be written in the main terms as:

$$\tilde{\sigma}+\tilde{s}_r\big|_{r=\overline{R}}=0, \tag{25}$$

$$\tilde{s}_{rz}\big|_{r=\overline{R}}=\overline{\sigma}_z\psi'_z(z,H), \tag{26}$$

where $\tilde{\sigma}$ is the first order approximation for pressure. Note that the boundary condition (25) has to be considered as the initial condition in order to solve Cauchy problem for the following equilibrium equation:

$$\frac{\partial}{\partial r}\overline{\lambda}\tilde{\sigma}=\frac{\partial}{\partial r}\overline{\lambda}\tilde{s}_z-\frac{1}{r}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\tilde{u}_r-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\tilde{u}_z}{\partial r}+\frac{\partial\tilde{u}_r}{\partial z}\right), \tag{27}$$

when the first term approximation for the velocities, strain rate tensor and the stress deviator is found, so that the right-hand side of (27) is known. Note also that the function $\psi(z,H)$ satisfies the following additional condition due to material incompressibility:

$$\int_0^H \psi(z,H)dz=0. \tag{28}$$

To solve the boundary value problem (21) – (24), Eqn. (21) should be transformed, taking into account incompressibility condition:

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\tilde{u}_r+\frac{\partial}{\partial z}\tilde{u}_z=0, \tag{29}$$

to the form:

$$\frac{\partial^4}{\partial z^4}\tilde{u}_r+\frac{\partial^2}{\partial z^2}M\tilde{u}_r+M^2\tilde{u}_r=0, \tag{30}$$

Here we have introduced a new notation for the differential operator M by the formula:

$$M=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}. \qquad (23) \tag{31}$$

Incorporating a well known property of the Bessel functions ,

(24)

$MJ_1(mr)=-m^2J_1(mr)$, it is natural to seek for the solution to the problem by Fourier decomposition method in form:

$$\tilde{u}_r(r,z)=J_1(mr)U_r(z). \tag{32}$$

For the function $U_r(z)$ one immediately obtains the following partial differential equation:

$$\frac{\partial^4}{\partial z^4}U_r-m^2\frac{\partial^2}{\partial z^2}U_r+m^4U_r=0, \tag{33}$$

which has four linearly independent solutions of the form $U_r(z)=e^{az}$. Here parameters $a_j(m)=m\alpha_j+im\beta_j$, $j=1,...,4$ can be easily calculated:

$$a_1=m\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right),\ a_2=m\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right),$$

$$a_3=m\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right),$$

$$a_4=m\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right).$$

As a result, the solution to equation (24) has the form:

$$\tilde{u}_r(r,z) = J_1(mr)\left[\begin{array}{l} e^{\frac{mz\sqrt{3}}{2}}\left(C_1 \cos\frac{mz}{2} + C_2 \sin\frac{mz}{2}\right) \\ + e^{-\frac{mz\sqrt{3}}{2}}\left(C_3 \cos\frac{mz}{2} + C_4 \sin\frac{mz}{2}\right) \end{array}\right].$$

Equation (29) together with boundary conditions $(22)_1$ and $(23)_1$ allows us to find the velocity component:

$$\tilde{u}_z(r,z) = F_k(r)\left[\begin{array}{l} e^{\frac{m_k z\sqrt{3}}{2}} D_1^{(k)} \sin\frac{m_k z}{2} \\ + e^{-\frac{m_k z\sqrt{3}}{2}} D_2^{(k)} \sin\frac{m_k z}{2} \end{array}\right] \quad (34)$$

where

$$m_k = \frac{2\pi k}{H}, \quad k = 1,2,.... \quad (35)$$

Incorporating all other homogeneous boundary conditions (22) - (24) one can finally obtain the solution to the problem in terms of velocities:

$$\tilde{u}_z(r,z) = -\sum_{k=1}^{\infty} D_k \cosh\left(\frac{m_k z\sqrt{3}}{2}\right), \quad (36)$$
$$* \sin\left(\frac{m_k z}{2}\right)\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) J_1(m_k r)$$

$$\tilde{u}_r(r,z) = \sum_{k=1}^{\infty} \frac{m_k}{2} D_k \left(\begin{array}{l} \cosh\left(\frac{m_k z\sqrt{3}}{2}\right) \\ * \cos\left(\frac{m_k z}{2}\right) \\ + \sqrt{3}\sinh\left(\frac{m_k z\sqrt{3}}{2}\right) \\ * \sin\frac{m_k z}{2} \end{array}\right) J_1(m_k r).$$

where unknown up to now constants D_k have to be found from the given friction law $(22)_2$ using the following representation for the shear stress:

$$\tilde{s}_{rz} = \frac{1}{2\bar{\lambda}} \sum_{k=1}^{\infty} \frac{m_k^2\sqrt{3}}{2} D_k \left(\begin{array}{l} \sqrt{3}\cosh\left(\frac{m_k z\sqrt{3}}{2}\right) \\ * \sin\left(\frac{m_k z}{2}\right) \\ + \sinh\left(\frac{m_k z\sqrt{3}}{2}\right) \\ * \cos\frac{m_k z}{2} \end{array}\right) J_1(m_k r)$$

or in term of the given function φ:

$$\varphi(r,H) = \frac{\sqrt{3}}{4\bar{\lambda}(H)} \sum_{k=1}^{\infty} (-1)^k m_k^2 D_k$$
$$* \quad \sinh\left(\frac{m_k H\sqrt{3}}{2}\right) J_1(m_k r)$$

When the constants D_k are calculated, the deviation of the compression can be found from (26) with the use of (28):

$$\psi(z,H) = -\frac{1}{2u_0\sqrt{3}} \sum_{k=1}^{\infty} (-1)^k D_k J_1(m_k \bar{R}(H))$$
$$* \sinh\left(\frac{m_k H\sqrt{3}}{2}\right) - \frac{H}{u_0\sqrt{3}} \sum_{k=1}^{\infty} m_k D_k$$
$$* \sinh\left(\frac{m_k z\sqrt{3}}{2}\right) \sin\left(\frac{m_k z}{2}\right) J_1(m_k \bar{R}(H))$$

5.Conclusion

The steady-state approximation (SSA) method has been applied to account the axial compression of tall solid cylinders with small friction . From through the application of this technique on the problem appeared easy, effective and accurate in the calculation of the compression with a small friction.

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استخدام تقريب الحالة الثابتة لإيجاد الضغط المحوري للاسطوانات الصلبة الطويلة مع وجود الاحتكاك الصغير

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الخلاصة

في هذا البحث طبقت طريقة تقريب الحالة الثابتة لتحديد مدى تأثير الاحتكاك الصغير على تشويه الاسطوانات الصلبة الطويلة عند الضغط المحوري ، حيث ان حل الرتبة الاولى للنظام يمثل الضغط المحوري للأسطوانة الصلبة وحل الرتبة الصفرية (بدون احتكاك) يعتبر حل تافه حيث يمكن الحصول عليه من القانون العام بدون صعوبة .